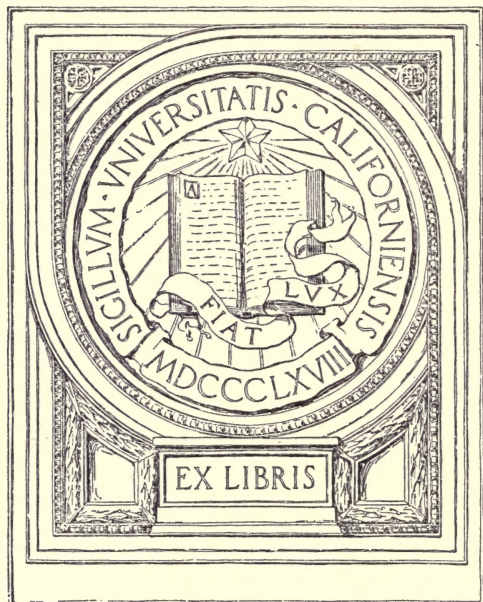


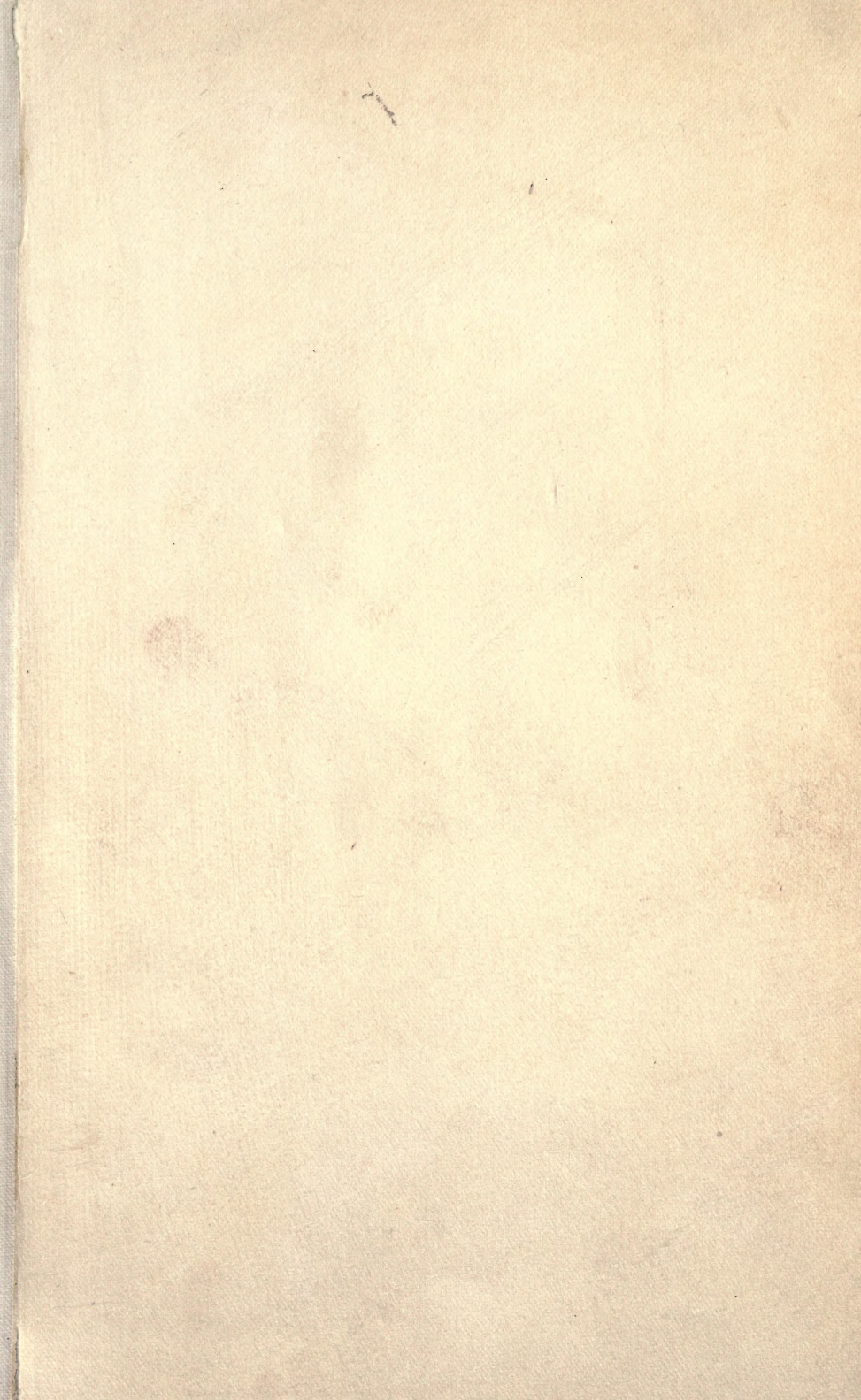
~~1000~~
600

June 29, 1924

GIFT OF
John S. Prell



EX LIBRIS



THEORETICAL ASTRONOMY

RELATING TO THE

MOTIONS OF THE HEAVENLY BODIES

REVOLVING AROUND THE SUN IN ACCORDANCE WITH
THE LAW OF UNIVERSAL GRAVITATION

EMBRACING

A SYSTEMATIC DERIVATION OF THE FORMULÆ FOR THE CALCULATION OF THE GEOCENTRIC AND HELIO-
CENTRIC PLACES, FOR THE DETERMINATION OF THE ORBITS OF PLANETS AND COMETS, FOR
THE CORRECTION OF APPROXIMATE ELEMENTS, AND FOR THE COMPUTATION OF
SPECIAL PERTURBATIONS; TOGETHER WITH THE THEORY OF THE COMBI-
NATION OF OBSERVATIONS AND THE METHOD OF LEAST SQUARES.

With Numerical Examples and Auxiliary Tables

JOHN S. PRELL

Civil & Mechanical Engineer.

SAN FRANCISCO, CAL.

BY

JAMES C. WATSON

DIRECTOR OF THE OBSERVATORY AT ANN ARBOR, AND PROFESSOR OF ASTRONOMY IN THE
UNIVERSITY OF MICHIGAN

PHILADELPHIA:

J. B. LIPPINCOTT COMPANY.

LONDON: 36 SOUTHAMPTON STREET, COVENT GARDEN.

1900.

Entered, according to Act of Congress, in the year 1868

J. B. LIPPINCOTT & CO.,

in the Clerk's Office of the District Court of the United States for the Eastern District
of Pennsylvania.

Copyright, 1896, by ANNETTE H. WATSON.

Adm Lib - MC

GIFT

Pres

QB355
W3
1896

JOHN S. PRELL
Civil & Mechanical Engineer.
SAN FRANCISCO, CAL.

P R E F A C E.

THE discovery of the great law of nature, the law of gravitation, by NEWTON, prepared the way for the brilliant achievements which have distinguished the history of astronomical science. A first essential, however, to the solution of those recondite problems which were to exhibit the effect of the mutual attraction of the bodies of our system, was the development of the infinitesimal calculus; and the labors of those who devoted themselves to pure analysis have contributed a most important part in the attainment of the high degree of perfection which characterizes the results of astronomical investigations. Of the earlier efforts to develop the great results following from the law of gravitation, those of EULER stand pre-eminent, and the memoirs which he published have, in reality, furnished the germ of all subsequent investigations in celestial mechanics. In this connection also the names of BERNOULLI, CLAIRAUT, and D'ALEMBERT deserve the most honorable mention as having contributed also, in a high degree, to give direction to the investigations which were to unfold so many mysteries of nature. By means of the researches thus inaugurated, the great problems of mechanics were successfully solved, many beautiful theorems relating to the planetary motions demonstrated, and many useful formulæ developed.

It is true, however, that in the early stage of the science methods were developed which have since been found to be impracticable, even if not erroneous; still, enough was effected to direct attention in the proper channel, and to prepare the way for the more complete labors of LAGRANGE and LAPLACE. The genius and the analytical skill of these extraordinary men gave to the progress of Theoretical Astronomy the most rapid strides; and the intricate investigations which they successfully performed, served constantly to educe new discoveries, so that of all the problems relating to the mutual attraction of the several planets

but little more remained to be accomplished by their successors than to develop and simplify the methods which they made known, and to introduce such modifications as should be indicated by experience or rendered possible by the latest discoveries in the domain of pure analysis.

The problem of determining the elements of the orbit of a comet moving in a parabola, by means of observed places, which had been considered by NEWTON, EULER, BOSCOVICH, LAMBERT, and others, received from LAGRANGE and LAPLACE the most careful consideration in the light of all that had been previously done. The solution given by the former is analytically complete, but far from being practically complete; that given by the latter is especially simple and practical so far as regards the labor of computation; but the results obtained by it are so affected by the unavoidable errors of observation as to be often little more than rude approximations. The method which was found to answer best in actual practice, was that proposed by OLBERS in his work entitled *Leichteste und bequemste Methode die Bahn eines Cometen zu berechnen*, in which, by making use of a beautiful theorem of parabolic motion demonstrated by EULER and also by LAMBERT, and by adopting a method of trial and error in the numerical solution of certain equations, he was enabled to effect a solution which could be performed with remarkable ease. The accuracy of the results obtained by OLBERS's method, and the facility of its application, directed the attention of LEGENDRE, IVORY, GAUSS, and ENCKE to this subject, and by them the method was extended and generalized, and rendered applicable in the exceptional cases in which the other methods failed.

It should be observed, however, that the knowledge of one element, the eccentricity, greatly facilitated the solution; and, although elliptic elements had been computed for some of the comets, the first hypothesis was that of parabolic motion, so that the subsequent process required simply the determination of the corrections to be applied to these elements in order to satisfy the observations. The more difficult problem of determining all the elements of planetary motion directly from three observed places, remained unsolved until the discovery of *Ceres* by PIAZZI in 1801, by which the attention of GAUSS was directed to this subject, the result of which was the subsequent publication of his *Theoria Motus Corporum Cœlestium*, a most able work, in which he gave to the world, in a finished form, the results of many years of attention

to the subject of which it treats. His method for determining all the elements directly from given observed places, as given in the *Theoria Motus*, and as subsequently given in a revised form by ENCKE, leaves scarcely any thing to be desired on this topic. In the same work he gave the first explanation of the method of least squares, a method which has been of inestimable service in investigations depending on observed data.

The discovery of the minor planets directed attention also to the methods of determining their perturbations, since those applied in the case of the major planets were found to be inapplicable. For a long time astronomers were content simply to compute the special perturbations of these bodies from epoch to epoch, and it was not until the commencement of the brilliant researches by HANSEN that serious hopes were entertained of being able to compute successfully the general perturbations of these bodies. By devising an entirely new mode of considering the perturbations, namely, by determining what may be called the perturbations of the time, and thus passing from the undisturbed place to the disturbed place, and by other ingenious analytical and mechanical devices, he succeeded in effecting a solution of this most difficult problem, and his latest works contain all the formulæ which are required for the cases actually occurring. The refined and difficult analysis and the laborious calculations involved were such that, even after HANSEN's methods were made known, astronomers still adhered to the method of special perturbations by the variation of constants as developed by LAGRANGE.

The discovery of *Astræa* by HENCKE was speedily followed by the discovery of other planets, and fortunately indeed it so happened that the subject of special perturbations was to receive a new improvement. The discovery by BOND and ENCKE of a method by which we determine at once the variations of the rectangular co-ordinates of the disturbed body by integrating the fundamental equations of motion by means of mechanical quadrature, directed the attention of HANSEN to this phase of the problem, and soon after he gave formulæ for the determination of the perturbations of the latitude, the mean anomaly, and the logarithm of the radius-vector, which are exceedingly convenient in the process of integration, and which have been found to give the most satisfactory results. The formulæ for the perturbations of the latitude,

true longitude, and radius-vector, to be integrated in the same manner, were afterwards given by BRÜNNOW.

Having thus stated briefly a few historical facts relating to the problems of theoretical astronomy, I proceed to a statement of the object of this work. The discovery of so many planets and comets has furnished a wide field for exercise in the calculations relating to their motions, and it has occurred to me that a work which should contain a development of all the formulæ required in determining the orbits of the heavenly bodies directly from given observed places, and in correcting these orbits by means of more extended discussions of series of observations, including also the determination of the perturbations, together with a complete collection of auxiliary tables, and also such practical directions as might guide the inexperienced computer, might add very materially to the progress of the science by attracting the attention of a greater number of competent computers. Having carefully read the works of the great masters, my plan was to prepare a complete work on this subject, commencing with the fundamental principles of dynamics, and systematically treating, from one point of view, all the problems presented. The scope and the arrangement of the work will be best understood after an examination of its contents; and let it suffice to add that I have endeavored to keep constantly in view the wants of the computer, providing for the exceptional cases as they occur, and giving all the formulæ which appeared to me to be best adapted to the problems under consideration. I have not thought it worth while to trace out the geometrical signification of many of the auxiliary quantities introduced. Those who are curious in such matters may readily derive many beautiful theorems from a consideration of the relations of some of these auxiliaries. For convenience, the formulæ are numbered consecutively through each chapter, and the references to those of a preceding chapter are defined by adding a subscript figure denoting the number of the chapter.

Besides having read the works of those who have given special attention to these problems, I have consulted the *Astronomische Nachrichten*, the *Astronomical Journal*, and other astronomical periodicals, in which is to be found much valuable information resulting from the experience of those who have been or are now actively engaged in astronomical pursuits. I must also express my obligations to the publishers,

Messrs. J. B. LIPPINCOTT & Co., for the generous interest which they have manifested in the publication of the work, and also to Dr. B. A. GOULD, of Cambridge, Mass., and to Dr. OPPOLZER, of Vienna, for valuable suggestions.

For the determination of the time from the perihelion and of the true anomaly in very eccentric orbits I have given the method proposed by BESSEL in the *Monatliche Correspondenz*, vol. xii.,—the tables for which were subsequently given by BRÜNNOW in his *Astronomical Notices*,—and also the method proposed by GAUSS, but in a more convenient form. For obvious reasons, I have given the solution for the special case of parabolic motion before completing the solution of the general problem of finding all of the elements of the orbit by means of three observed places. The differential formulæ and the other formulæ for correcting approximate elements are given in a form convenient for application, and the formulæ for finding the chord or the time of describing the subtended arc of the orbit, in the case of very eccentric orbits, will be found very convenient in practice.

I have given a pretty full development of the application of the theory of probabilities to the combination of observations, endeavoring to direct the attention of the reader, as far as possible, to the sources of error to be apprehended and to the most advantageous method of treating the problem so as to eliminate the effects of these errors. For the rejection of doubtful observations, according to theoretical considerations, I have given the simple formula, suggested by CHAUVENET, which follows directly from the fundamental equations for the probability of errors, and which will answer for the purposes here required as well as the more complete criterion proposed by PEIRCE. In the chapter devoted to the theory of special perturbations I have taken particular pains to develop the whole subject in a complete and practical form, keeping constantly in view the requirements for accurate and convenient numerical application. The time is adopted as the independent variable in the determination of the perturbations of the elements directly, since experience has established the convenience of this form; and should it be desired to change the independent variable and to use the differential coefficients with respect to the eccentric anomaly, the equations between this function and the mean motion will enable us to effect readily the required transformation.

The numerical examples involve data derived from actual observations, and care has been taken to make them complete in every respect, so as to serve as a guide to the efforts of those not familiar with these calculations; and when different fundamental planes are spoken of, it is presumed that the reader is familiar with the elements of spherical astronomy, so that it is unnecessary to state, in all cases, whether the centre of the sphere is taken at the centre of the earth, or at any other point in space.

The preparation of the Tables has cost me a great amount of labor, logarithms of ten decimals being employed in order to be sure of the last decimal given. Several of those in previous use have been recomputed and extended, and others here given for the first time have been prepared with special care. The adopted value of the constant of the solar attraction is that given by GAUSS, which, as will appear, is not accurately in accordance with the adoption of the mean distance of the earth from the sun as the unit of space; but until the absolute value of the earth's mean motion is known, it is best, for the sake of uniformity and accuracy, to retain GAUSS's constant.

The preparation of this work has been effected amid many interruptions, and with other labors constantly pressing me, by which the progress of its publication has been somewhat delayed, even since the stereotyping was commenced, so that in some cases I have been anticipated in the publication of formulæ which would have here appeared for the first time. I have, however, endeavored to perform conscientiously the self-imposed task, seeking always to secure a logical sequence in the development of the formulæ, to preserve uniformity and elegance in the notation, and to elucidate the successive steps in the analysis, so that the work may be read by those who, possessing a respectable mathematical education, desire to be informed of the means by which astronomers are enabled to arrive at so many grand results connected with the motions of the heavenly bodies, and by which the grandeur and sublimity of creation are unveiled. The labor of the preparation of the work will have been fully repaid if it shall be the means of directing a more general attention to the study of the wonderful mechanism of the heavens, the contemplation of which must ever serve to impress upon the mind the reality of the perfection of the OMNIPOTENT, the LIVING GOD!

CONTENTS.

THEORETICAL ASTRONOMY.

CHAPTER I.

INVESTIGATION OF THE FUNDAMENTAL EQUATIONS OF MOTION, AND OF THE FORMULÆ FOR DETERMINING, FROM KNOWN ELEMENTS, THE HELIOCENTRIC AND GEOCENTRIC PLACES OF A HEAVENLY BODY, ADAPTED TO NUMERICAL COMPUTATION FOR CASES OF ANY ECCENTRICITY WHATEVER.

| | PAGE |
|--|------|
| Fundamental Principles | 15 |
| Attraction of Spheres..... | 19 |
| Motions of a System of Bodies..... | 23 |
| Invariable Plane of the System..... | 29 |
| Motion of a Solid Body..... | 31 |
| The Units of Space, Time, and Mass..... | 36 |
| Motion of a Body relative to the Sun..... | 38 |
| Equations for Undisturbed Motion | 42 |
| Determination of the Attractive Force of the Sun | 49 |
| Determination of the Place in an Elliptic Orbit | 53 |
| Determination of the Place in a Parabolic Orbit..... | 59 |
| Determination of the Place in a Hyperbolic Orbit | 65 |
| Methods for finding the True Anomaly and the Time from the Perihelion in the case of Orbits of Great Eccentricity | 70 |
| Determination of the Position in Space..... | 81 |
| Heliocentric Longitude and Latitude..... | 83 |
| Reduction to the Ecliptic | 85 |
| Geocentric Longitude and Latitude..... | 86 |
| Transformation of Spherical Co-ordinates..... | 87 |
| Direct Determination of the Geocentric Right Ascension and Declination..... | 90 |
| Reduction of the Elements from one Epoch to another | 99 |
| Numerical Examples | 103 |
| Interpolation | 112 |
| Time of Opposition | 114 |

CHAPTER II.

INVESTIGATION OF THE DIFFERENTIAL FORMULÆ WHICH EXPRESS THE RELATION BETWEEN THE GEOCENTRIC OR HELIOCENTRIC PLACES OF A HEAVENLY BODY AND THE VARIATIONS OF THE ELEMENTS OF ITS ORBIT.

| | PAGE |
|--|------|
| Variation of the Right Ascension and Declination..... | 118 |
| Case of Parabolic Motion | 125 |
| Case of Hyperbolic Motion..... | 128 |
| Case of Orbits differing but little from the Parabola..... | 130 |
| Numerical Examples..... | 135 |
| Variation of the Longitude and Latitude..... | 143 |
| The Elements referred to the same Fundamental Plane as the Geocentric Places | 149 |
| Numerical Example..... | 150 |
| Plane of the Orbit taken as the Fundamental Plane to which the Geocentric Places are referred | 153 |
| Numerical Example..... | 159 |
| Variation of the Auxiliaries for the Equator..... | 163 |

CHAPTER III.

INVESTIGATION OF FORMULÆ FOR COMPUTING THE ORBIT OF A COMET MOVING IN A PARABOLA, AND FOR CORRECTING APPROXIMATE ELEMENTS BY THE VARIATION OF THE GEOCENTRIC DISTANCE.

| | |
|--|-----|
| Correction of the Observations for Parallax..... | 167 |
| Fundamental Equations..... | 169 |
| Particular Cases..... | 172 |
| Ratio of Two Curtate Distances..... | 178 |
| Determination of the Curtate Distances..... | 181 |
| Relation between Two Radii-Vectores, the Chord joining their Extremities, and the Time of describing the Parabolic Arc..... | 184 |
| Determination of the Node and Inclination..... | 192 |
| Perihelion Distance and Longitude of the Perihelion | 194 |
| Time of Perihelion Passage | 195 |
| Numerical Example..... | 199 |
| Correction of Approximate Elements by varying the Geocentric Distance | 208 |
| Numerical Example..... | 213 |

CHAPTER IV.

DETERMINATION, FROM THREE COMPLETE OBSERVATIONS, OF THE ELEMENTS OF THE ORBIT OF A HEAVENLY BODY, INCLUDING THE ECCENTRICITY OR FORM OF THE CONIC SECTION.

| | |
|-------------------------------|-----|
| Reduction of the Data..... | 220 |
| Corrections for Parallax..... | 223 |

| | PAGE |
|--|------|
| Fundamental Equations..... | 225 |
| Formulæ for the Curtate Distances | 228 |
| Modification of the Formulæ in Particular Cases..... | 231 |
| Determination of the Curtate Distance for the Middle Observation..... | 236 |
| Case of a Double Solution..... | 239 |
| Position indicated by the Curvature of the Observed Path of the Body | 242 |
| Formulæ for a Second Approximation..... | 243 |
| Formulæ for finding the Ratio of the Sector to the Triangle..... | 247 |
| Final Correction for Aberration..... | 257 |
| Determination of the Elements of the Orbit..... | 259 |
| Numerical Example..... | 264 |
| Correction of the First Hypothesis..... | 278 |
| Approximate Method of finding the Ratio of the Sector to the Triangle..... | 279 |

CHAPTER V.

DETERMINATION OF THE ORBIT OF A HEAVENLY BODY FROM FOUR OBSERVATIONS, OF WHICH THE SECOND AND THIRD MUST BE COMPLETE.

| | |
|---|-----|
| Fundamental Equations..... | 282 |
| Determination of the Curtate Distances..... | 289 |
| Successive Approximations..... | 293 |
| Determination of the Elements of the Orbit..... | 294 |
| Numerical Example..... | 294 |
| Method for the Final Approximation..... | 307 |

CHAPTER VI.

INVESTIGATION OF VARIOUS FORMULÆ FOR THE CORRECTION OF THE APPROXIMATE ELEMENTS OF THE ORBIT OF A HEAVENLY BODY.

| | |
|--|-----|
| Determination of the Elements of a Circular Orbit | 311 |
| Variation of Two Geocentric Distances..... | 313 |
| Differential Formulæ..... | 318 |
| Plane of the Orbit taken as the Fundamental Plane..... | 320 |
| Variation of the Node and Inclination..... | 324 |
| Variation of One Geocentric Distance..... | 328 |
| Determination of the Elements of the Orbit by means of the Co-ordinates and Velocities | 332 |
| Correction of the Ephemeris..... | 335 |
| Final Correction of the Elements | 338 |
| Relation between Two Places in the Orbit..... | 339 |
| Modification when the Semi-Transverse Axis is very large..... | 341 |
| Modification for Hyperbolic Motion | 346 |
| Variation of the Semi-Transverse Axis and Ratio of Two Curtate Distances..... | 349 |

| | PAGE |
|---|------|
| Variation of the Geocentric Distance and of the Reciprocal of the Semi-Transverse Axis..... | 352 |
| Equations of Condition | 353 |
| Orbit of a Comet..... | 355 |
| Variation of Two Radii-Vectores | 357 |

CHAPTER VII.

METHOD OF LEAST SQUARES, THEORY OF THE COMBINATION OF OBSERVATIONS, AND DETERMINATION OF THE MOST PROBABLE SYSTEM OF ELEMENTS FROM A SERIES OF OBSERVATIONS.

| | |
|--|-----|
| Statement of the Problem | 360 |
| Fundamental Equations for the Probability of Errors..... | 362 |
| Determination of the Form of the Function which expresses the Probability ... | 363 |
| The Measure of Precision, and the Probable Error..... | 366 |
| Distribution of the Errors..... | 367 |
| The Mean Error, and the Mean of the Errors..... | 368 |
| The Probable Error of the Arithmetical Mean..... | 370 |
| Determination of the Mean and Probable Errors of Observations..... | 371 |
| Weights of Observed Values..... | 372 |
| Equations of Condition | 376 |
| Normal Equations | 378 |
| Method of Elimination | 380 |
| Determination of the Weights of the Resulting Values of the Unknown Quantities | 386 |
| Separate Determination of the Unknown Quantities and of their Weights..... | 392 |
| Relation between the Weights and the Determinants..... | 396 |
| Case in which the Problem is nearly Indeterminate..... | 398 |
| Mean and Probable Errors of the Results..... | 399 |
| Combination of Observations..... | 401 |
| Errors peculiar to certain Observations | 403 |
| Rejection of Doubtful Observations..... | 410 |
| Correction of the Elements..... | 412 |
| Arrangement of the Numerical Operations..... | 415 |
| Numerical Example..... | 418 |
| Case of very Eccentric Orbits..... | 423 |

CHAPTER VIII.

INVESTIGATION OF VARIOUS FORMULE FOR THE DETERMINATION OF THE SPECIAL PERTURBATIONS OF A HEAVENLY BODY.

| | |
|---------------------------------|-----|
| Fundamental Equations..... | 426 |
| Statement of the Problem..... | 428 |
| Variation of Co-ordinates | 429 |

| | PAGE |
|--|------|
| Mechanical Quadrature..... | 433 |
| The Interval for Quadrature..... | 443 |
| Mode of effecting the Integration..... | 445 |
| Perturbations depending on the Squares and Higher Powers of the Masses | 446 |
| Numerical Example..... | 448 |
| Change of the Equinox and Ecliptic | 455 |
| Determination of New Osculating Elements..... | 459 |
| Variation of Polar Co-ordinates..... | 462 |
| Determination of the Components of the Disturbing Force..... | 467 |
| Determination of the Heliocentric or Geocentric Place..... | 471 |
| Numerical Example | 474 |
| Change of the Osculating Elements..... | 477 |
| Variation of the Mean Anomaly, the Radius-Vector, and the Co-ordinate z | 480 |
| Fundamental Equations..... | 483 |
| Determination of the Components of the Disturbing Force..... | 489 |
| Case of very Eccentric Orbits..... | 493 |
| Determination of the Place of the Disturbed Body..... | 495 |
| Variation of the Node and Inclination..... | 502 |
| Numerical Example..... | 505 |
| Change of the Osculating Elements | 510 |
| Variation of Constants..... | 516 |
| Case of very Eccentric Orbits..... | 523 |
| Variation of the Periodic Time..... | 526 |
| Numerical Example..... | 529 |
| Formulae to be used when the Eccentricity or the Inclination is small..... | 533 |
| Correction of the Assumed Value of the Disturbing Mass..... | 535 |
| Perturbations of Comets | 536 |
| Motion about the Common Centre of Gravity of the Sun and Planet..... | 537 |
| Reduction of the Elements to the Common Centre of Gravity of the Sun and Planet | 538 |
| Reduction by means of Differential Formulae | 540 |
| Near Approach of a Comet to a Planet | 546 |
| The Sun may be regarded as the Disturbing Body..... | 548 |
| Determination of the Elements of the Orbit about the Planet..... | 550 |
| Subsequent Motion of the Comet..... | 551 |
| Effect of a Resisting Medium in Space..... | 552 |
| Variation of the Elements on account of the Resisting Medium | 554 |
| Method to be applied when no Assumption is made in regard to the Density of the Ether..... | 556 |



TABLES.

| | PAGE |
|---|------|
| I. Angle of the Vertical and Logarithm of the Earth's Radius | 561 |
| II. For converting Intervals of Mean Solar Time into Equivalent Intervals of Sidereal Time..... | 563 |
| III. For converting Intervals of Sidereal Time into Equivalent Intervals of Mean Solar Time..... | 564 |
| IV. For converting Hours, Minutes, and Seconds into Decimals of a Day... | 565 |
| V. For finding the Number of Days from the Beginning of the Year | 565 |
| VI. For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit | 566 |
| VII. For finding the True Anomaly in a Parabolic Orbit when τ is nearly 180° | 611 |
| VIII. For finding the Time from the Perihelion in a Parabolic Orbit..... | 612 |
| IX. For finding the True Anomaly or the Time from the Perihelion in Orbits of Great Eccentricity..... | 614 |
| X. For finding the True Anomaly or the Time from the Perihelion in El- liptic and Hyperbolic Orbits..... | 618 |
| XI. For the Motion in a Parabolic Orbit..... | 619 |
| XII. For the Limits of the Roots of the Equation $\sin(z' - \zeta) = m_0 \sin^4 z' \dots$ | 622 |
| XIII. For finding the Ratio of the Sector to the Triangle..... | 624 |
| XIV. For finding the Ratio of the Sector to the Triangle..... | 629 |
| XV. For Elliptic Orbits of Great Eccentricity..... | 632 |
| XVI. For Hyperbolic Orbits..... | 632 |
| XVII. For Special Perturbations..... | 633 |
| XVIII. Elements of the Orbits of the Comets which have been observed..... | 638 |
| XIX. Elements of the Orbits of the Minor Planets..... | 646 |
| XX. Elements of the Orbits of the Major Planets | 648 |
| XXI. Constants, &c..... | 649 |
| EXPLANATION OF THE TABLES..... | 651 |
| APPENDIX.—Precession..... | 657 |
| Nutation | 658 |
| Aberration | 659 |
| Intensity of Light..... | 660 |
| Numerical Calculations..... | 662 |

THEORETICAL ASTRONOMY.

CHAPTER I.

INVESTIGATION OF THE FUNDAMENTAL EQUATIONS OF MOTION, AND OF THE FORMULÆ FOR DETERMINING, FROM KNOWN ELEMENTS, THE HELIOCENTRIC AND GEOCENTRIC PLACES OF A HEAVENLY BODY, ADAPTED TO NUMERICAL COMPUTATION FOR CASES OF ANY ECCENTRICITY WHATEVER.

1. THE study of the motions of the heavenly bodies does not require that we should know the ultimate limit of divisibility of the matter of which they are composed,—whether it may be subdivided indefinitely, or whether the limit is an indivisible, impenetrable atom. Nor are we concerned with the relations which exist between the separate atoms or molecules, except so far as they form, in the aggregate, a definite body whose relation to other bodies of the system it is required to investigate. On the contrary, in considering the operation of the laws in obedience to which matter is aggregated into single bodies and systems of bodies, it is sufficient to conceive simply of its divisibility to a limit which may be regarded as infinitesimal compared with the finite volume of the body, and to regard the magnitude of the element of matter thus arrived at as a mathematical point.

An element of matter, or a material body, cannot give itself motion; neither can it alter, in any manner whatever, any motion which may have been communicated to it. This tendency of matter to resist all changes of its existing state of rest or motion is known as *inertia*, and is the fundamental law of the motion of bodies. Experience invariably confirms it as a law of nature; the continuance of motion as resistances are removed, as well as the sensibly unchanged motion of the heavenly bodies during many centuries, affording the

most convincing proof of its universality. Whenever, therefore, a material point experiences any change of its state as respects rest or motion, the cause must be attributed to the operation of something external to the element itself, and which we designate by the word *force*. The nature of forces is generally unknown, and we estimate them by the effects which they produce. They are thus rendered comparable with some unit, and may be expressed by abstract numbers.

2. If a material point, free to move, receives an impulse by virtue of the action of any force, or if, at any instant, the force by which motion is communicated shall cease to act, the subsequent motion of the point, according to the law of inertia, must be rectilinear and *uniform*, equal spaces being described in equal times. Thus, if s , v , and t represent, respectively, the *space*, the *velocity*, and the *time*, the measure of v being the space described in a unit of time, we shall have, in this case,

$$s = vt.$$

It is evident, however, that the space described in a unit of time will vary with the intensity of the force to which the motion is due, and, the nature of the force being unknown, we must necessarily compare the velocities communicated to the point by different forces, in order to arrive at the relation of their effects. We are thus led to regard the force as proportional to the velocity; and this also has received the most indubitable proof as being a law of nature. Hence, the principles of the composition and resolution of forces may be applied also to the composition and resolution of velocities.

If the force acts incessantly, the velocity will be accelerated, and the force which produces this motion is called an *accelerating* force. In regard to the mode of operation of the force, however, we may consider it as acting absolutely without cessation, or we may regard it as acting instantaneously at successive infinitesimal intervals represented by dt , and hence the motion as uniform during each of these intervals. The latter supposition is that which is best adapted to the requirements of the infinitesimal calculus; and, according to the fundamental principles of this calculus, the finite result will be the same as in the case of a force whose action is absolutely incessant. Therefore, if we represent the element of space by ds , and the element of time by dt , the instantaneous velocity will be

$$v = \frac{ds}{dt},$$

which will vary from one instant to another.

3. Since the force is proportional to the velocity, its measure at any instant will be determined by the corresponding velocity. If the accelerating force is constant, the motion will be uniformly accelerated; and if we designate the *acceleration* due to the force by f , the unit of f being the velocity generated in a unit of time, we shall have

$$v = ft.$$

If, however, the force be variable, we shall have, at any instant, the relation

$$f = \frac{dv}{dt},$$

the force being regarded as constant in its action during the element of time dt . The instantaneous value of v gives, by differentiation,

$$\frac{dv}{dt} = \frac{d^2s}{dt^2},$$

and hence we derive

$$f = \frac{d^2s}{dt^2}; \quad (1)$$

so that, in varied motion, the acceleration due to the force is measured by the second differential of the space divided by the square of the element of time.

4. By the *mass* of the body we mean its absolute quantity of matter. The *density* is the mass of a unit of volume, and hence the entire mass is equal to the volume multiplied by the density. If it is required to compare the forces which act upon different bodies, it is evident that the masses must be considered. If equal masses receive impulses by the action of instantaneous forces, the forces acting on each will be to each other as the velocities imparted; and if we consider as the unit of force that which gives to a unit of mass the unit of velocity, we have for the measure of a force F , denoting the mass by M ,

$$F = Mv.$$

This is called the *quantity of motion* of the body, and expresses its capacity to overcome inertia. By virtue of the inert state of matter, there can be no action of a force without an equal and contrary reaction; for, if the body to which the force is applied is fixed, the equilibrium between the resistance and the force necessarily implies the development of an equal and contrary force; and, if the body be free to move, in the change of state, its inertia will oppose equal and

contrary resistance. Hence, as a necessary consequence of inertia, it follows that action and reaction are simultaneous, equal, and contrary.

If the body is acted upon by a force such that the motion is varied, the accelerating force upon each element of its mass is represented by $\frac{dv}{dt}$, and the entire *motive force* F is expressed by

$$F = M \frac{dv}{dt},$$

M being the sum of all the elements, or the mass of the body. Since

$$v = \frac{ds}{dt},$$

this gives

$$F = M \frac{d^2s}{dt^2},$$

which is the expression for the intensity of the motive force, or of the force of inertia developed. For the unit of mass, the measure of the force is

$$\frac{d^2s}{dt^2};$$

and this, therefore, expresses that part of the intensity of the motive force which is impressed upon the unit of mass, and is what is usually called the *accelerating force*.

5. The force in obedience to which the heavenly bodies perform their journey through space, is known as the *attraction of gravitation*; and the law of the operation of this force, in itself simple and unique, has been confirmed and generalized by the accumulated researches of modern science. Not only do we find that it controls the motions of the bodies of our own solar system, but that the revolutions of binary systems of stars in the remotest regions of space proclaim the universality of its operation. It unfailingly explains all the phenomena observed, and, outstripping observation, it has furnished the means of predicting many phenomena subsequently observed. The law of this force is that *every particle of matter is attracted by every other particle by a force which varies directly as the mass and inversely as the square of the distance of the attracting particle*.

This reciprocal action is instantaneous, and is not modified, in any degree, by the interposition of other particles or bodies of matter. It is also absolutely independent of the nature of the molecules themselves, and of their aggregation.

If we consider two bodies the masses of which are m and m' , and whose magnitudes are so small, relatively to their mutual distance ρ , that we may regard them as material points, according to the law of gravitation, the action of m on each molecule or unit of m' will be $\frac{m}{\rho^2}$, and the total force on m' will be

$$m' \frac{m}{\rho^2}.$$

The action of m' on each molecule of m will be expressed by $\frac{m'}{\rho^2}$, and its total action by

$$m \frac{m'}{\rho^2}.$$

The absolute or moving force with which the masses m and m' tend toward each other is, therefore, the same on each body, which result is a necessary consequence of the equality of action and reaction. The velocities, however, with which these bodies would approach each other must be different, the velocity of the smaller mass exceeding that of the greater, and in the ratio of the masses moved. The expression for the velocity of m' , which would be generated in a unit of time if the force remained constant, is obtained by dividing the absolute force exerted by m by the mass moved, which gives

$$\frac{m}{\rho^2},$$

and this is, therefore, the measure of the acceleration due to the action of m at the distance ρ . For the acceleration due to the action of m' we derive, in a similar manner,

$$\frac{m'}{\rho^2}.$$

6. Observation shows that the heavenly bodies are nearly spherical in form, and we shall therefore, preparatory to finding the equations which express the relative motions of the bodies of the system, determine the attraction of a spherical mass of uniform density, or varying from the centre to the surface according to any law, for a point exterior to it.

If we suppose a straight line to be drawn through the centre of the sphere and the point attracted, the total action of the sphere on the point will be a force acting along this line, since the mass of the sphere is symmetrical with respect to it. Let dm denote an element

of the mass of the sphere, and ρ its distance from the point attracted; then will

$$\frac{dm}{\rho^2}$$

express the action of this element on the point attracted. If we suppose the density of the sphere to be constant, and equal to unity, the element dm becomes an element of volume, and will be expressed by

$$dm = dx dy dz;$$

x , y , and z being the co-ordinates of the element referred to a system of rectangular co-ordinates. If we take the origin of co-ordinates at the centre of the sphere, and introduce polar co-ordinates, so that

$$\begin{aligned} x &= r \cos \varphi \cos \theta, \\ y &= r \cos \varphi \sin \theta, \\ z &= r \sin \varphi, \end{aligned}$$

the expression for dm becomes

$$dm = r^2 \cos \varphi dr d\varphi d\theta;$$

and its action on the point attracted is

$$df = \frac{r^2 \cos \varphi dr d\varphi d\theta}{\rho^2}.$$

If we suppose the axis of z to be directed to the point attracted, the co-ordinates of this point will be

$$x' = 0, \quad y' = 0, \quad z' = a,$$

a being the distance of the point from the centre of the sphere, and, since

$$\rho^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

we shall have

$$\rho^2 = a^2 - 2ar \sin \varphi + r^2.$$

The component of the force df in the direction of the line a , joining the point attracted and the centre of the sphere, is

$$df \cos \gamma,$$

where γ is the angle at the point attracted between the element dm and the centre of the sphere. It is evident that the sum of all the components which act in the direction of the line a will express the total action of the sphere, since the sum of those which act perpen-

dicular to this line, taken so as to include the entire mass of the sphere, is zero.

But we have

$$a = z + \rho \cos \gamma,$$

and hence

$$\cos \gamma = \frac{a - r \sin \varphi}{\rho}.$$

The differentiation of the expression for ρ^2 , with respect to a , gives

$$\frac{d\rho}{da} = \frac{a - r \sin \varphi}{\rho} = \cos \gamma.$$

Therefore, if we denote the attraction of the sphere by A , we shall have, by means of the values of df and $\cos \gamma$,

$$dA = \frac{r^2 \cos \varphi \, dr \, d\varphi \, d\theta}{\rho^2} \cdot \frac{d\rho}{da},$$

or

$$dA = -r^2 \cos \varphi \, dr \, d\varphi \, d\theta \frac{d \frac{1}{\rho}}{da}.$$

The polar co-ordinates r , φ , and θ are independent of a , and hence

$$dA = - \frac{d \frac{r^2 \cos \varphi \, dr \, d\varphi \, d\theta}{\rho}}{da}.$$

Let us now put

$$dV = \frac{r^2 \cos \varphi \, dr \, d\varphi \, d\theta}{\rho}, \quad (2)$$

and we shall have

$$A = - \frac{dV}{da}.$$

Consequently, to find the total action of the sphere on the given point, we have only to find V by means of equation (2), the limits of the integration being taken so as to include the entire mass of the sphere, and then find its differential coefficient with respect to a .

If we integrate equation (2) first with reference to θ , for which ρ is constant, between the limits $\theta = 0$ and $\theta = 2\pi$, we get

$$V = 2\pi \iint \frac{r^2 \cos \varphi \, dr \, d\varphi}{\rho}.$$

This must be integrated between the limits $\varphi = +\frac{1}{2}\pi$ and $\varphi = -\frac{1}{2}\pi$;

but since ρ is a function of φ , if we differentiate the expression for ρ^2 with respect to φ , we have

$$r \cos \varphi \, d\varphi = -\frac{\rho}{a} d\rho,$$

and hence

$$V = -\frac{2\pi}{a} \iint r \, dr \, d\rho.$$

Corresponding to the limits of φ we have $\rho = a - r$, and $\rho = a + r$; and taking the integral with respect to ρ between these limits, we obtain

$$V = \frac{4\pi}{a} \int r^2 \, dr.$$

Integrating, finally, between the limits $r = 0$ and $r = r$, we get

$$V = \frac{4}{3} \frac{\pi r^3}{a},$$

r , being the radius of the sphere, and, if we denote its entire mass by m , this becomes

$$V = \frac{m}{a}.$$

Therefore,

$$A = -\frac{dV}{da} = \frac{m}{a^2},$$

from which it appears that the action of a homogeneous spherical mass on a point exterior to it, is the same as if the entire mass were concentrated at its centre. If, in the integration with respect to r , we take the limits r' and r'' , we obtain

$$A = \frac{4}{3} \frac{\pi (r''^3 - r'^3)}{a^2},$$

and, denoting by m_0 the mass of a spherical shell whose radii are r'' and r' , this becomes

$$A = \frac{m_0}{a^2}.$$

Consequently, the attraction of a homogeneous spherical shell on a point exterior to it, is the same as if the entire mass were concentrated at its centre.

The supposition that the point attracted is situated within a spherical shell of uniform density, does not change the form of the

general equation; but, in the integration with reference to ρ , the limits will be $\rho = r + a$, and $\rho = r - a$, which give

$$V = -4\pi \int r dr;$$

and this being independent of a , we have

$$A = -\frac{dV}{da} = 0.$$

Whence it follows that a point placed in the interior of a spherical shell is equally attracted in all directions, and that, if not subject to the action of any extraneous force, it will be in equilibrium in every position.

7. Whatever may be the law of the change of the density of the heavenly bodies from the surface to the centre, we may regard them as composed of homogeneous, concentric layers, the density varying only from one layer to another, and the number of the layers may be indefinite. The action of each of these will be the same as if its mass were united at the centre of the shell; and hence the total action of the body will be the same as if the entire mass were concentrated at its centre of gravity. The planets are indeed not exactly spheres, but oblate spheroids differing but little from spheres; and the error of the assumption of an exact spherical form, so far as it relates to their action upon each other, is extremely small, and is in fact compensated by the magnitude of their distances from each other; for, whatever may be the form of the body, if its dimensions are small in comparison with its distance from the body which it attracts, it is evident that its action will be sensibly the same as if its entire mass were concentrated at its centre of gravity. If we suppose a system of bodies to be composed of spherical masses, each unattended with any satellite, and if we suppose that the dimensions of the bodies are small in comparison with their mutual distances, the formation of the equations for the motion of the bodies of the system will be reduced to the consideration of the motions of simple points endowed with forces of attraction corresponding to the respective masses. Our solar system is, in reality, a compound system, the several systems of primary and satellites corresponding nearly to the case supposed; and, before proceeding with the formation of the equations which are applicable to the general case, we will consider, at first, those for a simple system of bodies, considered as points and subject to their mutual actions and the action of the forces which correspond to the

actual velocities of the different parts of the system for any instant. It is evident that we cannot consider the motion of any single body as free, and subject only to the action of the primitive impulsion which it has received and the accelerating forces which act upon it; but, on the contrary, the motion of each body will depend on the force which acts upon it directly, and also on the reaction due to the other bodies of the system. The consideration, however, of the variations of the motion of the several bodies of the system is reduced to the simple case of equilibrium by means of the general principle that, if we assign to the different bodies of the system motions which are modified by their mutual action, we may regard these motions as composed of those which the bodies actually have and of other motions which are destroyed, and which must therefore necessarily be such that, if they alone existed, the system would be in equilibrium. We are thus enabled to form at once the equations for the motion of a system of bodies. Let $m, m', m'',$ &c. be the masses of the several bodies of the system, and $x, y, z, x', y', z',$ &c. their co-ordinates referred to any system of rectangular axes. Further, let the components of the total force acting upon a unit of the mass of m , or of the accelerating force, resolved in directions parallel to the co-ordinate axes, be denoted by $X, Y,$ and $Z,$ respectively, then will

$$mX, \quad mY, \quad mZ,$$

be the forces which act upon the body in the same directions. The velocities of the body m at any instant, in directions parallel to the co-ordinate axes, will be

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt};$$

and the corresponding forces are

$$m \frac{dx}{dt}, \quad m \frac{dy}{dt}, \quad m \frac{dz}{dt}.$$

By virtue of the action of the accelerating force, these forces for the next instant become

$$m \frac{dx}{dt} + mXdt, \quad m \frac{dy}{dt} + mYdt, \quad m \frac{dz}{dt} + mZdt,$$

which may be written respectively:

$$\begin{aligned}
m \frac{dx}{dt} + md \frac{dx}{dt} - md \frac{dx}{dt} + mXd, \\
m \frac{dy}{dt} + md \frac{dy}{dt} - md \frac{dy}{dt} + mYd, \\
m \frac{dz}{dt} + md \frac{dz}{dt} - md \frac{dz}{dt} + mZd.
\end{aligned}$$

The actual velocities for this instant are

$$\frac{dx}{dt} + d \frac{dx}{dt}, \quad \frac{dy}{dt} + d \frac{dy}{dt}, \quad \frac{dz}{dt} + d \frac{dz}{dt},$$

and the corresponding forces are

$$m \frac{dx}{dt} + md \frac{dx}{dt}, \quad m \frac{dy}{dt} + md \frac{dy}{dt}, \quad m \frac{dz}{dt} + md \frac{dz}{dt}.$$

Comparing these with the preceding expressions for the forces, it appears that the forces which are destroyed, in directions parallel to the co-ordinate axes, are

$$\begin{aligned}
& -md \frac{dx}{dt} + mXd, \\
& -md \frac{dy}{dt} + mYd, \\
& -md \frac{dz}{dt} + mZd.
\end{aligned} \tag{3}$$

In the same manner we find for the forces which will be destroyed in the case of the body m' :

$$\begin{aligned}
& -m'd \frac{dx'}{dt} + m'X'dt, \\
& -m'd \frac{dy'}{dt} + m'Y'dt, \\
& -m'd \frac{dz'}{dt} + m'Z'dt;
\end{aligned}$$

and similarly for the other bodies of the system. According to the general principle above enunciated, the system under the action of these forces alone, will be in equilibrium. The conditions of equilibrium for a system of points of invariable but arbitrary form, and subject to the action of forces directed in any manner whatever, are

$$\begin{aligned}
\Sigma X, = 0, & \quad \Sigma Y, = 0, & \quad \Sigma Z, = 0, \\
\Sigma (Y_x - X_y) = 0, & \quad \Sigma (X_z - Z_x) = 0, & \quad \Sigma (Z_y - Y_z) = 0,
\end{aligned}$$

in which X , Y , Z , denote the components, resolved parallel to the

co-ordinate axes, of the forces acting on any point, and x, y, z , the co-ordinates of the point. These equations are equally applicable to the case of the equilibrium at any instant of a system of variable form; and substituting in them the expressions (3) for the forces destroyed in the case of a system of bodies, we shall have

$$\begin{aligned}
 \Sigma m \frac{d^2x}{dt^2} - \Sigma mX &= 0, \\
 \Sigma m \frac{d^2y}{dt^2} - \Sigma mY &= 0, \\
 \Sigma m \frac{d^2z}{dt^2} - \Sigma mZ &= 0, \\
 \Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) - \Sigma m (Yx - Xy) &= 0, \\
 \Sigma m \left(z \frac{d^2x}{dt^2} - x \frac{d^2z}{dt^2} \right) - \Sigma m (Xz - Zx) &= 0, \\
 \Sigma m \left(y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2} \right) - \Sigma m (Zy - Yz) &= 0;
 \end{aligned} \tag{4}$$

which are the general equations for the motions of a system of bodies.

8. Let x, y, z , be the co-ordinates of the centre of gravity of the system, and, by differentiation of the equations for the co-ordinates of the centre of gravity, which are

$$x = \frac{\Sigma mx}{\Sigma m}, \quad y = \frac{\Sigma my}{\Sigma m}, \quad z = \frac{\Sigma mz}{\Sigma m},$$

we get

$$\frac{d^2x}{dt^2} = \frac{\Sigma m \frac{d^2x}{dt^2}}{\Sigma m}, \quad \frac{d^2y}{dt^2} = \frac{\Sigma m \frac{d^2y}{dt^2}}{\Sigma m}, \quad \frac{d^2z}{dt^2} = \frac{\Sigma m \frac{d^2z}{dt^2}}{\Sigma m}.$$

Introducing these values into the first three of equations (4), they become

$$\frac{d^2x}{dt^2} = \frac{\Sigma mX}{\Sigma m}, \quad \frac{d^2y}{dt^2} = \frac{\Sigma mY}{\Sigma m}, \quad \frac{d^2z}{dt^2} = \frac{\Sigma mZ}{\Sigma m}; \tag{5}$$

from which it appears that the centre of gravity of the system moves in space as if the masses of the different bodies of which it is composed, were united in that point, and the forces directly applied to it.

If we suppose that the only accelerating forces which act on the bodies of the system, are those which result from their mutual action, we have the obvious relation :

$$mX = -m'X', \quad mY = -m'Y', \quad mZ = -m'Z'$$

and similarly for any two bodies ; and, consequently,

$$\Sigma mX = 0, \quad \Sigma mY = 0, \quad \Sigma mZ = 0;$$

so that equations (5) become

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = 0.$$

Integrating these once, and denoting the constants of integration by c, c', c'' , we find, by combining the results,

$$\frac{dx'^2 + dy'^2 + dz'^2}{dt^2} = v^2 = c^2 + c'^2 + c''^2;$$

and hence the absolute motion of the centre of gravity of the system, when subject only to the mutual action of the bodies which compose it, must be uniform and rectilinear. Whatever, therefore, may be the relative motions of the different bodies of the system, the motion of its centre of gravity is not thereby affected.

9. Let us now consider the last three of equations (4), and suppose the system to be submitted only to the mutual action of the bodies which compose it, and to a force directed toward the origin of co-ordinates. The action of m' on m , according to the law of gravitation, is expressed by $\frac{m'}{\rho^2}$, in which ρ denotes the distance of m from m' . To resolve this force in directions parallel to the three rectangular axes, we must multiply it by the cosine of the angle which the line joining the two bodies makes with the co-ordinate axes respectively, which gives

$$X = \frac{m' (x' - x)}{\rho^3}, \quad Y = \frac{m' (y' - y)}{\rho^3}, \quad Z = \frac{m' (z' - z)}{\rho^3}.$$

Further, for the components of the accelerating force of m on m' , we have

$$X' = \frac{m (x - x')}{\rho^3}, \quad Y' = \frac{m (y - y')}{\rho^3}, \quad Z' = \frac{m (z - z')}{\rho^3}.$$

Hence we derive

$$m (Yx - Xy) + m' (Y'x' - X'y') = 0,$$

and generally

$$\Sigma m (Yx - Xy) = 0. \quad (6)$$

In a similar manner, we find

$$\begin{aligned}\Sigma m (Xz - Zx) &= 0, \\ \Sigma m (Zy - Yz) &= 0.\end{aligned}\tag{7}$$

These relations will not be altered if, in addition to their reciprocal action, the bodies of the system are acted upon by forces directed to the origin of co-ordinates. Thus, in the case of a force acting upon m , and directed to the origin of co-ordinates, we have, for its action alone,

$$Yx = Xy, \quad Xz = Zx, \quad Zy = Yz,$$

and similarly for the other bodies. Hence these forces disappear from the equations, and, therefore, when the several bodies of the system are subject only to their reciprocal action and to forces directed to the origin of co-ordinates, the last three of equations (4) become

$$\begin{aligned}\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) &= 0, \\ \Sigma m \left(z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right) &= 0, \\ \Sigma m \left(y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right) &= 0,\end{aligned}$$

the integration of which gives

$$\begin{aligned}\Sigma m (xdy - ydx) &= cat, \\ \Sigma m (zdx - xdz) &= c'dt, \\ \Sigma m (ydz - zdy) &= c''dt,\end{aligned}\tag{8}$$

c , c' , and c'' being the constants of integration. Now, $xdy - ydx$ is double the area described about the origin of co-ordinates by the projection of the radius-vector, or line joining m with the origin of co-ordinates, on the plane of xy during the element of time dt ; and, further, $zdx - xdz$ and $ydz - zdy$ are respectively double the areas described, during the same time, by the projection of the radius-vector on the planes of xz and yz . The constant c , therefore, expresses double the sum of the products formed by multiplying the *areal velocity* of each body, in the direction of the co-ordinate plane xy , by its mass; and c' , c'' , express the same sum with reference to the co-ordinate planes xz and yz respectively. Hence the sum of the areal velocities of the several bodies of the system about the origin of co-ordinates, each multiplied by the corresponding mass, is constant; and the sum of the areas traced, each multiplied by the corresponding mass, is proportional to the time. If the only forces which operate, are those

resulting from the mutual action of the bodies which compose the system, this result is correct whatever may be the point in space taken as the origin of co-ordinates.

The areas described by the projection of the radius-vector of each body on the co-ordinate planes, are the projections, on these planes, of the areas actually described in space. We may, therefore, conceive of a resultant, or principal plane of projection, such that the sum of the areas traced by the projection of each radius-vector on this plane, when projected on the three co-ordinate planes, each being multiplied by the corresponding mass, will be respectively equal to the first members of the equations (8). Let α , β , and γ be the angles which this principal plane makes with the co-ordinate planes xy , xz , and yz , respectively; and let S denote the sum of the areas traced on this plane, in a unit of time, by the projection of the radius-vector of each of the bodies of the system, each area being multiplied by the corresponding mass. The sum S will be found to be a maximum, and its projections on the co-ordinate planes, corresponding to the element of time dt , are

$$S \cos \alpha \, dt, \quad S \cos \beta \, dt, \quad S \cos \gamma \, dt.$$

Therefore, by means of equations (8), we have

$$c = S \cos \alpha, \quad c' = S \cos \beta, \quad c'' = S \cos \gamma,$$

and, since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$,

$$S^2 = c^2 + c'^2 + c''^2.$$

Hence we derive

$$\begin{aligned} \cos \alpha &= \frac{c}{\sqrt{c^2 + c'^2 + c''^2}}, & \cos \beta &= \frac{c'}{\sqrt{c^2 + c'^2 + c''^2}}, \\ \cos \gamma &= \frac{c''}{\sqrt{c^2 + c'^2 + c''^2}} \end{aligned}$$

These angles, being therefore constant and independent of the time, show that this principal plane of projection remains constantly parallel to itself during the motion of the system in space, whatever may be the relative positions of the several bodies; and for this reason it is called the *invariable plane* of the system. Its position with reference to any known plane is easily determined when the velocities, in directions parallel to the co-ordinate axes, and the masses and co-ordinates of the several bodies of the system, are known. The values of c , c' , c'' are given by equations (8), and

hence the values of α , β , and γ , which determine the position of the invariable plane.

Since the positions of the co-ordinate planes are arbitrary, we may suppose that of xy to coincide with the invariable plane, which gives $\cos \beta = 0$ and $\cos \gamma = 0$, and, therefore, $c' = 0$ and $c'' = 0$. Further, since the positions of the axes of x and y in this plane are arbitrary, it follows that for every plane perpendicular to the invariable plane, the sum of the areas traced by the projections of the radii-vectores of the several bodies of the system, each multiplied by the corresponding mass, is zero. It may also be observed that the value of S is constant whatever may be the position of the co-ordinate planes, and that its value is necessarily greater than that of either of the quantities in the second member of the equation,

$$S^2 = c^2 + c'^2 + c''^2,$$

except when two of them are each equal to zero. It is, therefore, a maximum, and the invariable plane is also the plane of maximum areas.

10. If we suppose the origin of co-ordinates itself to move with uniform and rectilinear motion in space, the relations expressed by equations (8) will remain unchanged. Thus, let x, y, z , be the co-ordinates of the movable origin of co-ordinates, referred to a fixed point in space taken as the origin; and let $x_0, y_0, z_0, x_0', y_0', z_0'$, &c. be the co-ordinates of the several bodies referred to the movable origin. Then, since the co-ordinate planes in one system remain always parallel to those of the other system of co-ordinates, we shall have

$$x = x_0 + x, \quad y = y_0 + y, \quad z = z_0 + z,$$

and similarly for the other bodies of the system. Introducing these values of x, y , and z into the first three of equations (4), they become

$$\begin{aligned} \Sigma m \left(\frac{d^2 x}{dt^2} + \frac{d^2 x_0}{dt^2} \right) - \Sigma m X &= 0, \\ \Sigma m \left(\frac{d^2 y}{dt^2} + \frac{d^2 y_0}{dt^2} \right) - \Sigma m Y &= 0, \\ \Sigma m \left(\frac{d^2 z}{dt^2} + \frac{d^2 z_0}{dt^2} \right) - \Sigma m Z &= 0. \end{aligned}$$

The condition of uniform rectilinear motion of the movable origin gives

$$\frac{d^2 x_0}{dt^2} = 0, \quad \frac{d^2 y_0}{dt^2} = 0, \quad \frac{d^2 z_0}{dt^2} = 0,$$

and the preceding equations become

$$\begin{aligned}\Sigma m \frac{d^2 x_0}{dt^2} - \Sigma m X &= 0, \\ \Sigma m \frac{d^2 y_0}{dt^2} - \Sigma m Y &= 0, \\ \Sigma m \frac{d^2 z_0}{dt^2} - \Sigma m Z &= 0.\end{aligned}\tag{9}$$

Substituting the same values in the last three of equations (4), observing that the co-ordinates x_i, y_i, z_i are the same for all the bodies of the system, and reducing the resulting equations by means of equations (9), we get

$$\begin{aligned}\Sigma m \left(x_0 \frac{d^2 y_0}{dt^2} - y_0 \frac{d^2 x_0}{dt^2} \right) - \Sigma m (Y x_0 - X y_0) &= 0, \\ \Sigma m \left(z_0 \frac{d^2 x_0}{dt^2} - x_0 \frac{d^2 z_0}{dt^2} \right) - \Sigma m (X z_0 - Z x_0) &= 0, \\ \Sigma m \left(y_0 \frac{d^2 z_0}{dt^2} - z_0 \frac{d^2 y_0}{dt^2} \right) - \Sigma m (Z y_0 - Y z_0) &= 0.\end{aligned}\tag{10}$$

Hence it appears that the form of the equations for the motion of the system of bodies, remains unchanged when we suppose the origin of co-ordinates to move in space with a uniform and rectilinear motion.

11. The equations already derived for the motions of a system of bodies, considered as reduced to material points, enable us to form at once those for the motion of a solid body. The mutual distances of the parts of the system are, in this case, invariable, and the masses of the several bodies become the elements of the mass of the solid body. If we denote an element of the mass by dm , the equations (5) for the motion of the centre of gravity of the body become

$$m \frac{d^2 x}{dt^2} = \int X dm, \quad m \frac{d^2 y}{dt^2} = \int Y dm, \quad m \frac{d^2 z}{dt^2} = \int Z dm, \tag{11}$$

the summation, or integration with reference to dm , being taken so as to include the entire mass of the body, from which it appears that the centre of gravity of the body moves in space as if the entire mass were concentrated in that point, and the forces applied to it directly.

If we take the origin of co-ordinates at the centre of gravity of the body, and suppose it to have a rectilinear, uniform motion in space, and denote the co-ordinates of the element dm , in reference to this origin, by x_0, y_0, z_0 , we have, by means of the equations (10),

$$\begin{aligned}
& \int \left(x_0 \frac{d^2 y_0}{dt^2} - y_0 \frac{d^2 x_0}{dt^2} \right) dm - \int (Yx_0 - Xy_0) dm = 0, \\
& \int \left(z_0 \frac{d^2 x_0}{dt^2} - x_0 \frac{d^2 z_0}{dt^2} \right) dm - \int (Xz_0 - Zx_0) dm = 0, \quad (12) \\
& \int \left(y_0 \frac{d^2 z_0}{dt^2} - z_0 \frac{d^2 y_0}{dt^2} \right) dm - \int (Zy_0 - Yz_0) dm = 0,
\end{aligned}$$

the integration with respect to dm being taken so as to include the entire mass of the body. These equations, therefore, determine the motion of rotation of the body around its centre of gravity regarded as fixed, or as having a uniform rectilinear motion in space. Equations (11) determine the position of the centre of gravity for any instant, and hence for the successive instants at intervals equal to dt ; and we may consider the motion of the body during the element of time dt as rectilinear and uniform, whatever may be the form of its trajectory. Hence, equations (11) and (12) completely determine the position of the body in space,—the former relating to the motion of translation of the centre of gravity, and the latter to the motion of rotation about this point. It follows, therefore, that for any forces which act upon a body we can always decompose the actual motion into those of the translation of the centre of gravity in space, and of the motion of rotation around this point; and these two motions may be considered independently of each other, the motion of the centre of gravity being independent of the form and position of the body about this point.

If the only forces which act upon the body are the reciprocal action of the elements of its mass and forces directed to the origin of co-ordinates, the second terms of equations (12) become each equal to zero, and the results indicated by equations (8) apply in this case also. The parts of the system being invariably connected, the plane of maximum areas, or *invariable plane*, is evidently that which is perpendicular to the axis of rotation passing through the centre of gravity, and therefore, in the motion of translation of the centre of gravity in space, the axis of rotation remains constantly parallel to itself. Any extraneous force which tends to disturb this relation will necessarily develop a contrary reaction, and hence a rotating body resists any change of its plane of rotation not parallel to itself. We may observe, also, that on account of the invariability of the mutual distances of the elements of the mass, according to equations (8), the motion of rotation must be uniform.

12. We shall now consider the action of a system of bodies on a

distant mass, which we will denote by M . Let $x_0, y_0, z_0, x'_0, y'_0, z'_0$, &c. be the co-ordinates of the several bodies of the system referred to its centre of gravity as the origin of co-ordinates; x, y , and z , the co-ordinates of the centre of gravity of the system referred to the centre of gravity of the body M . The co-ordinates of the body m , of the system, referred to this origin, will therefore be

$$x = x_0 + x, \quad y = y_0 + y, \quad z = z_0 + z,$$

and similarly for the other bodies of the system. If we denote by r the distance of the centre of gravity of m from that of M , the accelerating force of the former on an element of mass at the centre of gravity of the latter, resolved parallel to the axis of x , will be

$$\frac{mx}{r^3},$$

and, therefore, that of the entire system on the element of M , resolved in the same direction, will be

$$\Sigma \frac{mx}{r^3}.$$

We have also

$$r^2 = (x_0 + x)^2 + (y_0 + y)^2 + (z_0 + z)^2,$$

and, if we denote by r , the distance of the centre of gravity of the system from M ,

$$r^2 = x_0^2 + y_0^2 + z_0^2.$$

Therefore

$$\frac{x}{r^3} = (x_0 + x) (r^2 + 2(x_0x + y_0y + z_0z) + r_0^2)^{-\frac{3}{2}}.$$

We shall now suppose the mutual distances of the bodies of the system to be so small in comparison with the distance r , of its centre of gravity from that of M , that terms of the order r_0^2 may be neglected; a condition which is actually satisfied in the case of the secondary systems belonging to the solar system. Hence, developing the second factor of the second member of the last equation, and neglecting terms of the order r_0^2 , we shall have

$$\frac{x}{r^3} = \frac{x_0}{r_0^3} + \frac{x}{r_0^3} - \frac{3x_0(x_0x + y_0y + z_0z)}{r_0^5},$$

and

$$\Sigma \frac{mx}{r^3} = x_0 \Sigma \frac{m}{r_0^3} + \frac{\Sigma mx_0}{r_0^3} - \frac{3x_0}{r_0^5} (x_0 \Sigma mx_0 + y_0 \Sigma my_0 + z_0 \Sigma mz_0).$$

But, since x_0, y_0, z_0 , are the co-ordinates in reference to the centre of gravity of the system as the origin, we have

$$\Sigma m x_0 = 0, \quad \Sigma m y_0 = 0, \quad \Sigma m z_0 = 0,$$

and the preceding equation reduces to

$$\Sigma \frac{mx}{r^3} = x, \frac{\Sigma m}{r_i^3}.$$

In a similar manner, we find

$$\Sigma \frac{my}{r^3} = y, \frac{\Sigma m}{r_i^3}, \quad \Sigma \frac{mz}{r^3} = z, \frac{\Sigma m}{r_i^3}.$$

The second members of these equations are the expressions for the total accelerating force due to the action of the bodies of the system on M , resolved parallel to the co-ordinate axes respectively, when we consider the several masses to be collected at the centre of gravity of the system. Hence we conclude that when an element of mass is attracted by a system of bodies so remote from it that terms of the order of the squares of the co-ordinates of the several bodies, referred to the centre of gravity of the system as the origin of co-ordinates, may be neglected in comparison with the distance of the system from the point attracted, the action of the system will be the same as if the masses were all united at its centre of gravity.

If we suppose the masses $m, m', m'',$ &c. to be the elements of the mass of a single body, the form of the equations remains unchanged; and hence it follows that the mass M is acted upon by another mass, or by a system of bodies, as if the entire mass of the body, or of the system, were collected at its centre of gravity. It is evident, also, that reciprocally in the case of two systems of bodies, in which the mutual distances of the bodies are small in comparison with the distance between the centres of gravity of the two systems, their mutual action is the same as if all the several masses in each system were collected at the common centre of gravity of that system; and the two centres of gravity will move as if the masses were thus united.

13. The results already obtained are sufficient to enable us to form the equations for the motions of the several bodies which compose the solar system. If these bodies were exact spheres, which could be considered as composed of homogeneous concentric spherical shells, the density varying only from one layer to another, the action of

each on an element of the mass of another would be the same as if the entire mass of the attracting body were concentrated at its centre of gravity. The slight deviation from this law, arising from the ellipsoidal form of the heavenly bodies, is compensated by the magnitude of their mutual distances; and, besides, these mutual distances are so great that the action of the attracting body on the entire mass of the body attracted, is the same as if the latter were concentrated at its centre of gravity. Hence the consideration of the reciprocal action of the single bodies of the system, is reduced to that of material points corresponding to their respective centres of gravity, the masses of which, however, are equivalent to those of the corresponding bodies. The mutual distances of the bodies composing the secondary systems of planets attended with satellites are so small, in comparison with the distances of the different systems from each other and from the other planets, that they act upon these, and are reciprocally acted upon, in nearly the same manner as if the masses of the secondary systems were united at their common centres of gravity, respectively. The motion of the centre of gravity of a system consisting of a planet and its satellites is not affected by the reciprocal action of the bodies of that system, and hence it may be considered independently of this action. The difference of the action of the other planets on a planet and its satellites will simply produce inequalities in the relative motions of the latter bodies as determined by their mutual action alone, and will not affect the motion of their common centre of gravity. Hence, in the formation of the equations for the motion of translation of the centres of gravity of the several planets or secondary systems which compose the solar system, we have simply to consider them as points endowed with attractive forces corresponding to the several single or aggregated masses. The investigation of the motion of the satellites of each of the planets thus attended, forms a problem entirely distinct from that of the motion of the common centre of gravity of such a system. The consideration of the motion of rotation of the several bodies of the solar system about their respective centres of gravity, is also independent of the motion of translation. If the resultant of all the forces which act upon a planet passed through the centre of gravity, the motion of rotation would be undisturbed; and, since this resultant in all cases very nearly satisfies this condition, the disturbance of the motion of rotation is very slight. The inequalities thus produced in the motion of rotation are, in fact, sensible, and capable of being indicated by observation, only in the case of the earth and moon. It has, indeed,

been rigidly demonstrated that the axis of rotation of the earth relative to the body itself is fixed, so that the poles of rotation and the terrestrial equator preserve constantly the same position in reference to the surface; and that also the velocity of rotation is constant. This assures us of the permanency of geographical positions, and, in connection with the fact that the change of the length of the mean solar day arising from the variation of the obliquity of the ecliptic and in the length of the tropical year, due to the action of the sun, moon, and planets upon the earth, is absolutely insensible,—amounting to only a small fraction of a second in a million of years,—assures us also of the permanence of the interval which we adopt as the unit of time in astronomical investigations.

14. Placed, as we are, on one of the bodies of the system, it is only possible to deduce from observation the relative motions of the different heavenly bodies. These relative motions in the case of the comets and primary planets are referred to the centre of the sun, since the centre of gravity of this body is near the centre of gravity of the system, and its preponderant mass facilitates the integration of the equations thus obtained. In the case, however, of the secondary systems, the motions of the satellites are considered in reference to the centre of gravity of their primaries. We shall, therefore, form the equations for the motion of the planets relative to the centre of gravity of the sun; for which it becomes necessary to consider more particularly the relation between the heterogeneous quantities, space, time, and mass, which are involved in them. Each denomination, being divided by the unit of its kind, is expressed by an abstract number; and hence it offers no difficulty by its presence in an equation. For the unit of space we may arbitrarily take the mean distance of the earth from the sun, and the mean solar day may be taken as the unit of time. But, in order that when the space is expressed by 1, and the time by 1, the force or velocity may also be expressed by 1, if the unit of space is first adopted, the relation of the time and the mass—which determines the measure of the force—will be such that the units of both cannot be arbitrarily chosen. Thus, if we denote by f the acceleration due to the action of the mass m on a material point at the distance a , and by f' the acceleration corresponding to another mass m' acting at the same distance, we have the relation

$$\frac{f}{f'} = \frac{m}{m'};$$

and hence, since the acceleration is proportional to the mass, it may be taken as the measure of the latter. But we have, for the measure of f ,

$$f = \frac{d^2s}{dt^2}.$$

Integrating this, regarding f as constant, and the point to move from a state of rest, we get

$$s = \frac{1}{2}ft^2. \quad (13)$$

The acceleration in the case of a variable force is, at any instant, measured by the velocity which the force acting at that instant would generate, if supposed to remain constant in its action, during a unit of time. The last equation gives, when $t = 1$,

$$f = 2s;$$

and hence the acceleration is also measured by double the space which would be described by a material point, from a state of rest, during a unit of time, the force being supposed constant in its action during this time. In each case the duration of the unit of time is involved in the measure of the acceleration, and hence in that of the mass on which the acceleration depends; and the unit of mass, or of the force, will depend on the duration which is chosen for the unit of time. In general, therefore, we regard as the unit of mass that which, acting constantly at a distance equal to unity on a material point free to move, will give to this point, in a unit of time, a velocity which, if the force ceased to act, would cause it to describe the unit of distance in the unit of time.

Let the unit of time be a mean solar day; k^2 the acceleration due to the force exerted by the mass of the sun at the unit of distance; and f the acceleration corresponding to the distance r ; then will

$$f = \frac{k^2}{r^2},$$

and k^2 becomes the measure of the mass of the sun. The unit of mass is, therefore, equal to the mass of the sun taken as many times as k^2 is contained in unity. Hence, when we take the mean solar day as the unit of time, the mass of the sun is measured by k^2 ; by which we are to understand that if the sun acted during a mean solar day, on a material point free to move, at a distance constantly equal to the mean distance of the earth from the sun, it would, at the end of that time, have communicated to the point a velocity which, if

the force did not thereafter act, would cause it to describe, in a unit of time, the space expressed by k^2 .

The acceleration due to the action of the sun at the unit of distance is designated by k^2 , since the square root of this quantity appears frequently in the formulæ which will be derived.

If we take arbitrarily the mass of the sun as the unit of mass, the unit of time must be determined. Let t denote the number of mean solar days which must be taken for the unit of time when the unit of mass is the mass of the sun. The space which the force due to this mass, acting constantly on a material point at a distance equal to the mean distance of the earth from the sun, would cause the point to describe in the time t , is, according to equation (13),

$$s = \frac{1}{2}k^2t^2.$$

But, since t expresses the number of mean solar days in the unit of time, the measure of the acceleration corresponding to this unit is $2s$, and this being the unit of force, we have

$$k^2t^2 = 1;$$

and hence

$$t = \frac{1}{k}.$$

Therefore, if the mass of the sun is regarded as the unit of mass, the number of mean solar days in the unit of time will be equal to unity divided by the square root of the acceleration due to the force exerted by this mass at the unit of distance. The numerical value of k will be subsequently found to be 0.0172021, which gives 58.13244 mean solar days for the unit of time, when the mass of the sun is taken as the unit of mass.

15. Let x, y, z be the co-ordinates of a heavenly body referred to the centre of gravity of the sun as the origin of co-ordinates; r its *radius-vector*, or distance from this origin; and let m denote the quotient obtained by dividing its mass by that of the sun; then, taking the mean solar day as the unit of time, the mass of the sun is expressed by k^2 , and that of the planet or comet by mk^2 . For a second body let the co-ordinates be x', y', z' ; the distance from the sun, r' ; and the mass, $m'k^2$; and similarly for the other bodies of the system. Let the co-ordinates of the centre of gravity of the sun referred to any fixed point in space be ξ, η, ζ , the co-ordinate planes being parallel to those of x, y , and z , respectively; then will the

acceleration due to the action of m on the sun be expressed by $\frac{mk^2}{r^3}$. and the three components of this force in directions parallel to the co-ordinate axes, respectively, will be

$$mk^2 \frac{x}{r^3}, \quad mk^2 \frac{y}{r^3}, \quad mk^2 \frac{z}{r^3}.$$

The action of m' on the sun will be expressed by

$$m'k^2 \frac{x'}{r'^3}, \quad m'k^2 \frac{y'}{r'^3}, \quad m'k^2 \frac{z'}{r'^3},$$

and hence the acceleration due to the combined and simultaneous action of the several bodies of the system on the sun, resolved parallel to the co-ordinate axes, will be

$$k^2 \Sigma \frac{mx}{r^3}, \quad k^2 \Sigma \frac{my}{r^3}, \quad k^2 \Sigma \frac{mz}{r^3}.$$

The motion of the centre of gravity of the sun, relative to the fixed origin, will, therefore, be determined by the equations

$$\frac{d^2 \xi}{dt^2} = k^2 \Sigma \frac{mx}{r^3}, \quad \frac{d^2 \eta}{dt^2} = k^2 \Sigma \frac{my}{r^3}, \quad \frac{d^2 \zeta}{dt^2} = k^2 \Sigma \frac{mz}{r^3}. \quad (14)$$

Let ρ denote the distance of m from m' ; ρ' its distance from m'' , adding an accent for each successive body considered; then will the action of the bodies m' , m'' , &c. on m be

$$k^2 \Sigma \frac{m'}{\rho^2},$$

of which the three components parallel to the co-ordinate axes, respectively, are

$$k^2 \Sigma m' \frac{x' - x}{\rho^3}, \quad k^2 \Sigma m' \frac{y' - y}{\rho^3}, \quad k^2 \Sigma m' \frac{z' - z}{\rho^3}.$$

The action of the sun on m , resolved in the same manner, is expressed by

$$-\frac{k^2 x}{r^3}, \quad -\frac{k^2 y}{r^3}, \quad -\frac{k^2 z}{r^3},$$

which are negative, since the force tends to diminish the co-ordinates x , y , and z . The three components of the total action of the other bodies of the system on m are, therefore,

$$\begin{aligned}
& -\frac{k^2x}{r^3} + k^2\Sigma \frac{m'(x'-x)}{\rho^3}, \\
& -\frac{k^2y}{r^3} + k^2\Sigma \frac{m'(y'-y)}{\rho^3}, \\
& -\frac{k^2z}{r^3} + k^2\Sigma \frac{m'(z'-z)}{\rho^3};
\end{aligned}$$

and, since the co-ordinates of m referred to the fixed origin are

$$\xi + x, \quad \eta + y, \quad \zeta + z,$$

the equations which determine the absolute motion are

$$\begin{aligned}
\frac{d^2\xi}{dt^2} + \frac{d^2x}{dt^2} + \frac{k^2x}{r^3} &= k^2\Sigma \frac{m'(x'-x)}{\rho^3}, \\
\frac{d^2\eta}{dt^2} + \frac{d^2y}{dt^2} + \frac{k^2y}{r^3} &= k^2\Sigma \frac{m'(y'-y)}{\rho^3}, \\
\frac{d^2\zeta}{dt^2} + \frac{d^2z}{dt^2} + \frac{k^2z}{r^3} &= k^2\Sigma \frac{m'(z'-z)}{\rho^3},
\end{aligned} \tag{15}$$

the symbol of summation in the second members relating simply to the masses and co-ordinates of the several bodies which act on m , exclusive of the sun. Substituting for $\frac{d^2\xi}{dt^2}$, $\frac{d^2\eta}{dt^2}$, and $\frac{d^2\zeta}{dt^2}$ their values given by equations (14), we get

$$\begin{aligned}
\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3} &= k^2\Sigma m' \left(\frac{x'-x}{\rho^3} - \frac{x'}{r'^3} \right), \\
\frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3} &= k^2\Sigma m' \left(\frac{y'-y}{\rho^3} - \frac{y'}{r'^3} \right), \\
\frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} &= k^2\Sigma m' \left(\frac{z'-z}{\rho^3} - \frac{z'}{r'^3} \right).
\end{aligned} \tag{16}$$

Since x, y, z are the co-ordinates of m relative to the centre of gravity of the sun, these equations determine the motion of m relative to that point. The second members may be put in another form, which greatly facilitates the solution of some of the problems relating to the motion of m . Thus, let us put

$$u = \frac{m'}{1+m} \left(\frac{1}{\rho} - \frac{xx' + yy' + zz'}{r'^3} \right) + \frac{m''}{1+m} \left(\frac{1}{\rho'} - \frac{xx'' + yy'' + zz''}{r''^3} \right) + \dots \&c., \tag{17}$$

and we shall have for the partial differential coefficient of this with respect to x ,

$$\left(\frac{d\Omega}{dx} \right) = \frac{m'}{1+m} \left(-\frac{1}{\rho^2} \cdot \frac{d\rho}{dx} - \frac{x'}{r'^3} \right) + \frac{m''}{1+m} \left(-\frac{1}{\rho'^2} \frac{d\rho'}{dx} - \frac{x''}{r''^3} \right) + \dots \&c.$$

But, since

$$\begin{aligned}\rho^2 &= (x' - x)^2 + (y' - y)^2 + (z' - z)^2, \\ \rho'^2 &= (x'' - x)^2 + (y'' - y)^2 + (z'' - z)^2,\end{aligned}$$

we have

$$\frac{d\rho}{dx} = -\frac{x' - x}{\rho}, \quad \frac{d\rho'}{dx} = -\frac{x'' - x}{\rho'},$$

and hence we derive

$$\left(\frac{d\Omega}{dx}\right) = \frac{m'}{1+m} \left(\frac{x' - x}{\rho^3} - \frac{x'}{r^3}\right) + \frac{m''}{1+m} \left(\frac{x'' - x}{\rho'^3} - \frac{x''}{r'^3}\right) + \dots \&c.$$

or

$$(1+m) \left(\frac{d\Omega}{dx}\right) = \Sigma m' \left(\frac{x' - x}{\rho^3} - \frac{x'}{r^3}\right).$$

We find, also, in the same manner, for the partial differential coefficients with respect to y and z ,

$$\begin{aligned}(1+m) \left(\frac{d\Omega}{dy}\right) &= \Sigma m' \left(\frac{y' - y}{\rho^3} - \frac{y'}{r^3}\right), \\ (1+m) \left(\frac{d\Omega}{dz}\right) &= \Sigma m' \left(\frac{z' - z}{\rho^3} - \frac{z'}{r^3}\right).\end{aligned}$$

The equations (16), therefore, become

$$\begin{aligned}\frac{d^2x}{dt^2} + k^2(1+m) \frac{x}{r^3} &= k^2(1+m) \left(\frac{d\Omega}{dx}\right), \\ \frac{d^2y}{dt^2} + k^2(1+m) \frac{y}{r^3} &= k^2(1+m) \left(\frac{d\Omega}{dy}\right), \\ \frac{d^2z}{dt^2} + k^2(1+m) \frac{z}{r^3} &= k^2(1+m) \left(\frac{d\Omega}{dz}\right).\end{aligned} \tag{18}$$

It will be observed that the second members of equations (16) express the difference between the action of the bodies m' , m'' , &c. on m and on the sun, resolved parallel to the co-ordinate axes respectively. The mutual distances of the planets are such that these quantities are generally very small, and we may, therefore, in a first approximation to the motion of m relative to the sun, neglect the second members of these equations; and the integrals which may then be derived, express what is called the *undisturbed* motion of m . By means of the results thus obtained for the several bodies successively, the approximate values of the second members of equations (16) may be found, and hence a still closer approximation to the actual motion of m . The force whose components are expressed by the second members of these equations is called the *disturbing force*;

and, using the second form of the equations, the function \mathcal{Q} , which determines these components, is called the *perturbing function*. The complete solution of the problem is facilitated by an artifice of the infinitesimal calculus, known as the variation of parameters, or of constants, according to which the complete integrals of equations (16) are of the same form as those obtained by putting the second members equal to zero, the arbitrary constants, however, of the latter integration being regarded as variables. These constants of integration are the *elements* which determine the motion of m relative to the sun, and when the disturbing force is neglected the elements are pure constants. The variations of these, or of the co-ordinates, arising from the action of the disturbing force are, in almost all cases, very small, and are called the *perturbations*. The problem which first presents itself is, therefore, the determination of all the circumstances of the undisturbed motion of the heavenly bodies, after which the action of the disturbing forces may be considered.

It may be further remarked that, in the formation of the preceding equations, we have supposed the different bodies to be free to move, and, therefore, subject only to their mutual action. There are, indeed, facts derived from the study of the motion of the comets which seem to indicate that there exists in space a *resisting medium* which opposes the free motion of all the bodies of the system. If such a medium actually exists, its effect is very small, so that it can be sensible only in the case of rare and attenuated bodies like the comets, since the accumulated observations of the different planets do not exhibit any effect of such resistance. But, if we assume its existence, it is evidently necessary only to add to the second members of equations (16) a force which shall represent the effect of this resistance,—which, therefore, becomes a part of the disturbing force,—and the motion of m will be completely determined.

16. When we consider the undisturbed motion of a planet or comet relative to the sun, or simply the motion of the body relative to the sun as subject only to the reciprocal action of the two bodies, the equations (16) become

$$\begin{aligned}\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3} &= 0, \\ \frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3} &= 0, \\ \frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} &= 0.\end{aligned}\tag{19}$$

The equations for the undisturbed motion of a satellite relative to its primary are of the same form, the value of k^2 , however, being in this case the acceleration due to the force exerted by the mass of the primary at the unit of distance, and m the ratio of the mass of the satellite to that of the primary.

The integrals of these equations introduce six arbitrary constants of integration, which, when known, will completely determine the undisturbed motion of m relative to the sun.

If we multiply the first of these equations by y , and the second by x , and subtract the last product from the first, we shall find, by integrating the result,

$$\frac{xdy - ydx}{dt} = c,$$

c being an arbitrary constant.

In a similar manner, we obtain

$$\frac{xdz - zdx}{dt} = c', \quad \frac{ydz - zdy}{dt} = c''.$$

If we multiply these three equations respectively by z , $-y$, and x , and add the products, we obtain

$$cz - c'y + c''x = 0.$$

This, being the equation of a plane passing through the origin of co-ordinates, shows that the path of the body relative to the sun is a plane curve, and that *the plane of the orbit passes through the centre of the sun.*

Again, if we multiply the first of equations (19) by $2dx$, the second by $2dy$, and the third by $2dz$, take the sum and integrate, we shall find

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} + 2k^2(1+m) \int \frac{xdx + ydy + zdz}{r^3} = 0.$$

But, since $r^2 = x^2 + y^2 + z^2$, we shall have, by differentiation,

$$rdr = xdx + ydy + zdz.$$

Therefore, introducing this value into the preceding equation, we obtain

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2k^2(1+m)}{r} + h = 0, \quad (20)$$

h being an arbitrary constant.

If we add together the squares of the expressions for c , c' , and c'' and put $c^2 + c'^2 + c''^2 = 4f^2$, we shall have

$$\frac{(x^2 + y^2 + z^2)(dx^2 + dy^2 + dz^2)}{dt^2} - \frac{(xdx + ydy + zdz)^2}{dt^2} = 4f^2;$$

or

$$r^2 \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{r^2 dr^2}{dt^2} = 4f^2. \quad (21)$$

If we represent by dv the infinitely small angle contained between two consecutive radii-vectores r and $r + dr$, since $dx^2 + dy^2 + dz^2$ is the square of the element of path described by the body, we shall have

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 dv^2.$$

Substituting this value in the preceding equation, it becomes

$$r^2 dv = 2f dt. \quad (22)$$

The quantity $r^2 dv$ is double the area included by the element of path described in the element of time dt , and by the radii-vectores r and $r + dr$; and f , therefore, represents the *areal velocity*, which, being a constant, shows that the *radius-vector of a planet or comet describes equal areas in equal intervals of time*.

From the equations (20) and (21) we find, by elimination,

$$dt = \frac{r dr}{\sqrt{2rk^2(1+m) - hr^2 - 4f^2}}. \quad (23)$$

Substituting this value of dt in equation (22), we get

$$dv = \frac{2f dr}{r \sqrt{2rk^2(1+m) - hr^2 - 4f^2}}, \quad (24)$$

which gives, in order to find the maximum and minimum values of r ,

$$\frac{dr}{dv} = \frac{r \sqrt{2rk^2(1+m) - hr^2 - 4f^2}}{2f} = 0,$$

or

$$2rk^2(1+m) - hr^2 - 4f^2 = 0.$$

Therefore

$$\frac{k^2(1+m)}{h} + \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}},$$

and

$$\frac{k^2(1+m)}{h} - \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}},$$

are, respectively, the maximum and minimum values of r . The

points of the orbit, or trajectory of the body relative to the sun, corresponding to these values of r , are called the *apsides*; the former, the *aphelion*, and the latter, the *perihelion*. If we represent these values, respectively, by $a(1 + e)$ and $a(1 - e)$, we shall have

$$h = \frac{k^2(1 + m)}{a}; \quad 4f^2 = ak^2(1 + m)(1 - e^2) = k^2p(1 + m),$$

in which $p = a(1 - e^2)$. Introducing these values into the equation (24), it becomes

$$dv = \frac{\sqrt{p} dr}{r \sqrt{2r - \frac{1}{a}r^2 - p}} = - \frac{\frac{p}{e} d\frac{1}{r}}{\sqrt{1 - \left(\frac{p}{e} \cdot \frac{1}{r} - \frac{1}{e}\right)^2}},$$

the integral of which gives

$$v = \omega + \cos^{-1} \frac{1}{e} \left(\frac{p}{r} - 1 \right),$$

ω being an arbitrary constant. Therefore we shall have

$$\frac{1}{e} \left(\frac{p}{r} - 1 \right) = \cos(v - \omega),$$

from which we derive

$$r = \frac{p}{1 + e \cos(v - \omega)},$$

which is the polar equation of a conic section, the pole being at the focus, p being the semi-parameter, e the eccentricity, and $v - \omega$ the angle at the focus between the radius-vector and a fixed line, in the plane of the orbit, making the angle ω with the semi-transverse axis a .

If the angle $v - \omega$ is counted from the perihelion, we have $\omega = 0$, and

$$r = \frac{p}{1 + e \cos v}. \quad (25)$$

The angle v is called the *true anomaly*.

Hence we conclude that *the orbit of a heavenly body revolving around the sun is a conic section with the sun in one of the foci*. Observation shows that the planets revolve around the sun in ellipses, usually of small eccentricity, while the comets revolve either in ellipses of great eccentricity, in parabolas, or in hyperbolas, a circumstance which, as we shall have occasion to notice hereafter, greatly

lessens the amount of labor in many computations respecting their motion.

Introducing into equation (23) the values of h and $4f^2$ already found, we obtain

$$dt = \frac{\sqrt{a}}{k\sqrt{1+m}} \cdot \frac{rdr}{\sqrt{a^2e^2 - (a-r)^2}},$$

which may be written

$$dt = -\frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \cdot \frac{\left(1 - \frac{a-r}{a}\right) d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}},$$

or

$$dt = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left(\frac{-d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}} + e \frac{\frac{a-r}{ae} d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}} \right),$$

the integration of which gives

$$t = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left(\cos^{-1} \left(\frac{a-r}{ae} \right) - e \sqrt{1 - \left(\frac{a-r}{ae} \right)^2} \right) + C. \quad (26)$$

In the perihelion, $r = a(1-e)$, and the integral reduces to $t' = C$; therefore, if we denote the time from the perihelion by t_0 , we shall have

$$t_0 = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left(\cos^{-1} \left(\frac{a-r}{ae} \right) - e \sqrt{1 - \left(\frac{a-r}{ae} \right)^2} \right). \quad (27)$$

In the aphelion, $r = a(1+e)$; and therefore we shall have, for the time in which the body passes from the perihelion to the aphelion, $t_0 = \frac{1}{2}\tau$, or

$$\frac{1}{2}\tau = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \pi,$$

τ being the periodic time, or time of one revolution of the planet around the sun, a the semi-transverse axis of the orbit, or *mean distance* from the sun, and π the semi-circumference of a circle whose radius is unity. Therefore we shall have

$$\tau^2 = 4\pi^2 \frac{a^3}{k^2(1+m)}. \quad (28)$$

For a second planet, we shall have

$$\tau'^2 = 4\pi^2 \frac{a'^3}{k^2(1+m')};$$

and, consequently, between the mean distances and periodic times of any two planets, we have the relation

$$\frac{(1+m)\tau^2}{(1+m')\tau'^2} = \frac{a^3}{a'^3}. \quad (29)$$

If the masses of the two planets m and m' are very nearly the same, we may take $1+m=1+m'$; and hence, in this case, it follows that *the squares of the periodic times are to each other as the cubes of the mean distances from the sun*. The same result may be stated in another form, which is sometimes more convenient. Thus, since πab is the area of the ellipse, a and b representing the semi-axes, we shall have

$$\frac{\pi ab}{\tau} = f = \text{areal velocity};$$

and, since $b^2 = a^2(1-e^2)$, we have

$$f = \frac{\pi a^{\frac{3}{2}} a^{\frac{1}{2}} (1-e^2)^{\frac{1}{2}}}{\tau} = \frac{\pi a^{\frac{3}{2}} \sqrt{p}}{\tau},$$

which becomes, by substituting the value of τ already found,

$$f = \frac{1}{2} k \sqrt{p(1+m)}. \quad (30)$$

In like manner, for a second planet, we have

$$f' = \frac{1}{2} k \sqrt{p'(1+m')};$$

and, if the masses are such that we may take $1+m$ sensibly equal to $1+m'$, it follows that, in this case, *the areas described in equal times, in different orbits, are proportional to the square roots of their parameters*.

17. We shall now consider the signification of some of the constants of integration already introduced. Let i denote the inclination of the orbit of m to the plane of xy , which is thus taken as the plane of reference, and let Ω be the angle formed by the axis of x and the line of intersection of the plane of the orbit with the plane of xy ; then will the angles i and Ω determine the position of the plane of

the orbit in space. The constants c , c' , and c'' , involved in the equation

$$cz - c'y + c''x = 0,$$

are, respectively, double the projections, on the co-ordinate planes, xy , xz , and yz , of the areal velocity f ; and hence we shall have

$$c = 2f \cos i.$$

The projection of $2f$ on a plane passing through the intersection of the plane of the orbit with the plane of xy , and perpendicular to the latter, is

$$2f \sin i;$$

and the projection of this on the plane of xz , to which it is inclined at an angle equal to Ω , gives

$$c' = 2f \sin i \cos \Omega.$$

Its projection on the plane of yz gives

$$c'' = 2f \sin i \sin \Omega.$$

Hence we derive

$$z \cos i - y \sin i \cos \Omega + x \sin i \sin \Omega = 0, \quad (31)$$

which is the equation of the plane of the orbit; and, by means of the value of f in terms of p , and the values of c , c' , c'' , we derive, also,

$$\begin{aligned} x \frac{dy}{dt} - y \frac{dx}{dt} &= k \sqrt{p(1+m)} \cos i, \\ x \frac{dz}{dt} - z \frac{dx}{dt} &= k \sqrt{p(1+m)} \cos \Omega \sin i, \\ y \frac{dz}{dt} - z \frac{dy}{dt} &= k \sqrt{p(1+m)} \sin \Omega \sin i. \end{aligned} \quad (32)$$

These equations will enable us to determine Ω , i , and p , when, for any instant, the mass and co-ordinates of m , and the components of its velocity, in directions parallel to the co-ordinate axes, are known. The constants a and e are involved in the value of p , and hence four constants, or *elements*, are introduced into these equations, two of which, a and e , relate to the form of the orbit, and two, Ω and i , to the position of its plane in space. If we measure the angle $v - \omega$ from the point in which the orbit intersects the plane of xy , the constant ω will determine the position of the orbit in its own plane. Finally, the constant of integration C , in equation (26), is the time

of passage through the perihelion; and this determines the position of the body in its orbit. When these six constants are known, the undisturbed orbit of the body is completely determined.

Let V denote the velocity of the body in its orbit; then will equation (20) become

$$V^2 = k^2 (1 + m) \left(\frac{2}{r} - \frac{1}{a} \right).$$

At the perihelion, r is a minimum, and hence, according to this equation, the corresponding value of V is a maximum. At the aphelion, V is a minimum.

In the parabola, $a = \infty$; and hence

$$V = k \sqrt{1 + m} \sqrt{\frac{2}{r}},$$

which will determine the velocity at any instant, when r is known. It will be observed that the velocity, corresponding to the same value of r , in an elliptic orbit is less than in a parabolic orbit, and that, since a is negative in the hyperbola, the velocity in a hyperbolic orbit is still greater than in the case of the parabola. Further, since the velocity is thus found to be independent of the eccentricity, the direction of the motion has no influence on the species of conic section described.

If the position of a heavenly body at any instant, and the direction and magnitude of its velocity, are given, the relations already derived will enable us to determine the six constant elements of its orbit. But since we cannot know in advance the magnitude and direction of the primitive impulse communicated to the body, it is only by the aid of observation that these elements can be derived; and therefore, before considering the formulæ necessary to determine unknown elements by means of observed positions, we will investigate those which are necessary for the determination of the heliocentric and geocentric places of the body, assuming the elements to be known. The results thus obtained will facilitate the solution of the problem of finding the unknown elements from the data furnished by observation.

18. To determine the value of k , which is a constant for the solar system, we have, from equation (28),

$$k = \frac{2\pi}{\tau} \cdot \frac{a^{\frac{3}{2}}}{\sqrt{1 + m}}.$$

4

In the case of the earth, $\alpha = 1$, and therefore

$$k = \frac{2\pi}{\tau\sqrt{1+m}}.$$

In reducing this formula to numbers we should properly use, for τ , the absolute length of the sidereal year, which is invariable. The effect of the action of the other bodies of the system on the earth is to produce a very small secular change in its mean longitude corresponding to any fixed date taken as the epoch of the elements; and a correction corresponding to this secular variation should be applied to the value of τ derived from observation. The effect of this correction is slightly to increase the observed value of τ ; but to determine it with precision requires an exact knowledge of the masses of all the bodies of the system, and a complete theory of their relative motions,—a problem which is yet incompletely solved. Astronomical usage has, therefore, sanctioned the employment of the value of k found by means of the length of the sidereal year derived directly from observation. This is virtually adopting as the unit of space a distance which is very little less than the absolute, invariable mean distance of the earth from the sun; but, since this unit may be arbitrarily chosen, the accuracy of the results is not thereby affected.

The value of τ from which the adopted value of k has been computed, is 365.2563835 mean solar days; and the value of the combined mass of the earth and moon is

$$m = \frac{1}{354710}.$$

Hence we have $\log \tau = 2.5625978148$; $\log \sqrt{1+m} = 0.0000006122$; $\log 2\pi = 0.7981798684$; and, consequently,

$$\log k = 8.2355814414.$$

If we multiply this value of k by 206264.81, the number of seconds of arc corresponding to the radius of a circle, we shall obtain its value expressed in seconds of arc in a circle whose radius is unity, or on the orbit of the earth supposed to be circular. The value of k in seconds is, therefore,

$$\log k = 3.5500065746.$$

The quantity $\frac{2\pi}{\tau}$ expresses the mean angular motion of a planet in a mean solar day, and is usually designated by μ . We shall, therefore, have

$$\mu = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}, \quad (33)$$

for the expression for the *mean daily motion* of a planet.

Since, in the case of the earth, $\sqrt{1+m}$ differs very little from 1, it will be observed that k very nearly expresses the mean angular motion of the earth in a mean solar day.

In the case of a small planet or of a comet, the mass m is so small that it may, without sensible error, be neglected; and then we shall have

$$\mu = \frac{k}{a^{\frac{3}{2}}}. \quad (34)$$

For the old planets whose masses are considerable, the rigorous expression (33) must be used.

19. Let us now resume the polar equation of the ellipse, the pole being at the focus, which is

$$r = \frac{a(1-e^2)}{1+e\cos v}.$$

If we represent by φ the angle included between the conjugate axis and a line drawn from the extremity of this axis to the focus, we shall have

$$\sin \varphi = e;$$

and, since $a(1-e^2)$ is half the parameter of the transverse axis, which we have designated by p , we have

$$r = \frac{p}{1+\sin \varphi \cos v}.$$

The angle φ is called the *angle of eccentricity*.

Again, since $p = a(1-e^2) = a \cos^2 \varphi$, we have

$$r = \frac{a \cos^2 \varphi}{1+\sin \varphi \cos v} \quad (35)$$

It is evident, from this equation, that the maximum value of r in an elliptic orbit corresponds to $v = 180^\circ$, and that the minimum value of r corresponds to $v = 0$. It therefore increases from the perihelion to the aphelion, and then decreases as the planet approaches the perihelion.

In the case of the parabola, $\varphi = 90^\circ$, and $\sin \varphi = e = 1$; consequently,

$$r = \frac{p}{1 + \cos v}.$$

But, since $1 + \cos v = 2 \cos^2 \frac{1}{2}v$, if we put $q = \frac{1}{2}p$, we shall have

$$r = \frac{q}{\cos^2 \frac{1}{2}v}, \quad (36)$$

in which q is the *perihelion distance*. In this case, therefore, when $v = \pm 180^\circ$, r will be infinite, and the comet will never return, but course its way to other systems.

The angle φ cannot be applied to the case of the hyperbola, since in a hyperbolic orbit e is greater than 1; and, therefore, the eccentricity cannot be expressed by the sine of an arc. If, however, we designate by ψ the angle which the asymptote to the hyperbola makes with the transverse axis, we shall have

$$e \cos \psi = 1.$$

Introducing this value of e into the polar equation of the hyperbola, it becomes

$$r = \frac{p \cos \psi}{\cos v + \cos \psi}.$$

But, since $\cos v + \cos \psi = 2 \cos \frac{1}{2}(v + \psi) \cos \frac{1}{2}(v - \psi)$, this gives

$$r = \frac{p \cos \psi}{2 \cos \frac{1}{2}(v + \psi) \cos \frac{1}{2}(v - \psi)}. \quad (37)$$

It appears from this formula that r increases with v , and becomes infinite when $1 + e \cos v = 0$, or $\cos v = -\cos \psi$, in which case $v = 180^\circ - \psi$: consequently, the maximum positive value of v is represented by $180^\circ - \psi$, and the maximum negative value by $-(180^\circ - \psi)$. Further, it is evident that the orbit will be that branch of the hyperbola which corresponds to the focus in which the sun is placed, since, under the operation of an attractive force, the path of the body must be concave toward the centre of attraction. A body subject to a force of repulsion of the same intensity, and varying according to the same law, would describe the other branch of the curve.

The problem of finding the position of a heavenly body as seen from any point of reference, consists of two parts: first, the determination of the place of the body in its orbit; and then, by means of this and of the elements which fix the position of the plane of the

orbit, and that of the orbit in its own plane, the determination of the position in space.

In deriving the formulæ for finding the place of the body in its orbit, we will consider each species of conic section separately, commencing with the ellipse.

20. Since the value of $a - r$ can never exceed the limits $-ae$ and $+ae$, we may introduce an auxiliary angle such that we shall have

$$\frac{a - r}{ae} = \cos E.$$

This auxiliary angle E is called the *eccentric anomaly*; and its geometrical signification may be easily known from its relation to the true anomaly. Introducing this value of $\frac{a - r}{ae}$ into the equation (27) and writing $t - T$ in place of t_0 , T being the *time of perihelion passage*, and t the time for which the place of the planet in its orbit is to be computed, we obtain

$$\frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}(t - T) = E - e \sin E. \quad (38)$$

But $\frac{k\sqrt{1+m}}{a^{\frac{3}{2}}} = \text{mean daily motion of the planet} = \mu$; therefore

$$\mu(t - T) = E - e \sin E.$$

The quantity $\mu(t - T)$ represents what would be the angular distance from the perihelion if the planet had moved uniformly in a circular orbit whose radius is a , its mean distance from the sun. It is called the *mean anomaly*, and is usually designated by M . We shall, therefore, have

$$\begin{aligned} M &= \mu(t - T), \\ M &= E - e \sin E. \end{aligned} \quad (39)$$

When the planet or comet is in its perihelion, the true anomaly, mean anomaly, and eccentric anomaly are each equal to zero. All three of these increase from the perihelion to the aphelion, where they are each equal to 180° , and decrease from the aphelion to the perihelion, provided that they are considered negative. From the perihelion to the aphelion v is greater than E , and E is greater than M . The same relation holds true from the aphelion to the perihelion, if we regard, in this case, the values of v , E , and M as negative.

As soon as the auxiliary angle E is obtained by means of the mean motion and eccentricity, the values of r and v may be derived. For

this purpose there are various formulæ which may be applied in practice, and which we will now develop.

The equation

$$\frac{a-r}{ae} = \cos E,$$

gives

$$r = a(1 - e \cos E). \quad (40)$$

This also gives

$$\frac{a-r}{e} - ae = a \cos E - ae,$$

or

$$\frac{p-r}{e} = a \cos E - ae,$$

which, by means of equation (25), reduces to

$$r \cos v = a \cos E - ae. \quad (41)$$

If we square both members of equations (40) and (41), and subtract the latter result from the former, we get

$$r^2 \sin^2 v = a^2 (1 - e^2) \sin^2 E,$$

or

$$r \sin v = a \sqrt{1 - e^2} \sin E = b \sin E. \quad (42)$$

By means of the equations (41) and (42) it may be easily shown that the auxiliary angle E , or eccentric anomaly, is the angle at the centre of the ellipse between the semi-transverse axis, and a line drawn from the centre to the point where the prolongation of the ordinate perpendicular to this axis, and drawn through the place of the body, meets the circumference of the circumscribed circle.

Equations (40) and (41) give

$$r(1 \mp \cos v) = a(1 \pm e)(1 \mp \cos E).$$

By using first the upper sign, and then the lower sign, we obtain, by reduction,

$$\begin{aligned} \sqrt{r} \sin \tfrac{1}{2}v &= \sqrt{a(1+e)} \sin \tfrac{1}{2}E, \\ \sqrt{r} \cos \tfrac{1}{2}v &= \sqrt{a(1-e)} \cos \tfrac{1}{2}E, \end{aligned} \quad (43)$$

which are convenient for the calculation of r and v , and especially so when several places are required. By division, these equations give

$$\tan \tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2}E. \quad (44)$$

Since $e = \sin \varphi$, we have

$$\frac{1-e}{1+e} = \frac{1-\sin \varphi}{1+\sin \varphi} = \tan^2(45^\circ - \tfrac{1}{2}\varphi).$$

Consequently,

$$\tan \tfrac{1}{2}E = \tan(45^\circ - \tfrac{1}{2}\varphi) \tan \tfrac{1}{2}v. \quad (45)$$

Again,

$$\sqrt{1+e} = \sqrt{1+\sin \varphi} = \sqrt{1+2\sin \tfrac{1}{2}\varphi \cos \tfrac{1}{2}\varphi},$$

which may be written

$$\sqrt{1+e} = \sqrt{\sin^2 \tfrac{1}{2}\varphi + \cos^2 \tfrac{1}{2}\varphi + 2\sin \tfrac{1}{2}\varphi \cos \tfrac{1}{2}\varphi},$$

or

$$\sqrt{1+e} = \sin \tfrac{1}{2}\varphi + \cos \tfrac{1}{2}\varphi.$$

In a similar manner we find

$$\sqrt{1-e} = -\sin \tfrac{1}{2}\varphi + \cos \tfrac{1}{2}\varphi.$$

From these two equations we obtain

$$\begin{aligned} \sqrt{1+e} + \sqrt{1-e} &= 2 \cos \tfrac{1}{2}\varphi, \\ \sqrt{1+e} - \sqrt{1-e} &= 2 \sin \tfrac{1}{2}\varphi, \end{aligned} \quad (46)$$

which are convenient in many transformations of equations involving e or φ .

Equation (42) gives

$$\sin E = \frac{r \sin v}{b} = \frac{p \sin v}{b(1+e \cos v)};$$

but $p = a \cos^2 \varphi$, and $b = a \cos \varphi$, hence

$$\sin E = \frac{r \sin v}{a \cos \varphi} = \frac{\cos \varphi \sin v}{1+e \cos v}. \quad (47)$$

Equation (41) gives

$$\cos E = \frac{r \cos v + ae}{a} = \frac{p \cos v}{a(1+e \cos v)} + e,$$

or

$$\cos E = \frac{p \cos v + ae + ae^2 \cos v}{a(1+e \cos v)};$$

and, putting $a \cos^2 \varphi$ instead of p , and $\sin \varphi$ for e , we get

$$\cos E = \frac{\cos v + e}{1+e \cos v}. \quad (48)$$

If we multiply the first of equations (43) by $\cos \tfrac{1}{2}E$, and the

second by $\sin \frac{1}{2}E$, successively add and subtract the products, and reduce by means of the preceding equations, we obtain

$$\begin{aligned}\sin \frac{1}{2}(v + E) &= \sqrt{\frac{a}{r}} \cos \frac{1}{2}\varphi \sin E, \\ \sin \frac{1}{2}(v - E) &= \sqrt{\frac{a}{r}} \sin \frac{1}{2}\varphi \sin E.\end{aligned}\quad (49)$$

The perihelion distance, in an elliptic orbit, is given by the equation

$$q = a(1 - e).$$

21. The difference between the true and the mean anomaly, or $v - M$, is called the *equation of the centre*, and is positive from the perihelion to the aphelion, and negative from the aphelion to the perihelion. When the body is in either apsis, the equation of the centre will be equal to zero.

We have, from equation (39),

$$E = M + e \sin E.$$

Expanding this by Lagrange's theorem, we get

$$\begin{aligned}F(E) &= F(M) + \sin M \frac{dF(M)}{dM} \cdot \frac{e}{1} + \frac{d}{dM} \left(\sin^2 M \frac{dF(M)}{dM} \right) \frac{e^2}{1 \cdot 2} \\ &\quad + \frac{d^2}{dM^2} \left(\sin^3 M \frac{dF(M)}{dM} \right) \frac{e^3}{1 \cdot 2 \cdot 3} + \dots\end{aligned}\quad (50)$$

Let us now take, equation (40),

$$F(E) = (1 - e \cos E)^{-2} = \frac{a^2}{r^2},$$

and, consequently,

$$F(M) = (1 - e \cos M)^{-2}.$$

Therefore we shall have

$$\begin{aligned}\frac{a^2}{r^2} &= (1 - e \cos M)^{-2} - 2e^2 \sin^2 M (1 - e \cos M)^{-3} \\ &\quad - e^3 \frac{d}{dM} (\sin^3 M (1 - e \cos M)^{-3}) - \dots\end{aligned}$$

Expanding these terms, and performing the operations indicated, we get

$$\begin{aligned}\frac{a^2}{r^2} &= 1 + 2e \cos M + \frac{e^2}{2} (6 \cos^2 M - 4 \sin^2 M) \\ &\quad + \frac{e^3}{4} (16 \cos^3 M - 36 \sin^2 M \cos M) + \dots,\end{aligned}$$

which reduces to

$$\frac{a^2}{r^2} = 1 + 2e \cos M + \frac{e^2}{2}(1 + 5 \cos 2M) + \frac{e^3}{4}(13 \cos 3M + 3 \cos M) + \dots \quad (51)$$

Equation (22) gives

$$dv = \frac{2fdt}{r^2},$$

and, since $f = \frac{1}{2}k\sqrt{p(1+m)}$, we have

$$dv = \frac{k\sqrt{p(1+m)}}{r^2} dt, \quad (52)$$

or

$$dv = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}} \cdot \frac{a^2}{r^2} \sqrt{1-e^2} dt.$$

But $\frac{k\sqrt{1+m}}{a^{\frac{3}{2}}} = \mu$, and therefore

$$dv = \sqrt{1-e^2} \frac{a^2}{r^2} \mu dt = \sqrt{1-e^2} \frac{a^2}{r^2} dM.$$

By expanding the factor $\sqrt{1-e^2}$, we obtain

$$\sqrt{1-e^2} = 1 - \frac{1}{2}e^2 - \frac{1}{8}e^4 - \dots,$$

and hence

$$dv = (1 - \frac{1}{2}e^2 - \dots) \frac{a^2}{r^2} dM.$$

Substituting for $\frac{a^2}{r^2}$ its value from equation (51), and integrating, we get, since $v = 0$ when $M = 0$,

$$v - M = 2e \sin M + \frac{5}{4}e^2 \sin 2M + \frac{e^3}{12}(13 \sin 3M - 3 \sin M) + \dots \quad (53)$$

which is the expression for the *equation of the centre* to terms involving e^3 . In the same manner, this series may be extended to higher powers of e .

When the eccentricity is very small, this series converges very rapidly; and the value of $v - M$ for any planet may be arranged in a table with the argument M .

For the purpose, however, of computing the places of a heavenly body from the elements of its orbit, it is preferable to solve the equations which give v and E directly; and when the eccentricity is

very great, this mode is indispensable, since the series will not in that case be sufficiently convergent.

It will be observed that the formula which must be used in obtaining the eccentric anomaly from the mean anomaly is transcendental, and hence it can only be solved either by series or by trial. But fortunately, indeed, it so happens that the circumstances of the celestial motions render these approximations very rapid, the orbits being usually either nearly circular, or else very eccentric.

If, in equation (50), we put $F(E) = E$, and consequently $F(M) = M$, we shall have, performing the operations indicated and reducing,

$$E = M + e \sin M + \frac{1}{2}e^2 \sin 2M + \&c. \quad (54)$$

Let us now denote the approximate value of E computed from this equation by E_0 , then will

$$E_0 + \Delta E_0 = E,$$

in which ΔE_0 is the correction to be applied to the assumed value of E . Substituting this in equation (39), we get

$$M = E_0 + \Delta E_0 - e \sin E_0 - e \cos E_0 \Delta E_0;$$

and, denoting by M_0 the value of M corresponding to E_0 , we shall also have

$$M_0 = E_0 - e \sin E_0.$$

Subtracting this equation from the preceding one, we obtain

$$\frac{M - M_0}{1 - e \cos E_0} = \Delta E_0.$$

It remains, therefore, only to add the value of ΔE_0 found from this formula to the first assumed value of E , or to E_0 , and then, using this for a new value of E_0 , to proceed in precisely the same manner for a second approximation, and so on, until the correct value of E is obtained. When the values of E for a succession of dates, at equal intervals, are to be computed, the assumed values of E_0 may be obtained so closely by interpolation that the first approximation, in the manner just explained, will give the correct value; and in nearly every case two or three approximations in this manner will suffice.

Having thus obtained the value of E corresponding to M for any instant of time, we may readily deduce from it, by the formulæ already investigated, the corresponding values of r and v .

In the case of an ellipse of very great eccentricity, corresponding to the orbits of many of the comets, the most convenient method of

computing r and v , for any instant, is somewhat different. The manner of proceeding in the computation in such cases we shall consider hereafter; and we will now proceed to investigate the formulæ for determining r and v , when the orbit is a parabola, the formulæ for elliptic motion not being applicable, since, in the parabola, $a = \infty$, and $e = 1$.

22. Observation shows that the masses of the comets are insensible in comparison with that of the sun; and, consequently, in this case, $m = 0$ and equation (52), putting for p its value $2q$, becomes

$$k\sqrt{2q} \, dt = r^2 dv,$$

or

$$k\sqrt{2q} \, dt = \frac{2q^2}{\cos^{\frac{1}{2}} v} \frac{1}{2} dv,$$

which may be written

$$\frac{kdt}{\sqrt{2q}^{\frac{3}{2}}} = \frac{1}{2}(1 + \tan^2 \frac{1}{2}v) \sec^2 \frac{1}{2}v dv = (1 + \tan^2 \frac{1}{2}v) d \tan \frac{1}{2}v.$$

Integrating this expression between the limits T and t , we obtain

$$\frac{k(t-T)}{\sqrt{2q}^{\frac{3}{2}}} = \tan \frac{1}{2}v + \frac{1}{3} \tan^3 \frac{1}{2}v, \quad (55)$$

which is the expression for the relation between the true anomaly and the time from the perihelion, in a parabolic orbit.

Let us now represent by τ_0 the time of describing the arc of a parabola corresponding to $v = 90^\circ$; then we shall have

$$\frac{k\tau_0}{\sqrt{2q}^{\frac{3}{2}}} = \frac{4}{3},$$

or

$$\frac{3k}{\sqrt{2}} = \frac{4q^{\frac{3}{2}}}{\tau_0}.$$

Now, $\frac{3k}{\sqrt{2}}$ is constant, and its logarithm is 8.5621876983; and if we take $q = 1$, which is equivalent to supposing the comet to move in a parabola whose perihelion distance is equal to the semi-transverse axis of the earth's orbit, we find

$$\log \tau_0^{\text{days}} = 2.03987229, \text{ or } \tau_0 = 109.61558 \text{ days;}$$

that is, a comet moving in a parabola whose perihelion distance

is equal to the mean distance of the earth from the sun, requires 109.61558 days to describe an arc corresponding to $v = 90^\circ$.

Equation (55) contains only such quantities as are comparable with each other, and by it $t - T$, the time from the perihelion, may be readily found when the remaining terms are known; but, in order to find v from this formula, it will be necessary to solve the equation of the third degree, $\tan \frac{1}{2}v$ being the unknown quantity. If we put $x = \tan \frac{1}{2}v$, this equation becomes

$$x^3 + 3x - a = 0,$$

in which a is the known quantity, and is negative before, and positive after, the perihelion passage. According to the general principle in the theory of equations that in every equation, whether complete or incomplete, the number of positive roots cannot exceed the number of variations of sign, and that the number of negative roots cannot exceed the number of variations of sign, when the signs of the terms containing the odd powers of the unknown quantity are changed, it follows that when a is positive, there is one positive root and no negative root. When a is negative, there is one negative root and no positive root; and hence we conclude that equation (55) can have but one real root.

We may dispense with the direct solution of this equation by forming a table of the values of v corresponding to those of $t - T$ in a parabola whose perihelion distance is equal to the mean distance of the earth from the sun. This table will give the time corresponding to the anomaly v in any parabola, whose perihelion distance is q , by multiplying by $q^{\frac{3}{2}}$, the time which corresponds to the same anomaly in the table. We shall have the anomaly v corresponding to the time $t - T$ by dividing $t - T$ by $q^{\frac{3}{2}}$, and seeking in the table the anomaly corresponding to the time resulting from this division.

A more convenient method, however, of finding the true anomaly from the time, and the reverse, is to use a table of the form generally known as Barker's Table. The following will explain its construction:—

Multiplying equation (55) by 75, we obtain

$$\frac{75k}{\sqrt{2} q^{\frac{3}{2}}} (t - T) = 75 \tan \frac{1}{2}v + 25 \tan^3 \frac{1}{2}v.$$

Let us now put

$$M = 75 \tan \frac{1}{2}v + 25 \tan^3 \frac{1}{2}v,$$

and $C_0 = \frac{75k}{\sqrt{2}}$, which is a constant quantity; then will

$$\frac{C_0}{q^{\frac{3}{2}}}(t - T) = M.$$

The value of C_0 is

$$\log C_0 = 9.9601277069.$$

Again, let us take

$$m = \frac{C_0}{q^{\frac{3}{2}}},$$

which is called the mean daily motion in the parabola; then will

$$M = m(t - T) = 75 \tan^{\frac{1}{2}} v + 25 \tan^{\frac{3}{2}} v.$$

If we now compute the values of M corresponding to successive values of v from $v = 0^\circ$ to $v = 180^\circ$, and arrange them in a table with the argument v , we may derive at once, from this table, for the time $(t - T)$ either M when v is known, or v when $M = m(t - T)$ is known. It may also be observed that when $t - T$ is negative, the value of v is considered as being negative, and hence it is not necessary to pay any further attention to the algebraic sign of $t - T$ than to give the same sign to the value of v obtained from the table.

Table VI. gives the values of M for values of v from 0° to 180° , with differences for interpolation, the application of which will be easily understood.

23. When v approaches near to 180° , this table will be extremely inconvenient, since, in this case, the differences between the values of M for a difference of one minute in the value of v increase very rapidly; and it will be very troublesome to obtain the value of v from the table with the requisite degree of accuracy. To obviate the necessity of extending this table, we proceed in the following manner:—

Equation (55) may be written

$$\frac{k(t - T)}{\sqrt{2} q^{\frac{3}{2}}} = \frac{1}{3} \tan^{\frac{3}{2}} v (1 + 3 \cot^2 \frac{1}{2} v);$$

and, multiplying and dividing the second member by $(1 + \cot^2 \frac{1}{2} v)^3$, we shall have

$$\frac{k(t - T)}{\sqrt{2} q^{\frac{3}{2}}} = \frac{1}{3} \tan^{\frac{3}{2}} v (1 + \cot^2 \frac{1}{2} v)^3 \frac{1 + 3 \cot^2 \frac{1}{2} v}{(1 + \cot^2 \frac{1}{2} v)^3}.$$

But $1 + \cot^2 \frac{1}{2}v = \frac{2}{\sin v \tan \frac{1}{2}v}$ and consequently

$$\frac{k(t - T)}{\sqrt{2} q^{\frac{3}{2}}} = \frac{8}{3 \sin^3 v} \cdot \frac{1 + 3 \cot^2 \frac{1}{2}v}{(1 + \cot^2 \frac{1}{2}v)^3}.$$

Now, when v approaches near to 180° , $\cot \frac{1}{2}v$ will be very small, and the second factor of the second member of this equation will nearly $= 1$. Let us therefore denote by w the value of v on the supposition that this factor is equal to unity, which will be strictly true when $v = 180^\circ$, and we shall have, for the correct value of v , the following equation:

$$v = w + \Delta_0,$$

Δ_0 being a very small quantity. We shall therefore have

$$\frac{8}{\sin^3 w} = 3 \tan \frac{1}{2}(w + \Delta_0) + \tan^3 \frac{1}{2}(w + \Delta_0),$$

and, putting $\tan \frac{1}{2}w = \theta$, and $\tan \frac{1}{2}\Delta_0 = x$, we get, from this equation,

$$\frac{(1 + \theta^2)^3}{\theta^3} = 3 \frac{\theta + x}{1 - \theta x} + \frac{(\theta + x)^3}{(1 - \theta x)^3}.$$

Multiplying this through by $\theta^3(1 - \theta x)^3$, expanding and reducing, there results the following equation:

$$1 + 3\theta^2 = 3\theta(1 + 4\theta^2 + 2\theta^4 + \theta^6)x - 3\theta^2(1 + 4\theta^2 + 2\theta^4 + \theta^6)x^3 + \theta^3(2 + 6\theta^2 + 3\theta^4 + \theta^6)x^3.$$

Dividing through by the coefficient of x , we obtain

$$\frac{1 + 3\theta^2}{3\theta(1 + 4\theta^2 + 2\theta^4 + \theta^6)} = x - \theta x^2 + \frac{\theta^2(2 + 6\theta^2 + 3\theta^4 + \theta^6)x^3}{3(1 + 4\theta^2 + 2\theta^4 + \theta^6)}.$$

Let us now put

$$\frac{1 + 3\theta^2}{3\theta(1 + 4\theta^2 + 2\theta^4 + \theta^6)} = y;$$

then, substituting this in the preceding equation, inverting the series and reducing, we obtain finally

$$x = y + \theta y^2 + \frac{\theta^2(4 + 18\theta^2 + 9\theta^4 + 5\theta^6)}{3(1 + 4\theta^2 + 2\theta^4 + \theta^6)} y^3 + \&c.$$

But $\tan \frac{1}{2}\Delta_0 = x$, therefore

$$\Delta_0 = 2x - \frac{2}{3}x^3 + \dots$$

Substituting in this the value of x above found, and reducing, we obtain

$$\Delta_0 = 2y + 2\theta y^2 + \frac{-2 + 32\theta^4 + 16\theta^6 + 10\theta^8}{3(1 + 4\theta^2 + 2\theta^4 + \theta^6)} y^3 + \&c.$$

For all the cases in which this equation is to be applied, the third term of the second member will be insensible, and we shall have, to a sufficient degree of approximation,

$$\Delta_0 = 2y + 2\theta y^2.$$

Table VII. gives the values of Δ_0 , expressed in seconds of arc, corresponding to consecutive values of w from $w = 155^\circ$ to $w = 180^\circ$. In the application of this table, we have only to compute the value of M precisely as for the case in which Table VI. is to be used, namely,

$$M = m(t - T);$$

then will w be given by the formula

$$\sin w = \sqrt[3]{\frac{200}{M}},$$

since we have already found

$$\frac{k(t - T)}{\sqrt[3]{2} q^{\frac{3}{2}}} = \frac{8}{3 \sin^3 w},$$

or

$$\sin w = \sqrt[3]{\frac{8q^{\frac{3}{2}}\sqrt{2}}{3(t - T)k}} = \sqrt[3]{\frac{200}{M}}.$$

Having computed the value of w from this equation, Table VII. will furnish the corresponding value of Δ_0 ; and then we shall have, for the correct value of the true anomaly,

$$v = w + \Delta_0,$$

which will be precisely the same as that obtained directly from Table VI., when the second and higher orders of differences are taken into account.

If v is given and the time $t - T$ is required, the table will give, by inspection, an approximate value of Δ_0 , using v as argument, and then w is given by

$$w = v - \Delta_0.$$

The exact value of Δ_0 is then found from the table, and hence we derive that of w ; and finally $t - T$ from

$$t - T = \frac{200}{C_0} \cdot \frac{q^{\frac{3}{2}}}{\sin^3 w}.$$

24. The problem of finding the time $t - T$ when the true anomaly is given, may also be solved conveniently, and especially so when v is small, by the following process:—

Equation (55) is easily transformed into

$$\frac{3k(t - T)}{\sqrt{2} q^{\frac{3}{2}}} = \frac{\sin \frac{1}{2}v}{\cos^3 \frac{1}{2}v} (3 - 2 \sin^2 \frac{1}{2}v),$$

from which we obtain, since $q = r \cos^2 \frac{1}{2}v$,

$$\frac{3k(t - T)}{2 r^{\frac{3}{2}}} = 3 \left(\frac{\sin \frac{1}{2}v}{\sqrt{2}} \right) - 4 \left(\frac{\sin \frac{1}{2}v}{\sqrt{2}} \right)^3.$$

Let us now put

$$\sin x = \frac{\sin \frac{1}{2}v}{\sqrt{2}},$$

and we have

$$\frac{3k(t - T)}{2 r^{\frac{3}{2}}} = 3 \sin x - 4 \sin^3 x = \sin 3x.$$

Consequently,

$$t - T = \frac{2}{3k} r^{\frac{3}{2}} \sin 3x,$$

which admits of an accurate and convenient numerical solution. To facilitate the calculation we put

$$N = \frac{\sin 3x}{\sin v},$$

the values of which may be tabulated with the argument v . When $v = 0$, we shall have $N = \frac{1}{2}\sqrt{2}$, and when $v = 90^\circ$, we have $N = 1$; from which it appears that the value of N changes slowly for values of v from 0° to 90° . But when $v = 180^\circ$, we shall have $N = \infty$ and hence, when v exceeds 90° , it becomes necessary to introduce an auxiliary different from N . We shall, therefore, put in this case,

$$N' = N \sin v = \sin 3x;$$

from which it appears that $N' = 1$ when $v = 90^\circ$, and that $N' = \frac{1}{2}$ when $v = 180^\circ$. Therefore we have, finally, when v is less than 90° ,

$$t - T = \frac{2}{3k} N r^{\frac{3}{2}} \sin v,$$

and, when v is greater than 90° ,

$$t - T = \frac{2}{3k} N' r^{\frac{3}{2}},$$

in which $\log \frac{2}{3k} = 1.5883272995$, from which $t - T$ is easily derived when v is known.

Table VIII. gives the values of N , with differences for interpolation, for values of v from $v = 0^\circ$ to $v = 90^\circ$, and the values of N' for those of v from $v = 90^\circ$ to $v = 180^\circ$.

25. We shall now consider the case of the hyperbola, which differs from the ellipse only that e is greater than 1; and, consequently, the formulæ for elliptic and hyperbolic motion will differ from each other only that certain quantities which are positive in the ellipse are negative or imaginary in the hyperbola. We may, however, introduce auxiliary quantities which will serve to preserve the analogy between the two, and yet to mark the necessary distinctions.

For this purpose, let us resume the equation

$$r = \frac{p \cos \psi}{2 \cos \frac{1}{2}(v + \psi) \cos \frac{1}{2}(v - \psi)}.$$

When $v = 0$, the factors $\cos \frac{1}{2}(v + \psi)$ and $\cos \frac{1}{2}(v - \psi)$ in the denominator will be equal; and since the limits of the values of v are $180^\circ - \psi$ and $-(180^\circ - \psi)$, it follows that the first factor will vanish for the maximum positive value of v , and that the second factor will vanish for the maximum negative value of v , and, therefore, that, in either case, $r = \infty$.

In the hyperbola, the semi-transverse axis is negative, and, consequently, we have, in this case,

$$p = a(e^2 - 1), \quad \text{or } a = p \cot^2 \frac{1}{2} \psi.$$

We have, also, for the perihelion distance,

$$q = a(e - 1).$$

Let us now put

$$\tan \frac{1}{2} F = \tan \frac{1}{2} v \sqrt{\frac{e - 1}{e + 1}}, \quad (56)$$

which is analogous to the formula for the eccentric anomaly E in an ellipse; and, since $e = \frac{1}{\cos \psi}$, we shall have

$$\frac{e-1}{e+1} = \frac{1-\cos \psi}{1+\cos \psi} = \tan^2 \frac{1}{2} \psi,$$

and, consequently,

$$\tan \frac{1}{2} F = \tan \frac{1}{2} v \tan \frac{1}{2} \psi. \quad (57)$$

We shall now introduce an auxiliary quantity σ , such that

$$\sigma = \tan (45^\circ + \frac{1}{2} F) = \frac{1 + \tan \frac{1}{2} F}{1 - \tan \frac{1}{2} F},$$

whence we derive

$$\tan \frac{1}{2} F = \frac{\sigma - 1}{\sigma + 1}, \quad (58)$$

and also

$$\sigma = \frac{\cos \frac{1}{2} (v - \psi)}{\cos \frac{1}{2} (v + \psi)}. \quad (59)$$

This last equation shows that $\sigma = 1$ when the comet is in its perihelion; $\sigma = \infty$ when $v = 180^\circ - \psi$; and $\sigma = 0$ when $v = -(180^\circ - \psi)$.

Since $\tan F = \frac{2 \tan \frac{1}{2} F}{1 - \tan^2 \frac{1}{2} F}$, we shall have

$$\tan F = \frac{2 \left(\frac{\sigma - 1}{\sigma + 1} \right)}{1 - \left(\frac{\sigma - 1}{\sigma + 1} \right)^2} = \frac{1}{2} \left(\sigma - \frac{1}{\sigma} \right). \quad (60)$$

Squaring this equation, adding 1 to both members, and reducing we obtain

$$\frac{1}{\cos F} = \frac{1}{2} \left(\sigma + \frac{1}{\sigma} \right). \quad (61)$$

Replacing σ in this equation by its value from equation (59), we get

$$\frac{1}{\cos F} = \frac{\cos^2 \frac{1}{2} (v + \psi) + \cos^2 \frac{1}{2} (v - \psi)}{2 \cos \frac{1}{2} (v + \psi) \cos \frac{1}{2} (v - \psi)},$$

or

$$\frac{1}{\cos F} = \frac{1 + \cos v \cos \psi}{2 \cos \frac{1}{2} (v + \psi) \cos \frac{1}{2} (v - \psi)} = \frac{(e + \cos v) \cos \psi}{2 \cos \frac{1}{2} (v + \psi) \cos \frac{1}{2} (v - \psi)}$$

which reduces to

$$\frac{1}{\cos F} = \frac{r(e + \cos v)}{p}. \quad (62)$$

If we add ∓ 1 to both members of this equation, we shall have

$$\frac{1 \mp \cos F}{\cos F} = \frac{r(e \mp 1)(1 \mp \cos v)}{p}.$$

Taking first the upper sign, and then the lower sign, and reducing, we get

$$\begin{aligned}\sqrt{r} \sin \tfrac{1}{2}v &= \frac{\sqrt{a(e+1)}}{\sqrt{\cos F}} \sin \tfrac{1}{2}F, \\ \sqrt{r} \cos \tfrac{1}{2}v &= \frac{\sqrt{a(e-1)}}{\sqrt{\cos F}} \cos \tfrac{1}{2}F.\end{aligned}\quad (63)$$

These equations for finding r and v , it will be observed, are analogous to those previously investigated for an elliptic orbit. These equations give, by division,

$$\tan \tfrac{1}{2}v = \sqrt{\frac{e+1}{e-1}} \tan \tfrac{1}{2}F,$$

which is identical with the equation (56), and may be employed to verify the computation of r and v .

Multiplying the last of equations (63) by the first, putting for $e-1$ its value $\tan^2 \psi$, and reducing, we obtain

$$r \sin v = a \tan \psi \tan F = \tfrac{1}{2}a \tan \psi \left(\sigma - \frac{1}{\sigma} \right). \quad (64)$$

Further, we have

$$r \cos v = \frac{p \cos v}{1 + e \cos v} = ae - \frac{ar(e + \cos v)}{p},$$

which, combined with equation (62), gives

$$r \cos v = a \left(e - \frac{1}{\cos F} \right) = \tfrac{1}{2}a \left(2e - \sigma - \frac{1}{\sigma} \right). \quad (65)$$

If we square these values of $r \sin v$ and $r \cos v$, add the results together, reduce, and extract the square root, we find

$$r = a \left(\frac{e}{\cos F} - 1 \right) = \tfrac{1}{2}a \left(e \left(\sigma + \frac{1}{\sigma} \right) - 2 \right). \quad (66)$$

We might also introduce the auxiliary quantity σ into the equations (63); but such a transformation is hardly necessary, and, if at all desirable, it can be easily effected by means of the formulæ which we have already derived.

26. Let us now resume the equation

$$\sigma = \frac{\cos \frac{1}{2}(v - \psi)}{\cos \frac{1}{2}(v + \psi)}.$$

Differentiating this, regarding ψ as constant, we have

$$d\sigma = \frac{\sin \psi}{2 \cos^2 \frac{1}{2}(v + \psi)} dv,$$

and, dividing this equation by the preceding one, we get

$$\frac{d\sigma}{\sigma} = \frac{\sin \psi}{2 \cos \frac{1}{2}(v + \psi) \cos \frac{1}{2}(v - \psi)} dv.$$

But

$$r = \frac{p \cos \psi}{2 \cos \frac{1}{2}(v + \psi) \cos \frac{1}{2}(v - \psi)},$$

consequently,

$$\frac{d\sigma}{\sigma} = \frac{r \tan \psi}{p} dv,$$

which gives

$$r^2 dv = \frac{pr}{\sigma \tan \psi} d\sigma.$$

Substituting this value of $r^2 dv$ in equation (22), and putting instead of $2f$ its value $k\sqrt{p}$, from equation (30), the mass being considered as insensible in comparison with that of the sun, we get

$$k\sqrt{p} dt = \frac{pr}{\sigma \tan \psi} d\sigma.$$

Then, substituting for r its value from equation (66), and for p its value $a \tan^2 \psi$, we have

$$k\sqrt{p} dt = a^2 \tan \psi \left(\frac{1}{2} e \left(1 + \frac{1}{\sigma^2} \right) - \frac{1}{\sigma} \right) d\sigma.$$

Integrating this between the limits T and t , we obtain

$$k\sqrt{p}(t - T) = a^2 \tan \psi \left(\frac{1}{2} e \left(\sigma - \frac{1}{\sigma} \right) - \log_e \sigma \right), \quad (67)$$

in which $\log_e \sigma$ is the Naperian or hyperbolic logarithm of σ . Since $\sqrt{p} = \sqrt{a} \tan \psi$, if we put

$$\nu = \frac{k}{a^{\frac{3}{2}}},$$

in which ν is the mean daily motion; and if we also put

$$\nu(t - T) = N_0,$$

in which N_0 corresponds to the mean anomaly M in an ellipse, we shall have, from equation (67),

$$N_0 = \frac{1}{2}e \left(\sigma - \frac{1}{\sigma} \right) - \log_e \sigma. \quad (68)$$

If we multiply both members of this equation by $\lambda = 0.434294482$, the modulus of the common system of logarithms, and put

$$N = N_0 \lambda = \frac{\lambda k}{a^{\frac{3}{2}}} (t - T),$$

we shall have

$$N = \frac{1}{2}e\lambda \left(\sigma - \frac{1}{\sigma} \right) - \log \sigma,$$

wherein $\log \lambda = 9.6377843113$, and $\log \lambda k = 7.8733657527$.

Let us now introduce F into this formula; and for this purpose we have

$$\tan F = \frac{1}{2} \left(\sigma - \frac{1}{\sigma} \right),$$

and also

$$\log \sigma = \log \tan (45^\circ + \frac{1}{2}F).$$

Therefore we obtain

$$N = e\lambda \tan F - \log \tan (45^\circ + \frac{1}{2}F). \quad (69)$$

This equation will give, directly, the time $t - T$ from the perihelion, when a , e , and F are known; but, since it is transcendental, in the solution of the inverse problem, that of finding the true anomaly and radius-vector from the time, the value of F can only be found by successive approximations.

If we differentiate the last equation, regarding N and F as variable, we get

$$dN = \frac{\lambda}{\cos^2 F} (e - \cos F) dF.$$

Hence, if we denote an approximate value of F by F_1 , and the corresponding value of N by N_1 , the correction ΔF , to the assumed value of F may be computed by the formula

$$\Delta F = \frac{(N - N_1) \cos^2 F_1}{\lambda (e - \cos F_1)}.$$

This correction being applied to F , a nearer approximation to the true value of F will be obtained; and by repeating the operation there results a still closer approximation. This process may be continued until the exact value of F is found, and, when several successive places are required, the first assumed value may be estimated, in advance, so closely that a very few trials will suffice. In practice, however, cases will rarely occur in which this formula will be applied, since the probability of hyperbolic motion is small, and, whenever any positive indication of an eccentricity greater than 1 has been found to exist, it has only been after a very accurate series of observations has been introduced as the basis of the calculation. For a majority of the cases which do really occur, the most accurate and convenient method of finding r and v will be explained hereafter.

27. If we consider the equation

$$M = E - e \sin E,$$

we shall see that, when logarithms of six or seven decimals are used, the error which may exist in the determination of E when M and e are given, will increase as e increases, but in a much greater ratio; and, when the eccentricity becomes nearly equal to that of the parabola, the error may be very great. In the case of hyperbolic motion, also, the numerical solution of equation (69), when $e - 1$ is very small, and with the ordinary logarithmic tables, becomes very uncertain. This can only be remedied, when equations (39) and (69) are employed, by using more extended logarithmic tables; and when the orbit differs only in an extremely slight degree from a parabola, even with the most extended logarithmic tables which have been constructed, the error may be very large. For this reason we have recourse to other methods, which will give the required accuracy without introducing inconveniences which are proportionally great.

We shall, therefore, now proceed to develop the formulæ for finding the true anomaly in ellipses and hyperbolas which differ but little from the parabola, such that they will furnish the required accuracy, when the exact solution of equations (39) or (69) with the logarithmic tables in common use is impossible.

For this purpose, let us resume equation (22), which, by substituting for $2f$ its value $k\sqrt{p}$, the mass of the comet being neglected in comparison with that of the sun, becomes

$$k\sqrt{p} \, dt = r^2 dv,$$

or

$$k\sqrt{p} dt = p^2 \frac{dv}{(1 + e \cos v)^2}.$$

Let us now put $u = \tan \frac{1}{2}v$, and we shall have

$$\cos v = \frac{1 - u^2}{1 + u^2}; \quad dv = \frac{2du}{1 + u^2}.$$

Substituting these values in the preceding equation, and putting $\frac{1 - e}{1 + e} = i$, we get

$$k\sqrt{p} dt = \frac{2p^2}{(1 + e)^2} \frac{(1 + u^2) du}{(1 + iu^2)^2},$$

or, since $p = q(1 + e)$,

$$\frac{k\sqrt{1 + e} dt}{2q^{\frac{3}{2}}} = \frac{(1 + u^2) du}{(1 + iu^2)^2}.$$

Let us now develop the second member into a series. This may be written thus:

$$du(1 + u^2)(1 + iu^2)^{-2};$$

and developing the last factor into a series, we obtain

$$(1 + iu^2)^{-2} = 1 - 2iu^2 + 3i^2u^4 - 4i^3u^6 + \&c.$$

Consequently,

$$(1 + u^2)(1 + iu^2)^{-2} = 1 + u^2 - 2i(u^2 + u^4) + 3i^2(u^4 + u^6) - 4i^3(u^6 + u^8) + \dots$$

Multiplying this equation through by du , and integrating between the limits T and t , the result is

$$\frac{k(t - T)\sqrt{1 + e}}{2q^{\frac{3}{2}}} = u + \frac{1}{3}u^3 - 2i(\frac{1}{3}u^3 + \frac{1}{5}u^5) + 3i^2(\frac{1}{5}u^5 + \frac{1}{7}u^7) - 4i^3(\frac{1}{7}u^7 + \frac{1}{9}u^9) + \&c. \quad (70)$$

In the case of the parabola, $e = 1$ and $i = 0$, and this equation becomes identical with (55).

Let us now put

$$\frac{k(t - T)\sqrt{1 + e}}{2q^{\frac{3}{2}}} = U + \frac{1}{3}U^3, \quad (71)$$

and also

$$U = \tan \frac{1}{2} V;$$

then the angle V will not be the true anomaly in the parabola, but an angle derived from the solution of a cubic equation of the same form as that for finding the parabolic anomaly; and its value may be found by means of Table VI., if we use for M the value computed from

$$M = \frac{75k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} \cdot \sqrt{\frac{1+e}{2}}.$$

Let U be expanded into a series of the form

$$U = u + \alpha i + \beta i^2 + \gamma i^3 + \dots,$$

which is evidently admissible, $\alpha, \beta, \gamma, \dots$ being functions of u and independent of i . It remains now to determine the values of the coefficients α, β, γ , &c., and, in doing so, it will only be necessary to consider terms of the third order, or those involving i^3 , since, for nearly all of those cases in which the eccentricity is such that terms of the order i^4 will sensibly affect the result, the general formulæ already derived, with the ordinary means of solution, will give the required accuracy. We shall, therefore, have

$$U + \frac{1}{3}U^3 = u + \alpha i + \beta i^2 + \gamma i^3 + \frac{1}{3}(u + \alpha i + \beta i^2 + \gamma i^3)^3,$$

or, again neglecting terms of the order i^4 ,

$$U + \frac{1}{3}U^3 = u + \frac{1}{3}u^3 + i(1+u^2)\alpha + i^2(ua^2 + (1+u^2)\beta) + i^3(\frac{1}{3}a^3 + 2u\alpha\beta + (1+u^2)\gamma).$$

But we have already found, (70),

$$\frac{k(t-T)\sqrt{1+e}}{2q^{\frac{3}{2}}} = U + \frac{1}{3}U^3 = u + \frac{1}{3}u^3 - 2i(\frac{1}{3}u^3 + \frac{1}{6}u^5) + 3i^2(\frac{1}{5}u^5 + \frac{1}{7}u^7) - 4i^3(\frac{1}{7}u^7 + \frac{1}{9}u^9).$$

Since the first members of these equations are identical, it follows, by the principle of indeterminate coefficients, that the coefficients of the like powers of i are equal, and we shall, therefore, have

$$\begin{aligned} (1+u^2)\alpha &= -2(\frac{1}{3}u^3 + \frac{1}{6}u^5), \\ ua^2 + (1+u^2)\beta &= +3(\frac{1}{5}u^5 + \frac{1}{7}u^7), \\ \frac{1}{3}a^3 + 2u\alpha\beta + (1+u^2)\gamma &= -4(\frac{1}{7}u^7 + \frac{1}{9}u^9). \end{aligned}$$

From the first of these equations we find

$$\alpha = -\frac{2(\frac{1}{3}u^3 + \frac{1}{5}u^5)}{1+u^2}.$$

The second equation gives

$$\beta = \frac{3(\frac{1}{5}u^5 + \frac{1}{7}u^7) - \alpha^2}{1+u^2},$$

or, substituting for α its value just found, and reducing,

$$\beta = \frac{3(\frac{1}{5}u^5 + \frac{37}{94}\frac{3}{5}u^7 + \frac{97}{315}u^9 + \frac{47}{525}u^{11})}{(1+u^2)^3}.$$

We have also

$$\gamma = \frac{-4(\frac{1}{7}u^7 + \frac{1}{5}u^9) - \frac{1}{3}\alpha^3 - 2\alpha\beta u}{1+u^2};$$

and hence, substituting the values of α and β already found, and reducing, we obtain finally

$$\gamma = \frac{-4(\frac{1}{7}u^7 + \frac{129}{2835}u^9 + \frac{10174}{14175}u^{11} + \frac{196}{315}u^{13} + \frac{2213}{7875}u^{15} + \frac{82}{1575}u^{17})}{(1+u^2)^5}.$$

Again, we have

$$\tan^{-1}U = \tan^{-1}(u + \alpha i + \beta i^2 + \gamma i^3).$$

Developing this, and neglecting terms of the order i^4 , we get

$$\begin{aligned} \tan^{-1}U = \tan^{-1}u + \frac{1}{1+u^2}(\alpha i + \beta i^2 + \gamma i^3) - \frac{u}{(1+u^2)^2}(\alpha^2 i^2 + 2\alpha\beta i^3) \\ + \frac{u^2 - \frac{1}{3}}{(1+u^2)^3}\alpha^3 i^3. \end{aligned}$$

Now, since $u = \tan \frac{1}{2}v$ and $U = \tan \frac{1}{2}V$, we shall have

$$V = v + \frac{2}{1+u^2}(\alpha i + \beta i^2 + \gamma i^3) - \frac{2u}{(1+u^2)^2}(\alpha^2 i^2 + 2\alpha\beta i^3) + \frac{2(u^2 - \frac{1}{3})}{(1+u^2)^3}\alpha^3 i^3,$$

or

$$\begin{aligned} V = v + \frac{2\alpha}{1+u^2}i + \left(\frac{2\beta}{1+u^2} - \frac{2\alpha^2 u}{(1+u^2)^2}\right)i^2 \\ + \left(\frac{2\gamma}{1+u^2} - \frac{4\alpha\beta u}{(1+u^2)^2} + \frac{2(u^2 - \frac{1}{3})}{(1+u^2)^3}\alpha^3\right)i^3. \end{aligned} \quad (72)$$

Substituting in this equation the values of α , β , and γ already found, and reducing, we obtain finally

$$\begin{aligned} V = v - \frac{\frac{4}{3}u^3 + \frac{4}{5}u^5}{(1+u^2)^2}i + \frac{\frac{6}{5}u^5 + \frac{466}{315}u^7 + \frac{82}{105}u^9 + \frac{38}{175}u^{11}}{(1+u^2)^4}i^2 \\ - \frac{\frac{8}{7}u^7 + \frac{528}{2835}u^9 + \frac{26384}{14175}u^{11} + \frac{464}{315}u^{13} + \frac{512}{7875}u^{15} + \frac{904}{7875}u^{17}}{(1+u^2)^6}i^3. \end{aligned} \quad (73)$$

This equation can be used whenever the true anomaly in the ellipse or hyperbola is given, and the time from the perihelion is to be determined. Having found the value of V , we enter Table VI. with the argument V and take out the corresponding value of M ; and then we derive $t - T$ from

$$t - T = \frac{Mq^{\frac{3}{2}}}{C_0} \sqrt{\frac{2}{1+e}},$$

in which $\log C_0 = 9.96012771$.

For the converse of this, in which the time from the perihelion is given and the true anomaly is required, it is necessary to express the difference $v - V$ in a series of ascending powers of i , in which the coefficients are functions of U . Let us, therefore, put

$$u = U + \alpha' i + \beta' i^2 + \gamma' i^3 + \&c.$$

Substituting this value of u in equation (70), and neglecting terms multiplied by i^4 and higher powers of i , we get

$$\begin{aligned} \frac{k(t - T)\sqrt{1+e}}{2q^{\frac{3}{2}}} &= U + \frac{1}{3}U^3 + (\alpha'(1 + U^2) - \frac{2}{3}U^3 - \frac{2}{5}U^5) i \\ &+ (\beta'(1 + U^2) + U\alpha'^2 - 2U^2\alpha'(1 + U^2) + \frac{3}{5}U^5 + \frac{3}{7}U^7) i^2 \\ &+ (\gamma'(1 + U^2) + \frac{1}{3}\alpha'^3 + 2U\alpha'\beta' + 3U^2\alpha'(1 + U^2) - 2\beta'U^2(1 + U^2) \\ &- 4U^3\alpha'^2 - 2U\alpha'^2 - \frac{4}{7}U^7 - \frac{4}{9}U^9) i^3. \end{aligned}$$

But, since the first member of this equation is equal to $U + \frac{1}{3}U^3$, we shall have, by the principle of indeterminate coefficients,

$$\begin{aligned} \alpha'(1 + U^2) - \frac{2}{3}U^3 - \frac{2}{5}U^5 &= 0, \\ \beta'(1 + U^2) + U\alpha'^2 - 2U^2\alpha'(1 + U^2) + \frac{3}{5}U^5 + \frac{3}{7}U^7 &= 0, \\ \gamma'(1 + U^2) + \frac{1}{3}\alpha'^3 + 2U\alpha'\beta' + 3U^2\alpha'(1 + U^2) - 2\beta'U^2(1 + U^2) \\ - 4U^3\alpha'^2 - 2U\alpha'^2 - \frac{4}{7}U^7 - \frac{4}{9}U^9 &= 0. \end{aligned}$$

From these equations, we find

$$\begin{aligned} \alpha' &= \frac{\frac{2}{3}U^3 + \frac{2}{5}U^5}{1 + U^2}, \\ \beta' &= \frac{\frac{1}{15}U^5 + \frac{4}{315}U^7 + \frac{3}{35}U^9 + \frac{37}{175}U^{11}}{(1 + U^2)^3}, \\ \gamma' &= \frac{\frac{292}{315}U^7 + \frac{7928}{2835}U^9 + \frac{10328}{2835}U^{11} + \frac{432}{175}U^{13} + \frac{6692}{7875}U^{15} + \frac{184}{1575}U^{17}}{(1 + U^2)^5}. \end{aligned}$$

If we interchange v and V in equation (72), it becomes, writing α' , β' , γ' for α , β , γ ,

$$v = V + \frac{2\alpha'}{1+U^2}i + \left(\frac{2\beta'}{1+U^2} - \frac{2\alpha'^2 U}{(1+U^2)^2} \right)i^2 \\ + \left(\frac{2\gamma'}{1+U^2} - \frac{4\alpha'\beta' U}{(1+U^2)^2} + \frac{2(U^2 - \frac{1}{3})}{(1+U^2)^3} \alpha'^3 \right)i^3.$$

Substituting in this equation the above values of α' , β' , and γ' , and reducing, we obtain, finally,

$$v = V + \frac{\frac{4}{3}U^3 + \frac{4}{5}U^5}{(1+U^2)^2}i + \frac{\frac{22}{15}U^5 + \frac{598}{315}U^7 + \frac{86}{105}U^9 + \frac{18}{175}U^{11}}{(1+U^2)^4}i^2 \\ + \frac{\frac{584}{315}U^7 + \frac{9752}{2835}U^9 + \frac{37328}{14175}U^{11} + \frac{1648}{1575}U^{13} + \frac{1768}{7875}U^{15} + \frac{184}{7875}U^{17}}{(1+U^2)^6}i^3, \quad (74)$$

by means of which v may be determined, the angle V being taken from Table VI., so as to correspond with the value of M derived from

$$M = (t - T) \frac{C_0}{q^{\frac{3}{2}}} \cdot \sqrt{\frac{1+e}{2}}.$$

Equations (73) and (74) are applicable, without any modification, to the case of a hyperbolic orbit which differs but little from the parabola. In this case, however, e is greater than unity, and, consequently, i is negative.

28. In order to render these formulæ convenient in practice, tables may be constructed in the following manner:—

Let $x = v$ or V , and $\tan \frac{1}{2}x = \theta$, and let us put

$$A = \frac{\frac{4}{3}\theta^3 + \frac{4}{5}\theta^5}{100(1+\theta^2)^2} s, \\ B = \frac{\frac{22}{15}\theta^5 + \frac{598}{315}\theta^7 + \frac{86}{105}\theta^9 + \frac{18}{175}\theta^{11}}{10000(1+\theta^2)^4} s, \\ B' = \frac{\frac{6}{5}\theta^5 + \frac{466}{315}\theta^7 + \frac{82}{105}\theta^9 + \frac{38}{175}\theta^{11}}{10000(1+\theta^2)^4} s, \\ C = \frac{\frac{584}{315}\theta^7 + \frac{9752}{2835}\theta^9 + \frac{37328}{14175}\theta^{11} + \frac{1648}{1575}\theta^{13} + \frac{1768}{7875}\theta^{15} + \frac{184}{7875}\theta^{17}}{1000000(1+\theta^2)^6} s, \\ C' = \frac{\frac{8}{5}\theta^7 + \frac{5288}{2835}\theta^9 + \frac{26384}{14175}\theta^{11} + \frac{464}{315}\theta^{13} + \frac{5128}{7875}\theta^{15} + \frac{904}{7875}\theta^{17}}{1000000(1+\theta^2)^6} s,$$

wherein s expresses the number of seconds corresponding to the length of arc equal to the radius of a circle, or $\log s = 5.31412513$. We shall, therefore, have:—

When $x = V$,

$$v = V + A(100i) + B(100i)^2 + C(100i)^3;$$

and, when $x = v$,

$$V = v - A(100i) + B'(100i)^2 - C'(100i)^3.$$

Table IX. gives the values of A , B , B' , C , and C' for consecutive values of x from $x = 0^\circ$ to $x = 149^\circ$, with differences for interpolation.

When the value of v has been found, that of r may be derived from the formula

$$r = \frac{q(1+e)}{1+e \cos v}.$$

Similar expressions arranged in reference to the ascending powers of $(1-e)$ or of $\left(\left(\frac{2}{1+e}\right)^{\frac{1}{2}} - 1\right)$ may be derived, but they do not converge with sufficient rapidity; for, although $\left(\left(\frac{2}{1+e}\right)^{\frac{1}{2}} - 1\right)$ is less than i , yet the coefficients are, in each case, so much greater than those of the corresponding powers of i , that three terms will not afford the same degree of accuracy as the same number of terms in the expressions involving i .

29. Equations (73) and (74) will serve to determine v or $t - T$ in nearly all cases in which, with the ordinary logarithmic tables, the general methods fail. However, when the orbit differs considerably from a parabola, and when v is of considerable magnitude, the results obtained by means of these equations will not be sufficiently exact, and we must employ other methods of approximation in the case that the accurate numerical solution of the general formulæ is still impossible. It may be observed that when E or F exceeds 50° or 60° , the equations (39) and (69) will furnish accurate results, even when e differs but little from unity. Still, a case may occur in which the perihelion distance is very small and in which v may be very great before the disappearance of the comet, such that neither the general method, nor the special method already given, will enable us to determine v or $t - T$ with accuracy; and we shall, therefore, investigate another method, which will, in all cases, be sufficiently exact when the general formulæ are inapplicable directly. For this purpose, let us resume the equation

$$\frac{k(t-T)}{a^{\frac{3}{2}}} = E - e \sin E,$$

which, since $q = a(1 - e)$, may be written

$$\frac{k(t - T)\sqrt{1 - e}}{q^{\frac{3}{2}}} = \frac{1}{10}(9E + \sin E) + \frac{1}{10} \cdot \frac{1 + 9e}{1 - e}(E - \sin E).$$

If we put

$$A = 15 \frac{E - \sin E}{9E + \sin E},$$

we shall have

$$\frac{k(t - T)\sqrt{1 - e}}{2q^{\frac{3}{2}}} \cdot \frac{20\sqrt{A}}{9E + \sin E} = A^{\frac{1}{2}} + \frac{1}{3} \cdot \frac{1 + 9e}{5(1 - e)} A^{\frac{3}{2}}.$$

Let us now put

$$B = \frac{9E + \sin E}{20\sqrt{A}},$$

and

$$\tan^2 \frac{1}{2}w = \frac{1 + 9e}{5(1 - e)} A;$$

then we have

$$\frac{k(t - T)}{\sqrt{2}q^{\frac{3}{2}}} \cdot \frac{\sqrt{\frac{1}{10}(1 + 9e)}}{B} = \tan \frac{1}{2}w + \frac{1}{3} \tan^3 \frac{1}{2}w. \quad (75)$$

When B is known, the value of w may, according to this equation, be derived directly from Table VI. with the argument

$$M = \frac{75k(t - T)}{\sqrt{2}q^{\frac{3}{2}}} \cdot \frac{\sqrt{\frac{1}{10}(1 + 9e)}}{B},$$

and then from w we may find the value of A . It remains, therefore, to find the value of B ; and then that of v from the resulting value of A .

Now, we have

$$\sin E = \frac{2 \tan \frac{1}{2}E}{1 + \tan^2 \frac{1}{2}E},$$

and if we put $\tan^2 \frac{1}{2}E = \tau$, we get

$$\sin E = \frac{2\tau^{\frac{1}{2}}}{1 + \tau} = 2\tau^{\frac{1}{2}}(1 - \tau + \tau^2 - \tau^3 + \&c.).$$

We have, also,

$$E = 2 \tan^{-1} \tau^{\frac{1}{2}} = 2\tau^{\frac{1}{2}}(1 - \frac{1}{3}\tau + \frac{1}{5}\tau^2 - \frac{1}{7}\tau^3 + \&c.).$$

Therefore,

$$15(E - \sin E) = 2\tau^{\frac{1}{2}}(10\tau - \frac{6}{5}\tau^2 + \frac{9}{7}\tau^3 - \frac{12}{9}\tau^4 + \&c.),$$

and

$$9E + \sin E = 2\tau^{\frac{1}{2}}(10 - \frac{12}{3}\tau + \frac{14}{5}\tau^2 - \frac{16}{7}\tau^3 + \frac{18}{9}\tau^4 - \&c.).$$

Hence, by division,

$$15 \frac{E - \sin E}{9E + \sin E} = A = \tau - \frac{4}{5}\tau^2 + \frac{24}{35}\tau^3 - \frac{1592}{2625}\tau^4 + \frac{78856}{144375}\tau^5 \\ - \frac{10899688}{21856875}\tau^6 + \&c.;$$

and, inverting this series, we get

$$\frac{A}{\tau} = 1 - \frac{4}{5}A + \frac{8}{175}A^2 + \frac{8}{525}A^3 + \frac{31896}{336875}A^4 + \frac{28744}{13138125}A^5 + \&c.,$$

which converges rapidly, and from which the value of $\frac{A}{\tau}$ may be found.

Let us now put

$$\frac{A}{\tau} = \frac{1}{C^2},$$

then the values of C may be tabulated with the argument A ; and, besides, it is evident that as long as A is small C^2 will not differ much from $1 + \frac{4}{5}A$.

Next, to find B , we have

$$A^{\frac{1}{2}} = \tau^{\frac{1}{2}}(1 - \frac{2}{5}\tau + \frac{46}{175}\tau^2 - \frac{104}{525}\tau^3 + \frac{161002}{1010625}\tau^4 - \&c.),$$

and hence

$$\frac{\frac{1}{2}(9E + \sin E)}{\sqrt{A}} = B = 1 + \frac{3}{175}\tau^2 - \frac{62}{2625}\tau^3 + \frac{9907}{336875}\tau^4 - \&c.;$$

from which we easily find

$$B = 1 + \frac{3}{175}A^2 + \frac{2}{525}A^3 + \frac{471}{336875}A^4 + \&c.$$

If we compare equations (44) and (56), we get

$$\tan \frac{1}{2}E = \sqrt{-1} \tan \frac{1}{2}F.$$

Hence, in the case of a hyperbolic orbit, if we put $\tan^2 \frac{1}{2}F = \tau'$, we must write $-\tau'$ in place of τ in the formulæ already derived; and, from the series which gives A in terms of τ , it appears that A is in this case negative. Therefore, if we distinguish the equations for

hyperbolic motion from those for elliptic motion by writing A' , B' , and C' in place of A , B , and C , respectively, we shall have

$$\frac{1}{C'^2} = \frac{A'}{C'} = 1 + \frac{4}{5}A' + \frac{8}{175}A'^2 - \frac{8}{525}A'^3 + \frac{138896}{338875}A'^4 - \frac{28744}{13135125}A'^5 + \&c.,$$

$$B' = 1 + \frac{3}{175}A'^2 - \frac{2}{525}A'^3 + \frac{471}{338875}A'^4 - \&c.$$

Table X. contains the values of $\log B$ and $\log C$ for the ellipse and the hyperbola, with the argument A , from $A = 0$ to $A = 0.3$. For every case in which A exceeds 0.3, the general formulæ (39) and (69) may be conveniently applied, as already stated.

The equation

$$\tan \frac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2}E$$

gives

$$\tan^2 \frac{1}{2}v = \frac{1+e}{1-e} A C^2,$$

or, substituting the value of A in terms of w ,

$$\tan \frac{1}{2}v = C \tan \frac{1}{2}w \sqrt{\frac{5(1+e)}{1+9e}}. \quad (76)$$

The last of equations (43) gives

$$r \cos^2 \frac{1}{2}v = q \cos^2 \frac{1}{2}E = \frac{q}{1 + \tan^2 \frac{1}{2}E}$$

Hence we derive

$$r = \frac{q}{(1 + A C^2) \cos^2 \frac{1}{2}v}. \quad (77)$$

The equation for v in a hyperbolic orbit is of precisely the same form as (76), the accents being omitted, and the value of A being computed from

$$A = \frac{5(e-1)}{1+9e} \tan^2 \frac{1}{2}w. \quad (78)$$

For the radius-vector in a hyperbolic orbit, we find, by means of the last of equations (63),

$$r = \frac{q}{(1 - A C^2) \cos^2 \frac{1}{2}v}. \quad (79)$$

When $t - T$ is given and r and v are required, we first assume $B = 1$, and enter Table VI. with the argument

$$M = \frac{C_0(t-T) \sqrt{\frac{1}{10}(1+9e)}}{q^{\frac{3}{2}} B},$$

in which $\log C_0 = 9.96012771$, and take out the corresponding value of w . Then we derive A from the equation

$$A = \frac{5(1-e)}{1+9e} \tan^2 \frac{1}{2}w,$$

in the case of the ellipse, and from (78) in the case of a hyperbolic orbit. With the resulting value of A , we find from Table X. the corresponding value of $\log B$, and then, using this in the expression for M , we repeat the operation. The second result for A will not require any further correction, since the error of the first assumption of $B=1$ is very small; and, with this as argument, we derive the value of $\log C$ from the table, and then v and r by means of the equations (76) and (77) or (79).

When the true anomaly is given, and the time $t - T$ is required, we first compute τ from

$$\tau = \frac{1-e}{1+e} \tan^2 \frac{1}{2}v,$$

in the case of the ellipse, or from

$$\tau = \frac{e-1}{e+1} \tan^2 \frac{1}{2}v,$$

in the case of the hyperbola. Then, with the value of τ as argument, we enter the second part of Table X. and take out an approximate value of A , and, with this as argument, we find $\log B$ and $\log C$. The equation

$$A = \frac{\tau}{C^2}$$

will show whether the approximate value of A used in finding $\log C$ is sufficiently exact, and, hence, whether the latter requires any correction. Next, to find w , we have

$$\tan \frac{1}{2}w = \frac{\tan \frac{1}{2}v}{C} \cdot \sqrt{\frac{1+9e}{5(1+e)}};$$

and, with w as argument, we derive M from Table VI. Finally, we have

$$t - T = \frac{MBq^{\frac{3}{2}}}{C_0 \sqrt{\frac{1}{10}(1+9e)}}, \quad (80)$$

by means of which the time from the perihelion may be accurately determined.

30. We have thus far treated of the motion of the heavenly bodies, relative to the sun, without considering the positions of their orbits in space; and the elements which we have employed are the eccentricity and semi-transverse axis of the orbit, and the mean anomaly at a given epoch, or, what is equivalent, the time of passing the perihelion. These are the elements which determine the position of the body in its orbit at any given time. It remains now to fix its position in space in reference to some other point in space from which we conceive it to be seen. To accomplish this, the position of its orbit in reference to a known plane must be given; and the elements which determine this position are the longitude of the perihelion, the longitude of the ascending node, and the inclination of the plane of the orbit to the known plane, for which the plane of the ecliptic is usually taken. These three elements will enable us to determine the co-ordinates of the body in space, when its position in its orbit has been found by means of the formulæ already investigated.

The *longitude of the ascending node*, or longitude of the point through which the body passes from the south to the north side of the ecliptic, which we will denote by Ω , is the angular distance of this point from the vernal equinox. The line of intersection of the plane of the orbit with the fundamental plane is called the *line of nodes*.

The angle which the plane of the orbit makes with the plane of the ecliptic, which we will denote by i , is called the *inclination* of the orbit. It will readily be seen that, if we suppose the plane of the orbit to revolve about the line of nodes, when the angle i exceeds 180° , Ω will no longer be the longitude of the ascending node, but will become the longitude of the descending node, or of the point through which the planet passes from the north to the south side of the ecliptic, which is denoted by \mathfrak{S} , and which is measured, as in the case of Ω , from the vernal equinox.

It will easily be understood that, when seen from the sun, so long as the inclination of the orbit is less than 90° , the motion of the body will be in the same direction as that of the earth, and it is then said to be *direct*. When the inclination is 90° , the motion will be at right angles to that of the earth; and when i exceeds 90° , the motion in longitude will be in a direction opposite to that of the earth, and it is then called *retrograde*. It is customary, therefore, to extend the inclination of the orbit only to 90° , and if this angle exceeds a right angle, to regard its supplement as the inclination of the orbit, noting simply the distinction that the motion is *retrograde*.

The *longitude of the perihelion*, which is denoted by π , fixes the position of the orbit in its own plane, and is, in the case of direct motion, the sum of the longitude of the ascending node and the angular distance, measured in the direction of the motion, of the perihelion from this node. It is, therefore, the angular distance of the perihelion from a point in the orbit whose angular distance back from the ascending node is equal to the longitude of this node; or it may be measured on the ecliptic from the vernal equinox to the ascending node, then on the plane of the orbit from the node to the place of the perihelion.

In the case of retrograde motion, the longitudes of the successive points in the orbit, in the direction of the motion, decrease, and the point in the orbit from which these longitudes in the orbit are measured is taken at an angular distance from the ascending node equal to the longitude of that node, but taken, from the node, in the same direction as the motion. Hence, in this case, the longitude of the perihelion is equal to the longitude of the ascending node diminished by the angular distance of the perihelion from this node.

It may, perhaps, seem desirable that the distinctions, *direct* and *retrograde* motion, should be abandoned, and that the inclination of the orbit should be measured from 0° to 180° , since in this case one set of formulæ would be sufficient, while in the common form two sets are in part required. However, the custom of astronomers seems to have sanctioned these distinctions, and they may be perpetuated or not, as may seem advantageous.

Further, we may remark that in the case of direct motion the sum of the true anomaly and longitude of the perihelion is called the *true longitude in the orbit*; and that the sum of the mean anomaly and longitude of the perihelion is called the *mean longitude*, an expression which can occur only in the case of elliptic orbits.

In the case of retrograde motion the longitude in the orbit is equal to the longitude of the perihelion minus the true anomaly.

31. We will now proceed to derive the formulæ for determining the co-ordinates of a heavenly body in space, when its position in its orbit is known.

For the co-ordinates of the position of the body at the time t , we have

$$x = r \cos v,$$

$$y = r \sin v,$$

the line of apsides being taken as the axis of x , and the origin being taken at the centre of the sun.

If we take the line of nodes as the axis of x , we shall have

$$\begin{aligned}x &= r \cos (v + \omega), \\y &= r \sin (v + \omega),\end{aligned}$$

ω being the arc of the orbit intercepted between the place of the perihelion and of the node, or the angular distance of the perihelion from the node.

Now, we have $\omega = \pi - \Omega$ in the case of direct motion, and $\omega = \Omega - \pi$ in the case of retrograde motion; and hence the last equations become

$$\begin{aligned}x &= r \cos (v \pm \pi \mp \Omega) \\y &= r \sin (v \pm \pi \mp \Omega)\end{aligned}$$

the upper and lower signs being taken, respectively, according as the motion is direct or retrograde. The arc $v \pm \pi \mp \Omega = u$ is called the *argument of the latitude*.

Let us now refer the position of the body to three co-ordinate planes, the origin being at the centre of the sun, the ecliptic being taken as the plane of xy , and the axis of x , in the line of nodes. Then we shall have

$$\begin{aligned}x' &= r \cos u, \\y' &= \pm r \sin u \cos i, \\z' &= r \sin u \sin i.\end{aligned}$$

If we denote the heliocentric latitude and longitude of the body, at the time t , by b and l , respectively, we shall have

$$\begin{aligned}x' &= r \cos b \cos (l - \Omega), \\y' &= r \cos b \sin (l - \Omega), \\z' &= r \sin b,\end{aligned}$$

and, consequently,

$$\begin{aligned}\cos u &= \cos b \cos (l - \Omega), \\ \pm \sin u \cos i &= \cos b \sin (l - \Omega), \\ \sin u \sin i &= \sin b.\end{aligned}\tag{81}$$

From these we derive

$$\begin{aligned}\tan (l - \Omega) &= \pm \tan u \cos i, \\ \tan b &= \pm \tan i \sin (l - \Omega),\end{aligned}\tag{82}$$

which serve to determine l and b , when Ω , u , and i are given. Since

$\cos b$ is always positive, it follows that $l - \Omega$ and u must lie in the same quadrant when i is less than 90° ; but if i is greater than 90° , or the motion is retrograde, $l - \Omega$ and $360^\circ - u$ will belong to the same quadrant. Hence the ambiguity which the determination of $l - \Omega$ by means of its tangent involves, is wholly avoided.

If we use the distinction of retrograde motion, and consider i always less than 90° , $l - \Omega$ and $-u$ will lie in the same quadrant.

32. By multiplying the first of the equations (81) by $\sin u$, and the second by $\cos u$, and combining the results, considering only the upper sign, we derive

$$\cos b \sin(u - l + \Omega) = 2 \sin u \cos u \sin^2 \frac{1}{2}i,$$

or

$$\cos b \sin(u - l + \Omega) = \sin 2u \sin^2 \frac{1}{2}i.$$

In a similar manner, we find

$$\cos b \cos(u - l + \Omega) = \cos^2 u + \sin^2 u \cos i,$$

which may be written

$$\cos b \cos(u - l + \Omega) = \frac{1}{2}(1 + \cos 2u) + \frac{1}{2}(1 - \cos 2u) \cos i,$$

or

$$\cos b \cos(u - l + \Omega) = \frac{1}{2}(1 + \cos i) + \frac{1}{2}(1 - \cos i) \cos 2u;$$

and hence

$$\cos b \cos(u - l + \Omega) = \cos^2 \frac{1}{2}i + \sin^2 \frac{1}{2}i \cos 2u.$$

If we divide this equation by the value of $\cos b \sin(u - l + \Omega)$ already found, we shall have

$$\tan(u - l + \Omega) = \frac{\tan^2 \frac{1}{2}i \sin 2u}{1 + \tan^2 \frac{1}{2}i \cos 2u}. \quad (83)$$

The angle $u - l + \Omega$ is called the *reduction to the ecliptic*; and the expression for it may be arranged in a series which converges rapidly when i is small, as in the case of the planets. In order to effect this development, let us first take the equation

$$\tan y = \frac{n \sin x}{1 + n \cos x}.$$

Differentiating this, regarding y and n as variables, and reducing, we find

$$\frac{dy}{dn} = \frac{\sin x}{1 + 2n \cos x + n^2}.$$

which gives, by division, or by the method of indeterminate coefficients,

$$\frac{dy}{dn} = \sin x - n \sin 2x + n^2 \sin 3x - n^3 \sin 4x + \&c.$$

Integrating this expression, we get, since $y = 0$ when $x = 0$,

$$y = n \sin x - \frac{1}{2}n^2 \sin 2x + \frac{1}{3}n^3 \sin 3x - \frac{1}{4}n^4 \sin 4x + \dots, \quad (84)$$

which is the general form of the development of the above expression for $\tan y$. The assumed expression for $\tan y$ corresponds exactly with the formula for the reduction to the ecliptic by making $n = \tan^2 \frac{1}{2}i$ and $x = 2u$; and hence we obtain

$$\begin{aligned} u - l + \Omega &= \tan^2 \frac{1}{2}i \sin 2u - \frac{1}{2} \tan^4 \frac{1}{2}i \sin 4u + \frac{1}{3} \tan^6 \frac{1}{2}i \sin 6u \\ &\quad - \frac{1}{4} \tan^8 \frac{1}{2}i \sin 8u + \frac{1}{5} \tan^{10} \frac{1}{2}i \sin 10u - \&c. \end{aligned} \quad (85)$$

When the value of i does not exceed 10° or 12° , the first two terms of this development will be sufficient. To express $u - l + \Omega$ in seconds of arc, the value derived from the second member of this equation must be multiplied by 206264.81, the number of seconds corresponding to the radius of a circle.

If we denote by R_e the reduction to the ecliptic, we shall have

$$l = u + \Omega - R_e = v + \pi - R_e.$$

But we have $v = M$ + the equation of the centre; hence

$$l = M + \pi + \text{equation of the centre} - \text{reduction to the ecliptic},$$

and, putting $L = M + \pi = \text{mean longitude}$, we get

$$l = L + \text{equation of centre} - \text{reduction to ecliptic}. \quad (86)$$

In the tables of the motion of the planets, the equation of the centre (53) is given in a table with M as the argument; and the reduction to the ecliptic is given in a table in which i and u are the arguments.

33. In determining the place of a heavenly body directly from the elements of its orbit, there will be no necessity for computing the reduction to the ecliptic, since the heliocentric longitude and latitude may be readily found by the formulæ (82). When the heliocentric place has been found, we can easily deduce the corresponding geocentric place.

Let x, y, z be the rectangular co-ordinates of the planet or comet referred to the centre of the sun, the plane of xy being in the ecliptic,

the positive axis of x being directed to the vernal equinox, and the positive axis of z to the north pole of the ecliptic. Then we shall have

$$\begin{aligned}x &= r \cos b \cos l, \\y &= r \cos b \sin l, \\z &= r \sin b.\end{aligned}$$

Again, let X, Y, Z be the co-ordinates of the centre of the sun referred to the centre of the earth, the plane of XY being in the ecliptic, and the axis of X being directed to the vernal equinox; and let \odot denote the geocentric longitude of the sun, R its distance from the earth, and Σ its latitude. Then we shall have

$$\begin{aligned}X &= R \cos \Sigma \cos \odot, \\Y &= R \cos \Sigma \sin \odot, \\Z &= R \sin \Sigma.\end{aligned}$$

Let x', y', z' be the co-ordinates of the body referred to the centre of the earth; and let λ and β denote, respectively, the geocentric longitude and latitude, and Δ , the distance of the planet or comet from the earth. Then we obtain

$$\begin{aligned}x' &= \Delta \cos \beta \cos \lambda, \\y' &= \Delta \cos \beta \sin \lambda, \\z' &= \Delta \sin \beta.\end{aligned}\tag{87}$$

But, evidently, we also have

$$x' = x + X, \quad y' = y + Y, \quad z' = z + Z,$$

and, consequently,

$$\begin{aligned}\Delta \cos \beta \cos \lambda &= r \cos b \cos l + R \cos \Sigma \cos \odot, \\ \Delta \cos \beta \sin \lambda &= r \cos b \sin l + R \cos \Sigma \sin \odot, \\ \Delta \sin \beta &= r \sin b + R \sin \Sigma.\end{aligned}\tag{88}$$

If we multiply the first of these equations by $\cos \odot$, and the second by $\sin \odot$, and add the products; then multiply the first by $\sin \odot$, and the second by $\cos \odot$, and subtract the first product from the second, we get

$$\begin{aligned}\Delta \cos \beta \cos (\lambda - \odot) &= r \cos b \cos (l - \odot) + R \cos \Sigma, \\ \Delta \cos \beta \sin (\lambda - \odot) &= r \cos b \sin (l - \odot), \\ \Delta \sin \beta &= r \sin b + R \sin \Sigma.\end{aligned}\tag{89}$$

It will be observed that this transformation is equivalent to the supposition that the axis of x , in each of the co-ordinate systems, is

directed to a point whose longitude is \odot , or that the system has been revolved about the axis of z to a new position for which the axis of abscissas makes the angle \odot with that of the primitive system. We may, therefore, in general, in order to effect such a transformation in systems of equations thus derived, simply diminish the longitudes by the given angle.

The equations (89) will determine λ , β , and Δ when r , b , and l have been derived from the elements of the orbit, the quantities R , \odot , and Σ being furnished by the solar tables; or, when Δ , β , and λ are given, these equations determine l , b , and r . The latitude Σ of the sun never exceeds $\pm 0''.9$, and, therefore, it may in most cases be neglected, so that $\cos \Sigma = 1$ and $\sin \Sigma = 0$, and the last equations become

$$\begin{aligned}\Delta \cos \beta \cos (\lambda - \odot) &= r \cos b \cos (l - \odot) + R, \\ \Delta \cos \beta \sin (\lambda - \odot) &= r \cos b \sin (l - \odot), \\ \Delta \sin \beta &= r \sin b.\end{aligned}\tag{90}$$

If we suppose the axis of x to be directed to a point whose longitude is Ω , or to the ascending node of the planet or comet, the equations (88) become

$$\begin{aligned}\Delta \cos \beta \cos (\lambda - \Omega) &= r \cos u + R \cos \Sigma \cos (\odot - \Omega), \\ \Delta \cos \beta \sin (\lambda - \Omega) &= \pm r \sin u \cos i + R \cos \Sigma \sin (\odot - \Omega), \\ \Delta \sin \beta &= r \sin u \sin i + R \sin \Sigma,\end{aligned}\tag{91}$$

by means of which β and λ may be found directly from Ω , i , r , and u .

If it be required to determine the geocentric right ascension and declination, denoted respectively by α and δ , we may convert the values of β and λ into those of α and δ . To effect this transformation, denoting by ε the obliquity of the ecliptic, we have

$$\begin{aligned}\cos \delta \cos \alpha &= \cos \beta \cos \lambda, \\ \cos \delta \sin \alpha &= \cos \beta \sin \lambda \cos \varepsilon - \sin \beta \sin \varepsilon, \\ \sin \delta &= \cos \beta \sin \lambda \sin \varepsilon + \sin \beta \cos \varepsilon.\end{aligned}$$

Let us now take

$$\begin{aligned}n \sin N &= \sin \beta, \\ n \cos N &= \cos \beta \sin \lambda,\end{aligned}$$

and we shall have

$$\begin{aligned}\cos \delta \cos \alpha &= \cos \beta \cos \lambda, \\ \cos \delta \sin \alpha &= n \cos (N + \varepsilon), \\ \sin \delta &= n \sin (N + \varepsilon).\end{aligned}$$

Therefore, we obtain

$$\tan N = \frac{\tan \beta}{\sin \lambda}, \quad \tan \alpha = \frac{\cos(N + \varepsilon)}{\cos N} \tan \lambda, \quad (92)$$

$$\tan \delta = \tan(N + \varepsilon) \sin \alpha.$$

We also have

$$\frac{\cos(N + \varepsilon)}{\cos N} = \frac{\cos \delta \sin \alpha}{\cos \beta \sin \lambda},$$

which will serve to check the calculation of α and δ . Since $\cos \delta$ and $\cos \beta$ are always positive, $\cos \alpha$ and $\cos \lambda$ must have the same sign, and thus the quadrant in which α is to be taken, is determined.

For the solution of the inverse problem, in which α and δ are given and the values of λ and β are required, it is only necessary to interchange, in these equations, α and λ , δ and β , and to write $-\varepsilon$ in place of ε .

34. Instead of pursuing the tedious process, when several places are required, of computing first the heliocentric place, then the geocentric place referred to the ecliptic, and, finally, the geocentric right ascension and declination, we may derive formulæ which, when certain constant auxiliaries have once been computed, enable us to derive the geocentric place directly, referred either to the ecliptic or to the equator.

We will first consider the case in which the ecliptic is taken as the fundamental plane. Let us, therefore, resume the equations

$$\begin{aligned} x' &= r \cos u, \\ y' &= \pm r \sin u \cos i, \\ z' &= r \sin u \sin i, \end{aligned}$$

in which the axis of x is supposed to be directed to the ascending node of the orbit of the body. If we now pass to a new system x, y, z ,—the origin and the axis of z remaining the same,—in which the axis of x is directed to the vernal equinox, we shall move it back, in a negative direction, equal to the angle Ω , and, consequently,

$$\begin{aligned} x &= x' \cos \Omega - y' \sin \Omega, \\ y &= x' \sin \Omega + y' \cos \Omega, \\ z &= z'. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} x &= r(\cos u \cos \Omega \mp \sin u \cos i \sin \Omega), \\ y &= r(\pm \sin u \cos i \cos \Omega + \cos u \sin \Omega), \\ z &= r \sin u \sin i, \end{aligned} \quad (93)$$

which are the expressions for the heliocentric co-ordinates of a planet or comet referred to the ecliptic, the positive axis of x being directed to the vernal equinox. The upper sign is to be used when the motion is direct, and the lower sign when it is retrograde.

Let us now put

$$\begin{aligned}\cos \Omega &= \sin a \sin A, \\ \mp \cos i \sin \Omega &= \sin a \cos A, \\ \sin \Omega &= \sin b \sin B, \\ \pm \cos i \cos \Omega &= \sin b \cos B,\end{aligned}\tag{94}$$

in which $\sin a$ and $\sin b$ are positive, and the expressions for the co-ordinates become

$$\begin{aligned}x &= r \sin a \sin (A + u), \\ y &= r \sin b \sin (B + u), \\ z &= r \sin i \sin u.\end{aligned}\tag{95}$$

The auxiliary quantities a , b , A , and B , it will be observed, are functions of Ω and i , and, in computing an ephemeris, are constant so long as these elements are regarded as constant. They are called the *constants for the ecliptic*.

To determine them, we have, from equations (94),

$$\begin{aligned}\cot A &= \mp \tan \Omega \cos i, & \cot B &= \pm \cot \Omega \cos i, \\ \sin a &= \frac{\cos \Omega}{\sin A}, & \sin b &= \frac{\sin \Omega}{\sin B};\end{aligned}$$

the upper sign being used when the motion is direct, and the lower sign when it is retrograde.

The auxiliaries $\sin a$ and $\sin b$ are always positive, and, therefore, $\sin A$ and $\cos \Omega$, $\sin B$ and $\sin \Omega$, respectively, must have the same signs. The quadrants in which A and B are situated, are thus determined.

From the equations (94) we easily find

$$\begin{aligned}\cos a &= \sin i \sin \Omega, \\ \cos b &= -\sin i \cos \Omega.\end{aligned}\tag{96}$$

If we add to the heliocentric co-ordinates of the body the co-ordinates of the sun referred to the earth, for which the equations have already been given, we shall have

$$\begin{aligned}x + X &= \Delta \cos \beta \cos \lambda, \\ y + Y &= \Delta \cos \beta \sin \lambda, \\ z + Z &= \Delta \sin \beta,\end{aligned}\tag{97}$$

which suffice to determine λ , β , and A . The values of α and δ may be derived from these by means of the equations (92).

35. We shall now derive the formulæ for determining α and δ directly. For this purpose, let x, y, z be the heliocentric co-ordinates of the body referred to the equator, the positive axis of x being directed to the vernal equinox. To pass from the system of co-ordinates referred to the ecliptic to those referred to the equator as the fundamental plane, we must revolve the system negatively around the axis of x , so that the axes of z and y in the new system make the angle ϵ with those of the primitive system, ϵ being the obliquity of the ecliptic. In this case, we have

$$\begin{aligned}x'' &= x, \\y'' &= y \cos \epsilon - z \sin \epsilon, \\z'' &= y \sin \epsilon + z \cos \epsilon.\end{aligned}$$

Substituting for x, y , and z their values from equations (93), and omitting the accents, we get

$$\begin{aligned}x &= r \cos u \cos \Omega \mp r \sin u \cos i \sin \Omega, \\y &= r \cos u \sin \Omega \cos \epsilon + r \sin u (\pm \cos i \cos \Omega \cos \epsilon - \sin i \sin \epsilon), \\z &= r \cos u \sin \Omega \sin \epsilon + r \sin u (\pm \cos i \cos \Omega \sin \epsilon + \sin i \cos \epsilon).\end{aligned} \quad (98)$$

These are the expressions for the heliocentric co-ordinates of the planet or comet referred to the equator. To reduce them to a convenient form for numerical calculation, let us put

$$\begin{aligned}\cos \Omega &= \sin a \sin A, \\ \mp \cos i \sin \Omega &= \sin a \cos A, \\ \sin \Omega \cos \epsilon &= \sin b \sin B, \\ \pm \cos i \cos \Omega \cos \epsilon - \sin i \sin \epsilon &= \sin b \cos B, \\ \sin \Omega \sin \epsilon &= \sin c \sin C, \\ \pm \cos i \cos \Omega \sin \epsilon + \sin i \cos \epsilon &= \sin c \cos C;\end{aligned} \quad (99)$$

and the expressions for the co-ordinates reduce to

$$\begin{aligned}x &= r \sin a \sin (A + u), \\y &= r \sin b \sin (B + u), \\z &= r \sin c \sin (C + u).\end{aligned} \quad (100)$$

The auxiliary quantities, a, b, c, A, B , and C , are constant so long as Ω and i remain unchanged, and are called *constants for the equator*.

It will be observed that the equations involving a and A , regarding the motion as direct, correspond to the relations between the parts of a quadrantal triangle of which the sides are i and a , the

angle included between these sides being that which we designate by A , and the angle opposite the side a being $90^\circ - \Omega$. In the case of b and B , the relations are those of the parts of a spherical triangle of which the sides are b , i , and $90^\circ + \varepsilon$, B being the angle included by i and b , and $180^\circ - \Omega$ the angle opposite the side b . Further, in the case of c and C , the relations are those of the parts of a spherical triangle of which the sides are c , i , and ε , the angle C being that included by the sides i and c , and $180^\circ - \Omega$ that included by the sides i and ε . We have, therefore, the following additional equations:

$$\begin{aligned}\cos a &= \sin i \sin \Omega, \\ \cos b &= -\cos \Omega \sin i \cos \varepsilon - \cos i \sin \varepsilon, \\ \cos c &= -\cos \Omega \sin i \sin \varepsilon + \cos i \cos \varepsilon.\end{aligned}\tag{101}$$

In the case of retrograde motion, we must substitute in these $180^\circ - i$ in place of i .

The geometrical signification of the auxiliary constants for the equator is thus made apparent. The angles a , b , and c are those which a line drawn from the origin of co-ordinates perpendicular to the plane of the orbit on the north side, makes with the positive co-ordinate axes, respectively; and A , B , and C are the angles which the three planes, passing through this line and the co-ordinate axes, make with a plane passing through this line and perpendicular to the line of nodes.

In order to facilitate the computation of the constants for the equator, let us introduce another auxiliary quantity E_0 , such that

$$\begin{aligned}\sin i &= e_0 \sin E_0, \\ \pm \cos i \cos \Omega &= e_0 \cos E_0,\end{aligned}$$

e_0 being always positive. We shall, therefore, have

$$\tan E_0 = \pm \frac{\tan i}{\cos \Omega}.$$

Since both e_0 and $\sin i$ are positive, the angle E_0 cannot exceed 180° ; and the algebraic sign of $\tan E_0$ will show whether this angle is to be taken in the first or second quadrant.

The first two of equations (99) give

$$\cot A = \mp \tan \Omega \cos i;$$

and the first gives

$$\sin a = \frac{\cos \Omega}{\sin A}.$$

From the fourth of equations (99), introducing e_0 and E_0 , we get

$$\sin b \cos B = e_0 \cos E_0 \cos \epsilon - e_0 \sin E_0 \sin \epsilon = e_0 \cos (E_0 + \epsilon).$$

But

$$\sin b \sin B = \sin \Omega \cos \epsilon;$$

therefore

$$\cot B = \frac{e_0}{\sin \Omega} \cdot \frac{\cos (E_0 + \epsilon)}{\cos \epsilon} = \pm \frac{\cos i}{\tan \Omega \cos E_0} \cdot \frac{\cos (E_0 + \epsilon)}{\cos \epsilon}.$$

We have, also,

$$\sin b = \frac{\sin \Omega \cos \epsilon}{\sin B}.$$

In a similar manner, we find

$$\cot C = \pm \frac{\cos i}{\tan \Omega \cos E_0} \cdot \frac{\sin (E_0 + \epsilon)}{\sin \epsilon},$$

and

$$\sin c = \frac{\sin \Omega \sin \epsilon}{\sin C}.$$

The auxiliaries $\sin a$, $\sin b$, and $\sin c$ are always positive, and, therefore, $\sin A$ and $\cos \Omega$, $\sin B$ and $\sin \Omega$, and also $\sin C$ and $\sin \Omega$, must have the same signs, which will determine the quadrant in which each of the angles A , B , and C is situated.

If we multiply the last of equations (99) by the third, and the fifth of these equations by the fourth, and subtract the first product from the last, we get, by reduction,

$$\sin b \sin c \sin (C - B) = - \sin i \sin \Omega.$$

But

$$\sin a \cos A = \mp \cos i \sin \Omega;$$

and hence we derive

$$\frac{\sin b \sin c \sin (C - B)}{\sin a \cos A} = \pm \tan i,$$

which serves to check the accuracy of the numerical computation of the constants, since the value of $\tan i$ obtained from this formula must agree exactly with that used in the calculation of the values of these constants.

If we put $A' = A \pm \pi \mp \Omega$, $B' = B \pm \pi \mp \Omega$, and $C' = C \pm \pi \mp \Omega$, the upper or lower sign being used according as the motion is direct or retrograde, we shall have

$$\begin{aligned}x &= r \sin a \sin (A' + v), \\y &= r \sin b \sin (B' + v), \\z &= r \sin c \sin (C' + v),\end{aligned}\tag{102}$$

a transformation which is perhaps unnecessary, but which is convenient when a series of places is to be computed.

It will be observed that the formulæ for computing the constants $a, b, c, A, B,$ and $C,$ in the case of direct motion, are converted into those for the case in which the distinction of retrograde motion is adopted, by simply using $180^\circ - i$ instead of i .

36. When the heliocentric co-ordinates of the body have been found, referred to the equator as the fundamental plane, if we add to these the geocentric co-ordinates of the sun referred to the same fundamental plane, the sum will be the geocentric co-ordinates of the body referred also to the equator.

For the co ordinates of the sun referred to the centre of the earth, we have, neglecting the latitude of the sun,

$$\begin{aligned}X &= R \cos \odot, \\Y &= R \sin \odot \cos \epsilon, \\Z &= R \sin \odot \sin \epsilon = Y \tan \epsilon,\end{aligned}$$

in which R represents the radius-vector of the earth, \odot the sun's longitude, and ϵ the obliquity of the ecliptic.

We shall, therefore, have

$$\begin{aligned}x + X &= \Delta \cos \delta \cos \alpha, \\y + Y &= \Delta \cos \delta \sin \alpha, \\z + Z &= \Delta \sin \delta,\end{aligned}\tag{103}$$

which suffice to determine $\alpha, \delta,$ and Δ .

If we have regard to the latitude of the sun in computing its geocentric co-ordinates, the formulæ will evidently become

$$\begin{aligned}X &= R \cos \odot \cos \Sigma, \\Y &= R \sin \odot \cos \Sigma \cos \epsilon - R \sin \Sigma \sin \epsilon, \\Z &= R \sin \odot \cos \Sigma \sin \epsilon + R \sin \Sigma \cos \epsilon,\end{aligned}\tag{104}$$

in which, since Σ can never exceed $\pm 0''.9$, $\cos \Sigma$ is very nearly equal to 1, and $\sin \Sigma = \Sigma$.

The longitudes and latitudes of the sun may be derived from a solar ephemeris, or from the solar tables. The principal astronomical ephemerides, such as the *Berliner Astronomisches Jahrbuch*, the *Nautical Almanac*, and the *American Ephemeris and Nautical Al-*

manac, contain, for each year for which they are published, the equatorial co-ordinates of the sun, referred both to the mean equinox and equator of the beginning of the year, and to the apparent equinox of the date, taking into account the latitude of the sun.

37. In the case of an elliptic orbit, we may determine the co-ordinates directly from the eccentric anomaly in the following manner:—

The equations (102) give, accenting the letters a , b , and c ,

$$\begin{aligned}x &= r \cos v \sin a' \sin A' + r \sin v \sin a' \cos A', \\y &= r \cos v \sin b' \sin B' + r \sin v \sin b' \cos B', \\z &= r \cos v \sin c' \sin C' + r \sin v \sin c' \cos C'.\end{aligned}$$

Now, since $r \cos v = a \cos E - ae$, and $r \sin v = a \cos \varphi \sin E$, we shall have

$$\begin{aligned}x &= a \sin a' \sin A' \cos E - ae \sin a' \sin A' + a \cos \varphi \sin a' \cos A' \sin E, \\y &= a \sin b' \sin B' \cos E - ae \sin b' \sin B' + a \cos \varphi \sin b' \cos B' \sin E, \\z &= a \sin c' \sin C' \cos E - ae \sin c' \sin C' + a \cos \varphi \sin c' \cos C' \sin E.\end{aligned}$$

Let us now put

$$\begin{aligned}a \cos \varphi \sin a' \cos A' &= \lambda_x \cos L_x, \\a \sin a' \sin A' &= \lambda_x \sin L_x, \\-ae \sin a' \sin A' &= -e\lambda_x \sin L_x = \nu_x; \\a \cos \varphi \sin b' \cos B' &= \lambda_y \cos L_y, \\a \sin b' \sin B' &= \lambda_y \sin L_y, \\-ae \sin b' \sin B' &= -e\lambda_y \sin L_y = \nu_y; \\a \cos \varphi \sin c' \cos C' &= \lambda_z \cos L_z, \\a \sin c' \sin C' &= \lambda_z \sin L_z, \\-ae \sin c' \sin C' &= -e\lambda_z \sin L_z = \nu_z;\end{aligned}$$

in which $\sin a'$, $\sin b'$, and $\sin c'$ have the same values as in equations (102), the accents being added simply to mark the necessary distinction in the notation employed in these formulæ. We shall, therefore, have

$$\begin{aligned}x &= \lambda_x \sin (L_x + E) + \nu_x, \\y &= \lambda_y \sin (L_y + E) + \nu_y, \\z &= \lambda_z \sin (L_z + E) + \nu_z.\end{aligned}\tag{105}$$

By means of these formulæ, the co-ordinates are found directly from the eccentric anomaly, when the constants λ_x , λ_y , λ_z , L_x , L_y , L_z , ν_x , ν_y , and ν_z , have been computed from those already found, or from a , b , c , A , B , and C . This method is very convenient when a great

number of geocentric places are to be computed; but, when only a few places are required, the additional labor of computing so many auxiliary quantities will not be compensated by the facility afforded in the numerical calculation, when these constants have been determined. Further, when the ephemeris is intended for the comparison of a series of observations in order to determine the corrections to be applied to the elements by means of the differential formulæ which we shall investigate in the following chapter, it will always be advisable to compute the co-ordinates by means of the radius-vector and true anomaly, since both of these quantities will be required in finding the differential coefficients.

38. In the case of a hyperbolic orbit, the co-ordinates may be computed directly from F , since we have

$$\begin{aligned} r \cos v &= a(e - \sec F), \\ r \sin v &= a \tan \psi \tan F; \end{aligned}$$

and, consequently,

$$\begin{aligned} x &= ae \sin a' \sin A' - a \sec F \sin a' \sin A' + a \tan \psi \tan F \sin a' \cos A', \\ y &= ae \sin b' \sin B' - a \sec F \sin b' \sin B' + a \tan \psi \tan F \sin b' \cos B', \\ z &= ae \sin c' \sin C' - a \sec F \sin c' \sin C' + a \tan \psi \tan F \sin c' \cos C'. \end{aligned}$$

Let us now put

$$\begin{aligned} ae \sin a' \sin A' &= \lambda_x, \\ -a \sin a' \sin A' &= \mu_x, \\ a \tan \psi \sin a' \cos A' &= \nu_x; \\ ae \sin b' \sin B' &= \lambda_y, \\ -a \sin b' \sin B' &= \mu_y, \\ a \tan \psi \sin b' \cos B' &= \nu_y; \\ ae \sin c' \sin C' &= \lambda_z, \\ -a \sin c' \sin C' &= \mu_z, \\ a \tan \psi \sin c' \cos C' &= \nu_z. \end{aligned}$$

Then we shall have

$$\begin{aligned} x &= \lambda_x + \mu_x \sec F + \nu_x \tan F, \\ y &= \lambda_y + \mu_y \sec F + \nu_y \tan F, \\ z &= \lambda_z + \mu_z \sec F + \nu_z \tan F. \end{aligned} \tag{103}$$

In a similar manner we may derive expressions for the co-ordinates, in the case of a hyperbolic orbit, when the auxiliary quantity σ is used instead of F .

39. If we denote by π' , Ω' , and i' the elements which determine the position of the orbit in space when referred to the equator as the

fundamental plane, and by ω_0 the angular distance between the ascending node of the orbit on the ecliptic and its ascending node on the equator, being measured positively from the equator in the direction of the motion, we shall have

$$\pi' = \pi - \Omega + \Omega' + \omega_0.$$

To find Ω' and i' , we have, from the spherical triangle formed by the intersection of the planes of the orbit, ecliptic, and equator with the celestial vault,

$$\begin{aligned}\cos i' &= \cos i \cos \varepsilon - \sin i \sin \varepsilon \cos \Omega, \\ \sin i' \sin \Omega' &= \sin i \sin \Omega, \\ \sin i' \cos \Omega' &= \cos i \sin \varepsilon + \sin i \cos \varepsilon \cos \Omega.\end{aligned}$$

Let us now put

$$\begin{aligned}n \sin N &= \cos i, \\ n \cos N &= \sin i \cos \Omega,\end{aligned}$$

and these equations reduce to

$$\begin{aligned}\cos i' &= n \sin (N - \varepsilon), \\ \sin i' \sin \Omega' &= \sin i \sin \Omega, \\ \sin i' \cos \Omega' &= n \cos (N - \varepsilon);\end{aligned}$$

from which we find

$$\begin{aligned}\tan N &= \frac{\cot i}{\cos \Omega}, & \tan \Omega' &= \frac{\cos N}{\cos (N - \varepsilon)} \tan \Omega, \\ \cot i' &= \tan (N - \varepsilon) \cos \Omega'.\end{aligned}\tag{107}$$

Since $\sin i$ is always positive, $\cos N$ and $\cos \Omega$ must have the same signs. To prove the numerical calculation, we have

$$\frac{\sin i \cos \Omega}{\sin i' \cos \Omega'} = \frac{\cos N}{\cos (N - \varepsilon)},$$

the value of the second member of which must agree with that used in computing Ω' .

In order to find ω_0 , we have, from the same triangle,

$$\begin{aligned}\sin \omega_0 \sin i' &= \sin \Omega \sin \varepsilon, \\ \cos \omega_0 \sin i' &= \cos \varepsilon \sin i + \sin \varepsilon \cos i \cos \Omega.\end{aligned}$$

Let us now take

$$\begin{aligned}m \sin M &= \cos \varepsilon, \\ m \cos M &= \sin \varepsilon \cos \Omega;\end{aligned}$$

and we obtain

$$\begin{aligned}\cot M &= \tan \varepsilon \cos \Omega, \\ \tan \omega_0 &= \frac{\cos M}{\cos (M-i)} \tan \Omega,\end{aligned}\quad (108)$$

and, also, to check the calculation,

$$\frac{\sin \varepsilon \cos \Omega}{\sin i' \cos \omega_0} = \frac{\cos M}{\cos (M-i)}.$$

If we apply Gauss's analogies to the same spherical triangle, we get

$$\begin{aligned}\cos \tfrac{1}{2}i' \sin \tfrac{1}{2}(\Omega' + \omega_0) &= \sin \tfrac{1}{2}\Omega \cos \tfrac{1}{2}(i - \varepsilon), \\ \cos \tfrac{1}{2}i' \cos \tfrac{1}{2}(\Omega' + \omega_0) &= \cos \tfrac{1}{2}\Omega \cos \tfrac{1}{2}(i + \varepsilon), \\ \sin \tfrac{1}{2}i' \sin \tfrac{1}{2}(\Omega' - \omega_0) &= \sin \tfrac{1}{2}\Omega \sin \tfrac{1}{2}(i - \varepsilon), \\ \sin \tfrac{1}{2}i' \cos \tfrac{1}{2}(\Omega' - \omega_0) &= \cos \tfrac{1}{2}\Omega \sin \tfrac{1}{2}(i + \varepsilon).\end{aligned}\quad (109)$$

The quadrant in which $\tfrac{1}{2}(\Omega' + \omega_0)$ or $\tfrac{1}{2}(\Omega - \omega_0)$ is situated, must be so taken that $\sin \tfrac{1}{2}i'$ and $\cos \tfrac{1}{2}i'$ shall be positive; and the agreement of the values of the latter two quantities, computed by means of the value of $\tfrac{1}{2}i'$ derived from $\tan \tfrac{1}{2}i'$, will serve to check the accuracy of the numerical calculation.

For the case in which the motion is regarded as retrograde, we must use $180^\circ - i$ instead of i in these equations, and we have, also,

$$\pi' = \pi - \Omega + \Omega' - \omega_0.$$

We may thus find the elements π' , Ω' , and i' , in reference to the equator, from the elements referred to the ecliptic; and using the elements so found instead of π , Ω , and i , and using also the places of the sun referred to the equator, we may derive the heliocentric and geocentric places with respect to the equator by means of the formulæ already given for the ecliptic as the fundamental plane.

If the position of the orbit with respect to the equator is given, and its position in reference to the ecliptic is required, it is only necessary to interchange Ω and Ω' , as well as i and $180^\circ - i'$, ε remaining unchanged, in these equations. These formulæ may also be used to determine the position of the orbit in reference to any plane in space; but the longitude Ω must then be measured from the place of the descending node of this plane on the ecliptic. The value of Ω , therefore, which must be used in the solution of the equations is, in this case, equal to the longitude of the ascending node of the orbit on the ecliptic diminished by the longitude of the descending node of the new plane of reference on the ecliptic. The quantities Ω' , i' , and ω_0 will have the same signification in reference

to this plane that they have in reference to the equator, with this distinction, however, that Ω' is measured from the descending node of this new plane of reference on the ecliptic; and ε will in this case denote the inclination of the ecliptic to this plane.

40. We have now derived all the formulæ which can be required in the case of undisturbed motion, for the computation of the heliocentric or geocentric place of a heavenly body, referred either to the ecliptic or equator, or to any other known plane, when the elements of its orbit are known; and the formulæ which have been derived are applicable to every variety of conic section, thus including all possible forms of undisturbed orbits consistent with the law of universal gravitation. The circle is an ellipse of which the eccentricity is zero, and, consequently, $M = v = u$, and $r = a$, for every point of the orbit. There is no instance of a circular orbit yet known; but in the case of the discovery of the asteroid planets between Mars and Jupiter it is sometimes thought advisable, in order to facilitate the identification of comparison stars for a few days succeeding the discovery, to compute circular elements, and from these an ephemeris.

The elements which determine the form of the orbit remain constant so long as the system of elements is regarded as unchanged; but those which determine the position of the orbit in space, π , Ω , and i , vary from one epoch to another on account of the change of the relative position of the planes to which they are referred. Thus the inclination of the orbit will vary slowly, on account of the change of the position of the ecliptic in space, arising from the perturbations of the earth by the other planets; while the longitude of the perihelion and the longitude of the ascending node will vary, both on account of this change of the position of the plane of the ecliptic, and also on account of precession and nutation. If π , Ω , and i are referred to the true equinox and ecliptic of any date, the resulting heliocentric places will be referred to the same equinox and ecliptic; and, further, in the computation of the geocentric places, the longitudes of the sun must be referred to the same equinox, so that the resulting geocentric longitudes or right ascensions will also be referred to that equinox. It will appear, therefore, that, on account of these changes in the values of π , Ω , and i , the auxiliaries $\sin a$, $\sin b$, $\sin c$, A , B , and C , introduced into the formulæ for the coordinates, will not be constants in the computation of the places for a series of dates, unless the elements are referred constantly, in the calculation, to a fixed equinox and ecliptic. It is customary, there-

fore, to reduce the elements to the ecliptic and mean equinox of the beginning of the year for which the ephemeris is required, and then to compute the places of the planet or comet referred to this equinox, using, in the case of the right ascension and declination, the mean obliquity of the ecliptic for the date of the fixed equinox adopted, in the computation of the auxiliary constants and of the co-ordinates of the sun. The places thus found may be reduced to the true equinox of the date by the well-known formulæ for precession and nutation. Thus, for the reduction of the right ascension and declination from the mean equinox and equator of the beginning of the year to the apparent or true equinox and equator of any date, usually the date to which the co-ordinates of the body belong, we have

$$\begin{aligned}\Delta\alpha &= f + g \sin(G + \alpha) \tan \delta, \\ \Delta\delta &= g \cos(G + \alpha),\end{aligned}\tag{110}$$

for which the quantities f , g , and G are derived from the data given either in the solar and lunar tables, or in astronomical ephemerides, such as have already been mentioned.

The problem of reducing the elements from the ecliptic of one date t to that of another date t' may be solved by means of equations (109), making, however, the necessary distinction in regard to the point from which Ω and Ω' are measured. Let θ denote the longitude of the descending node of the ecliptic of t' on that of t , and let η denote the angle which the planes of the two ecliptics make with each other, then, in the equations (109), instead of Ω we must write $\Omega - \theta$, and, in order that Ω' shall be measured from the vernal equinox, we must also write $\Omega' - \theta$ in place of Ω' . Finally, we must write η instead of ε , and $\Delta\omega$ for ω_0 , which is the variation in the value of ω in the interval $t' - t$ on account of the change of the position of the ecliptic; then the equations become

$$\begin{aligned}\cos \tfrac{1}{2}i' \sin \tfrac{1}{2}(\Omega' - \theta + \Delta\omega) &= \sin \tfrac{1}{2}(\Omega - \theta) \cos \tfrac{1}{2}(i - \eta), \\ \cos \tfrac{1}{2}i' \cos \tfrac{1}{2}(\Omega' - \theta + \Delta\omega) &= \cos \tfrac{1}{2}(\Omega - \theta) \cos \tfrac{1}{2}(i + \eta), \\ \sin \tfrac{1}{2}i' \sin \tfrac{1}{2}(\Omega' - \theta - \Delta\omega) &= \sin \tfrac{1}{2}(\Omega - \theta) \sin \tfrac{1}{2}(i - \eta), \\ \sin \tfrac{1}{2}i' \cos \tfrac{1}{2}(\Omega' - \theta - \Delta\omega) &= \cos \tfrac{1}{2}(\Omega - \theta) \sin \tfrac{1}{2}(i + \eta).\end{aligned}\tag{111}$$

These equations enable us to determine accurately the values of Ω' , i' , and $\Delta\omega$, which give the position of the orbit in reference to the ecliptic corresponding to the time t' , when θ and η are known. The longitudes, however, will still be referred to the same mean equinox as before, which we suppose to be that of t ; and, in order to refer

them to the mean equinox of the epoch t' , the amount of the precession in longitude during the interval $t' - t$ must also be applied.

If the changes in the values of the elements are not of considerable magnitude, it will be unnecessary to apply these rigorous formulæ, and we may derive others sufficiently exact, and much more convenient in application. Thus, from the spherical triangle formed by the intersection of the plane of the orbit and of the planes of the two ecliptics with the celestial vault, we get

$$\sin \gamma \cos (\Omega - \theta) = -\cos i' \sin i + \sin i' \cos i \cos \Delta\omega,$$

from which we easily derive

$$\sin (i' - i) = \sin \gamma \cos (\Omega - \theta) + 2 \sin i' \cos i \sin^2 \frac{1}{2} \Delta\omega. \quad (112)$$

We have, further,

$$\sin \Delta\omega \sin i' = \sin \gamma \sin (\Omega - \theta),$$

or

$$\sin \Delta\omega = \sin \gamma \frac{\sin (\Omega - \theta)}{\sin i'}. \quad (113)$$

We have, also, from the same triangle,

$$\begin{aligned} \sin \Delta\omega \cos i' &= -\cos (\Omega - \theta) \sin (\Omega' - \theta) \\ &+ \sin (\Omega - \theta) \cos (\Omega' - \theta) \cos \gamma, \end{aligned}$$

which gives

$$\sin (\Omega' - \Omega) = -\sin \Delta\omega \cos i' - 2 \sin (\Omega - \theta) \cos (\Omega' - \theta) \sin^2 \frac{1}{2} \gamma,$$

or

$$\begin{aligned} \sin (\Omega' - \Omega) &= -\sin \gamma \sin (\Omega - \theta) \cot i' \\ &- 2 \sin (\Omega - \theta) \cos (\Omega' - \theta) \sin^2 \frac{1}{2} \gamma. \end{aligned} \quad (114)$$

Finally, we have

$$\pi' - \pi = \Omega' - \Omega + \Delta\omega.$$

Since γ is very small, these equations give, if we apply also the precession in longitude so as to reduce the longitudes to the mean equinox of the date t' ,

$$\begin{aligned} \Delta\omega &= \gamma \frac{\sin (\Omega - \theta)}{\sin i}, \\ i' &= i + \gamma \cos (\Omega - \theta) + \frac{1}{4} \frac{\Delta\omega^2}{s} \sin 2i, \\ \Omega' &= \Omega + (t' - t) \frac{dl}{dt} - \gamma \sin (\Omega - \theta) \cot i' - \frac{1}{4} \frac{\gamma^2}{s} \sin 2(\Omega - \theta), \\ \pi' &= \pi + (t' - t) \frac{dl}{dt} + \gamma \sin (\Omega - \theta) \tan \frac{1}{2} i' - \frac{1}{4} \frac{\gamma^2}{s} \sin 2(\Omega - \theta); \end{aligned} \quad (115)$$

in which $\frac{dl}{dt}$ is the annual precession in longitude, and in which $s = 206264''.8$. In most cases, the last terms of the expressions for i' , Ω' , and π' , being of the second order, may be neglected.

For the case in which the motion is regarded as retrograde, we must put $180^\circ - i$ and $180^\circ - i'$, instead of i and i' , respectively, in the equations for $\Delta\omega$, i' , and Ω' ; and for π' , in this case, we have

$$\pi' - \pi = \Omega' - \Omega - \Delta\omega,$$

which gives

$$\pi' = \pi + (t' - t) \frac{dl}{dt} - \eta \sin(\Omega - \theta) \tan \frac{1}{2} i' - \frac{1}{4} \frac{\eta^2}{s} \sin 2(\Omega - \theta).$$

If we adopt Bessel's determination of the luni-solar precession and of the variation of the mean obliquity of the ecliptic, we have, at the time $1750 + \tau$,

$$\frac{dl}{dt} = 50''.21129 + 0''.0002442966\tau,$$

$$\frac{d\eta}{dt} = 0''.48892 - 0''.000006143\tau,$$

and, consequently,

$$\eta = (0''.48892 - 0''.000006143\tau) (t' - t);$$

and in the computation of the values of these quantities we must put $\tau = \frac{1}{2}(t' + t) - 1750$, t and t' being expressed in years.

The longitude of the descending node of the ecliptic of the time t on the ecliptic of 1750.0 is also found to be

$$351^\circ 36' 10'' - 5''.21(t - 1750),$$

which is measured from the mean equinox of the beginning of the year 1750.

The longitude of the descending node of the ecliptic of t' on that of t , measured from the same mean equinox, is equal to this value diminished by the angular distance between the descending node of the ecliptic of t on that of 1750 and the descending node of the ecliptic of t' on that of t , which distance is, neglecting terms of the second order,

$$5''.21(t' - 1750);$$

and the result is

$$351^\circ 36' 10'' - 5''.21(t - 1750) - 5''.21(t' - 1750),$$

or

$$351^\circ 36' 10'' - 10''.42(t - 1750) - 5''.21(t' - t).$$

To reduce this longitude to the mean equinox at the time t , we must add the general precession during the interval $t - 1750$, or

$$50''.21(t - 1750),$$

so that we have, finally,

$$\theta = 351^\circ 36' 10'' + 39''.79(t - 1750) - 5''.21(t' - t).$$

When the elements π , Ω , and i have been thus reduced from the ecliptic and mean equinox to which they are referred, to those of the date for which the heliocentric or geocentric place is required, they may be referred to the apparent equinox of the date by applying the nutation in longitude. Then, in the case of the determination of the right ascension and declination, using the apparent obliquity of the ecliptic in the computation of the co-ordinates, we directly obtain the place of the body referred to the apparent equinox. But, in computing a series of places, the changes which thus take place in the elements themselves from date to date induce corresponding changes in the auxiliary quantities a , b , c , A , B , and C , so that these are no longer to be considered as constants, but as continually changing their values by small differences. The differential formulæ for the computation of these changes, which are easily derived from the equations (99), will be given in the next chapter; but they are perhaps unnecessary, since it is generally most convenient, in the cases which occur, to compute the auxiliaries for the extreme dates for which the ephemeris is required, and to interpolate their values for intermediate dates.

It is advisable, however, to reduce the elements to the ecliptic and mean equinox of the beginning of the year for which the ephemeris is required, and using the mean obliquity of the ecliptic for that epoch, in the computation of the auxiliary constants for the equator, the resulting geocentric right ascensions and declinations will be referred to the same equinox, and they may then be reduced to the apparent equinox of the date by applying the corrections for precession and nutation.

The places which thus result are *free from parallax and aberration*. In comparing observations with an ephemeris, the correction for parallax is applied directly to the observed apparent places, since this correction varies for different places on the earth's surface. The correction for aberration may be applied in two different modes. We may subtract from the time of observation the time in which the light from the planet or comet reaches the earth, and the true place for this reduced time is identical with the apparent place for the time

of observation; or, in case we know the daily or hourly motion of the body in right ascension and declination, we may compute the motion during the interval which is required for the light to pass from the body to the earth, which, being applied to the observed place, gives the true place for the time of observation.

We may also include the aberration directly in the ephemeris by using the time $t - 497^{\circ}.78\Delta$ in computing the geocentric places for the time t , or by subtracting from the place free from aberration, computed for the time t , the motion in α and δ during the interval $497^{\circ}.78\Delta$, in which expression Δ is the distance of the body from the earth, and 497.78 the number of seconds in which light traverses the mean distance of the earth from the sun.

It is customary, however, to compute the ephemeris free from aberration and to subtract the *time of aberration*, $497^{\circ}.78\Delta$, from the time of observation when comparing observations with an ephemeris, according to the first method above mentioned. The places of the sun used in computing its co-ordinates must also be free from aberration; and if the longitudes derived from the solar tables include aberration, the proper correction must be applied, in order to obtain the true longitude required.

41. EXAMPLES.—We will now collect together, in the proper order for numerical calculation, some of the principal formulæ which have been derived, and illustrate them by numerical examples, commencing with the case of an elliptic orbit. Let it be required to find the geocentric right ascension and declination of the planet *Eurynome* $\textcircled{19}$, for mean midnight at Washington, for the date 1865 February 24, the elements of the orbit being as follows:—

Epoch = 1864 Jan. 1.0 Greenwich mean time.

$M = 1^{\circ} 29' 40''.21$

$\pi = 44 \quad 20 \quad 33 \quad .09$

$\Omega = 206 \quad 42 \quad 40 \quad .13$

$i = 4 \quad 36 \quad 50 \quad .51$

$\varphi = 11 \quad 15 \quad 51 \quad .02$

$\log a = 0.3881319$

$\log \mu = 2.9678088$

$\mu = 928''.5574b$

} Ecliptic and Mean
Equinox, 1864.0.

When a series of places is to be computed, the first thing to be done is to compute the auxiliary constants used in the expressions for the co-ordinates, and although but a single place is required in the problem proposed, yet we will proceed in this manner, in order to

exhibit the application of the formulæ. Since the elements π , Ω , and i are referred to the ecliptic and mean equinox of 1864.0, we will first reduce them to the ecliptic and mean equinox of 1865.0. For this reduction we have $t = 1864.0$, and $t' = 1865.0$, which give

$$\frac{dl}{dt} = 50''.239, \quad \theta = 352^\circ 51' 41'', \quad \gamma = 0''.4882.$$

Substituting these values in the equations (115), we obtain

$$i' - i = \Delta i = -0''.40, \quad \Delta \Omega = +53''.61, \quad \Delta \pi = +50''.23;$$

and hence the elements which determine the position of the orbit in reference to the ecliptic of 1865.0 are

$$\pi = 44^\circ 21' 23''.32, \quad \Omega = 206^\circ 43' 33''.74, \quad i = 4^\circ 36' 50''.11.$$

For the same instant we derive, from the *American Ephemeris and Nautical Almanac*, the value of the mean obliquity of the ecliptic, which is

$$\varepsilon = 23^\circ 27' 24''.03.$$

The auxiliary constants for the equator are then found by means of the formulæ

$$\cot A = -\tan \Omega \cos i, \quad \tan E_0 = \frac{\tan i}{\cos \Omega},$$

$$\cot B = \frac{\cos i}{\tan \Omega \cos E_0} \cdot \frac{\cos(E_0 + \varepsilon)}{\cos \varepsilon},$$

$$\cot C = \frac{\cos i}{\tan \Omega \cos E_0} \cdot \frac{\sin(E_0 + \varepsilon)}{\sin \varepsilon},$$

$$\sin a = \frac{\cos \Omega}{\sin A}, \quad \sin b = \frac{\sin \Omega \cos \varepsilon}{\sin B}, \quad \sin c = \frac{\sin \Omega \sin \varepsilon}{\sin C}.$$

The angle E_0 is always less than 180° , and the quadrant in which it is to be taken, is indicated directly by the algebraic sign of $\tan E_0$. The values of $\sin a$, $\sin b$, and $\sin c$ are always positive, and, therefore, the angles A , B , and C must be so taken, with respect to the quadrant in which each is situated, that $\sin A$ and $\cos \Omega$, $\sin B$ and $\sin \Omega$, and also $\sin C$ and $\sin \Omega$, shall have the same signs. From these we derive

$$\begin{array}{ll} A = 296^\circ 39' 5''.07, & \log \sin a = 9.9997156, \\ B = 205 \quad 55 \quad 27.14, & \log \sin b = 9.9748254, \\ C = 212 \quad 32 \quad 17.74, & \log \sin c = 9.5222192. \end{array}$$

Finally, the calculation of these constants is proved by means of the formula

$$\tan i = \frac{\sin b \sin c \sin (C - B)}{\sin a \cos A},$$

which gives $\log \tan i = 8.9068875$, agreeing with the value 8.9068876 derived directly from i .

Next, to find r and u . The date 1865 February 24.5 mean time at Washington reduced to the meridian of Greenwich by applying the difference of longitude, $5^h 8^m 11^s.2$, becomes 1865 February 24.714018 mean time at Greenwich. The interval, therefore, from the epoch for which the mean anomaly is given and the date for which the geocentric place is required, is 420.714018 days; and multiplying the mean daily motion, $928''.55745$, by this number, and adding the result to the given value of M , we get the mean anomaly for the required place, or

$$M = 1^\circ 29' 40''.21 + 108^\circ 30' 57''.14 = 110^\circ 0' 37''.35.$$

The eccentric anomaly E is then computed by means of the equation

$$M = E - e \sin E,$$

the value of e being expressed in seconds of arc. For *Eurynome* we have $\log \sin \varphi = \log e = 9.2907754$, and hence the value of e expressed in seconds is

$$\log e = 4.6052005.$$

By means of the equation (54) we derive an approximate value of E , namely,

$$E_0 = 119^\circ 49' 24'',$$

the value of e^2 expressed in seconds being $\log e^2 = 3.895976$; and with this we get

$$M_0 = E_0 - e \sin E_0 = 110^\circ 6' 50''.$$

Then we have

$$\Delta E_0 = \frac{M - M_0}{1 - e \cos E_0} = -\frac{372''.7}{1.097} = -339''.7,$$

which gives, for a second approximation to the value of E ,

$$E_0 = 119^\circ 43' 44''.3.$$

This gives $M_0 = 110^\circ 0' 36''.98$, and hence

$$\Delta E_0 = +\frac{0''.37}{1.097} = +0''.34.$$

Therefore, we have, for a third approximation to the value of E ,

$$E = 119^\circ 43' 44''.64,$$

which requires no further correction, since it satisfies the equation between M and E .

To find r and v , we have

$$\begin{aligned}\sqrt{r} \sin \frac{1}{2}v &= \sqrt{a(1+e)} \sin \frac{1}{2}E, \\ \sqrt{r} \cos \frac{1}{2}v &= \sqrt{a(1-e)} \cos \frac{1}{2}E.\end{aligned}$$

The values of the first factors in the second members of these equations are: $\log \sqrt{a(1+e)} = 0.2328104$, and $\log \sqrt{a(1-e)} = 0.1468741$; and we obtain

$$v = 129^\circ 3' 50''.52, \quad \log r = 0.4282854$$

Since $\pi - \Omega = 197^\circ 37' 49''.58$, we have

$$u = v + \pi - \Omega = 326^\circ 41' 40''.10.$$

The heliocentric co-ordinates in reference to the equator as the fundamental plane are then derived from the equations

$$\begin{aligned}x &= r \sin a \sin (A + u), \\ y &= r \sin b \sin (B + u), \\ z &= r \sin c \sin (C + u),\end{aligned}$$

which give, for *Eurynome*,

$$x = -2.6611270, \quad y = +0.3250277, \quad z = +0.0119486.$$

The *American Nautical Almanac* gives, for the equatorial co-ordinates of the sun for 1865 February 24.5 mean time at Washington, referred to the mean equinox and equator of the beginning of the year,

$$X = +0.9094557, \quad Y = -0.3599298, \quad Z = -0.1561751.$$

Finally, the geocentric right ascension, declination, and distance are given by the equations

$$\tan \alpha = \frac{y+Y}{x+X}, \quad \tan \delta = \frac{z+Z}{y+Y} \sin \alpha = \frac{z+Z}{x+X} \cos \alpha, \quad \Delta = \frac{z+Z}{\sin \delta},$$

the first form of the equation for $\tan \delta$ being used when $\sin \alpha$ is greater than $\cos \alpha$.

The value of Δ must always be positive; and δ cannot exceed $\pm 90^\circ$, the minus sign indicating south declination. Thus, we obtain

$$\alpha = 181^{\circ} 8' 29''.29, \quad \delta = -4^{\circ} 42' 21''.56, \quad \log A = 0.2450054.$$

To reduce α and δ to the true equinox and equator of February 24.5, we have, from the *Nautical Almanac*,

$$f = +16''.80, \quad \log g = 1.0168, \quad G = 45^{\circ} 16';$$

and, substituting these values in equations (110), the result is

$$\Delta\alpha = +17''.42, \quad \Delta\delta = -7''.17.$$

Hence the geocentric place, referred to the true equinox and equator of the date, is

$$\alpha = 181^{\circ} 8' 46''.71, \quad \delta = -4^{\circ} 42' 28''.73, \quad \log A = 0.2450054.$$

When only a single place is required, it is a little more expeditious to compute r from

$$r = a(1 - e \cos E),$$

and then $v - E$ from

$$\sin \frac{1}{2}(v - E) = \sqrt{\frac{a}{r}} \sin \frac{1}{2}\varphi \sin E.$$

Thus, in the case of the required place of *Eurynome*, we get

$$\begin{aligned} \log r &= 0.4282852, & v - E &= 9^{\circ} 20' 5''.92, \\ v &= 129^{\circ} 3' 50''.56, \end{aligned}$$

agreeing with the values previously determined. The calculation may be proved by means of the formula

$$\sin \frac{1}{2}(v + E) = \sqrt{\frac{a}{r}} \cos \frac{1}{2}\varphi \sin E.$$

In the case of the values just found, we have

$$\frac{1}{2}(v + E) = 124^{\circ} 23' 47''.60, \quad \log \sin \frac{1}{2}(v + E) = 9.9165316.$$

while the second member of this equation gives

$$\log \sin \frac{1}{2}(v + E) = 9.9165316$$

In the calculation of a single place, it is also very little shorter to compute first the heliocentric longitude and latitude by means of the equations (82), then the geocentric latitude and longitude by means of (89) or (90), and finally convert these into right ascension and declination by means of (92). When a large number of places are to be computed, it is often advantageous to compute the heliocentric

co-ordinates directly from the eccentric anomaly by means of the equations (105).

The calculation of the geocentric place in reference to the ecliptic is, in all respects, similar to that in which the equator is taken as the fundamental plane, and does not require any further illustration.

The determination of the geocentric or heliocentric place in the cases of parabolic and hyperbolic motion differs from the process indicated in the preceding example only in the calculation of r and v . To illustrate the case of parabolic motion, let $t - T = 75.364$ days; $\log q = 9.9650486$; and let it be required to find r and v .

First, we compute m from

$$m = \frac{C_0}{q^{\frac{3}{2}}},$$

in which $\log C_0 = 9.9601277$, and the result is

$$\log m = 0.0125548.$$

Then we find M from

$$M = m(t - T),$$

which gives

$$\log M = 1.8897187.$$

From this value of $\log M$ we derive, by means of Table VI.,

$$v = 79^\circ 55' 57''.26.$$

Finally, r is found from

$$r = \frac{q}{\cos^2 \frac{1}{2}v},$$

which gives

$$\log r = 0.1961120.$$

For the case of hyperbolic motion, let there be given $t - T = 65.41236$ days; $\psi = 37^\circ 35' 0''.0$, or $\log e = 0.1010188$; and $\log a = 0.6020600$, to find r and v . First, we compute N from

$$N = \frac{\lambda k}{a^{\frac{3}{2}}}(t - T),$$

in which $\log \lambda = 9.6377843$, and we obtain

$$\log N = 8.7859356; \quad N = 0.06108514.$$

The value of F must now be found from the equation

$$N = e\lambda \tan F - \log \tan(45^\circ + \frac{1}{2}F).$$

If we assume $F = 30^\circ$, a more approximate value may be derived from

$$\tan F' = \frac{N + \log \tan 60^\circ}{e\lambda},$$

which gives $F' = 28^\circ 40' 23''$, and hence $N' = 0.072678$. Then we compute the correction to be applied to this value of F' , by means of the equation

$$\Delta F' = \frac{(N - N') \cos^2 F'}{\lambda(e - \cos F')} s,$$

wherein $s = 206264''.8$; and the result is

$$\Delta F' = 4.6097(N - N')s = -3^\circ 3' 43''.0.$$

Hence, for a second approximation to the value of F' , we have

$$F' = 25^\circ 36' 40''.0.$$

The corresponding value of N is $N' = 0.0617653$, and hence

$$\Delta F' = 5.199(N - N')s = -12' 9''.4.$$

The third approximation, therefore, gives $F' = 25^\circ 24' 30''.6$, and, repeating the operation, we get

$$F' = 25^\circ 24' 27''.74.$$

which requires no further correction.

To find r , we have

$$r = a \left(\frac{e}{\cos F'} - 1 \right),$$

which gives

$$\log r = 0.2008544.$$

Then, v is derived from

$$\tan \frac{1}{2}v = \cot \frac{1}{2}\psi \tan \frac{1}{2}F',$$

and we find

$$v = 67^\circ 3' 0''.0.$$

When several places are required, it is convenient to compute v and r by means of the equations

$$\sqrt{r} \sin \frac{1}{2}v = \frac{\sqrt{a(e+1)}}{\sqrt{\cos F'}} \sin \frac{1}{2}F',$$

$$\sqrt{r} \cos \frac{1}{2}v = \frac{\sqrt{a(e-1)}}{\sqrt{\cos F'}} \cos \frac{1}{2}F'.$$

For the given values of a and e we have $\log \sqrt{a(e+1)} = 0.4782649$, $\log \sqrt{a(e-1)} = 0.0100829$, and hence we derive

$$v = 67^\circ 2' 59''.92, \quad \log r = 0.2008545.$$

It remains yet to illustrate the calculation of v and r for elliptic and hyperbolic orbits in which the eccentricity differs but little from unity. First, in the case of elliptic motion, let $t - T = 68.25$ days; $e = 0.9675212$; and $\log q = 9.7668134$. We compute M from

$$M = (t - T) \frac{C_0}{q^{\frac{3}{2}}} \sqrt{\frac{1+e}{2}},$$

wherein $\log C_0 = 9.9601277$, which gives

$$\log M = 2.1404550.$$

With this as argument we get, from Table VI.,

$$V = 101^\circ 38' 3''.74,$$

and then with this value of V as argument we find, from Table IX.,

$$A = 1540''.08, \quad B = 9''.506, \quad C = 0''.062.$$

Then we have $\log i = \log \frac{1-e}{1+e} = 8.217680$, and from the equation

$$v = V + A(100i) + B(100i)^2 + C(100i)^3,$$

we get

$$v = V + 42' 22''.28 + 25''.90 + 0''.28 = 102^\circ 20' 52''.20.$$

The value of r is then found from

$$r = \frac{q(1+e)}{1+e \cos v},$$

namely,

$$\log r = 0.1614051.$$

We may also determine r and v by means of Table X. Thus, we first compute M from

$$M = \frac{C_0(t-T)}{q^{\frac{3}{2}}} \cdot \frac{\sqrt{\frac{1}{10}(1+9e)}}{B}.$$

Assuming $B = 1$, we get $\log M = 2.13757$, and, entering Table VI. with this as argument, we find $w = 101^\circ 25'$. Then we compute A from

$$A = \frac{5(1-e)}{1+9e} \tan^2 \frac{1}{2} w,$$

which gives $A = 0.024985$. With this value of A as argument, we find, from Table X.,

$$\log B = 0.0000047.$$

The exact value of M is then found to be

$$\log M = 2.1375635,$$

which, by means of Table VI., gives

$$w = 101^\circ 24' 36''.26.$$

By means of this we derive

$$A = 0.02497944,$$

and hence, from Table X.,

$$\log C = 0.0043771.$$

Then we have

$$\tan \frac{1}{2}v = C \tan \frac{1}{2}w \sqrt{\frac{5(1+e)}{1+9e}},$$

which gives

$$v = 102^\circ 20' 52''.20,$$

agreeing exactly with the value already found. Finally, r is given by

$$r = \frac{q}{(1 + AC^2) \cos^2 \frac{1}{2}v},$$

from which we get

$$\log r = 0.1614052.$$

Before the time of perihelion passage, $t - T$ is negative; but the value of v is computed as if this were positive, and is then considered as negative.

In the case of hyperbolic motion, i is negative, and, with this distinction, the process when Table IX. is used is precisely the same as for elliptic motion; but when table X. is used, the value of A must be found from

$$A = \frac{5(e-1)}{(1+9e)} \tan^2 \frac{1}{2}w,$$

and that of r from

$$r = \frac{q}{(1 - AC^2) \cos^2 \frac{1}{2}v},$$

the values of $\log B$ and $\log C$ being taken from the columns of the table which belong to hyperbolic motion.

In the calculation of the position of a comet in space, if the motion

is retrograde and the inclination is regarded as less than 90° , the distinctions indicated in the formulæ must be carefully noted.

42. When we have thus computed the places of a planet or comet for a series of dates equidistant, we may readily interpolate the places for intermediate dates by the usual formulæ for interpolation. The interval between the dates for which the direct computation is made should also be small enough to permit us to neglect the effect of the fourth differences in the process of interpolation. This, however, is not absolutely necessary, provided that a very extended series of places is to be computed, so that the higher orders of differences may be taken into account. To find a convenient formula for this interpolation, let us denote any date, or argument of the function, by $a + n\omega$, and the corresponding value of the co-ordinate, or of the function, for which the interpolation is to be made, by $f(a + n\omega)$. If we have computed the values of the function for the dates, or arguments, $a - \omega$, a , $a + \omega$, $a + 2\omega$, &c., we may assume that an expression for the function which exactly satisfies these values will also give the exact values corresponding to any intermediate value of the argument. If we regard n as variable, we may expand the function into the series

$$f(a + n\omega) = f(a) + An + Bn^2 + Cn^3 + \&c. \quad (116)$$

and if we regard the fourth differences as vanishing, it is only necessary to consider terms involving n^3 in the determination of the unknown coefficients A , B , and C . If we put n successively equal to -1 , 0 , 1 , and 2 , and then take the successive differences of these values, we get

| | I. Diff. | II. Diff. | III. Diff. |
|--|---------------|-----------|------------|
| $f(a - \omega) = f(a) - A + B - C$ | | | |
| $f(a) = f(a)$ | $A - B + C$ | $2B$ | |
| $f(a + \omega) = f(a) + A + B + C$ | $A + B + C$ | $2B + 6C$ | $6C$ |
| $f(a + 2\omega) = f(a) + 2A + 4B + 8C$ | $A + 3B + 7C$ | | |

If we symbolize, generally, the difference $f(a + n\omega) - f(a + (n-1)\omega)$ by $f'(a + (n-\frac{1}{2})\omega)$, the difference $f'(a + (n+\frac{1}{2})\omega) - f'(a + (n-\frac{1}{2})\omega)$ by $f''(a + n\omega)$, and similarly for the successive orders of differences, these may be arranged as follows:—

| Argument. | Function. | I. Diff. | II. Diff. | III. Diff. |
|---------------|------------------|-----------------------------|-------------------|-------------------------------|
| $a - \omega$ | $f(a - \omega)$ | | | |
| a | $f(a)$ | $f'(a - \frac{1}{2}\omega)$ | | |
| $a + \omega$ | $f(a + \omega)$ | $f'(a + \frac{1}{2}\omega)$ | $f''(a)$ | |
| $a + 2\omega$ | $f(a + 2\omega)$ | $f'(a + \frac{3}{2}\omega)$ | $f''(a + \omega)$ | $f'''(a + \frac{1}{2}\omega)$ |

Comparing these expressions for the differences with the above, we get

$$\begin{aligned} C &= \frac{1}{6} f'''(a + \frac{1}{2}\omega), & B &= \frac{1}{2} f''(a), \\ A &= f'(a + \frac{1}{2}\omega) - \frac{1}{2} f''(a) - \frac{1}{6} f'''(a + \frac{1}{2}\omega), \end{aligned}$$

which, from the manner in which the differences are formed, give

$$\begin{aligned} C &= \frac{1}{8} (f''(a + \omega) - f''(a)), & B &= \frac{1}{2} f''(a), \\ A &= f(a + \omega) - f(a) - \frac{1}{2} f''(a) - \frac{1}{8} (f''(a + \omega) - f''(a)). \end{aligned}$$

To find the value of the function corresponding to the argument $a + \frac{1}{2}\omega$, we have $n = \frac{1}{2}$, and, from (116),

$$f(a + \frac{1}{2}\omega) = f(a) + \frac{1}{2}A + \frac{1}{4}B + \frac{1}{8}C.$$

Substituting in this the values of A , B , and C , last found, and reducing, we get

$$f(a + \frac{1}{2}\omega) = \frac{1}{2} (f(a + \omega) + f(a)) - \frac{1}{8} (\frac{1}{2} (f''(a + \omega) + f''(a))),$$

in which only fourth differences are neglected, and, since the place of the argument for $n = 0$ is arbitrary, we have, therefore, generally,

$$\begin{aligned} f(a + (n + \frac{1}{2})\omega) &= \frac{1}{2} (f'(a + (n + 1)\omega) + f(a + n\omega)) \\ &\quad - \frac{1}{8} (\frac{1}{2} (f''(a + (n + 1)\omega) + f''(a + n\omega))). \end{aligned} \quad (117)$$

Hence, to interpolate the value of the function corresponding to a date midway between two dates, or values of the argument, for which the values are known, we take the arithmetical mean of these two known values, and from this we subtract one-eighth of the arithmetical mean of the second differences which are found on the same horizontal line as the two given values of the function.

By extending the analytical process here indicated so as to include the fourth and fifth differences, the additional term to be added to equation (117) is found to be

$$+ \frac{3}{128} (\frac{1}{2} (f^{(4)}(a + (n + 1)\omega) + f^{(4)}(a + n\omega))),$$

and the correction corresponding to this being applied, only sixth differences will be neglected.

It is customary in the case of the comets which do not move too rapidly, to adopt an interval of four days, and in the case of the asteroid planets, either four or eight days, between the dates for which the direct calculation is made. Then, by interpolating, in the case of an interval ω , equal to four days, for the intermediate dates, we obtain a series of places at intervals of two days; and, finally, inter-

polating for the dates intermediate to these, we derive the places at intervals of one day. When a series of places has been computed, the use of differences will serve as a check upon the accuracy of the calculation, and will serve to detect at once the place which is not correct, when any discrepancy is apparent. The greatest discordance will be shown in the differences on the same horizontal line as the erroneous value of the function; and the discordance will be greater and greater as we proceed successively to take higher orders of differences. In order to provide against the contingency of systematic error, duplicate calculation should be made of those quantities in which such an error is likely to occur.

The ephemerides of the planets, to be used for the comparison of observations, are usually computed for a period of a few weeks before and after the time of opposition to the sun; and the time of the opposition may be found in advance of the calculation of the entire ephemeris. Thus, we find first the date for which the mean longitude of the planet is equal to the longitude of the sun increased by 180° ; then we compute the equation of the centre at this time by means of the equation (53), using, in most cases, only the first term of the development, or

$$v - M = 2e \sin M,$$

e being expressed in seconds. Next, regarding this value as constant, we find the date for which

$$L + \text{equation of the centre}$$

is equal to the longitude of the sun increased by 180° ; and for this date, and also for another at an interval of a few days, we compute u , and hence the heliocentric longitudes by means of the equation

$$\tan(l - \Omega) = \tan u \cos i.$$

Let these longitudes be denoted by l and l' , the times to which they correspond by t and t' , and the longitudes of the sun for the same times by \odot and \odot' ; then for the time t_0 , for which the heliocentric longitudes of the planet and the earth are the same, we have

$$t_0 = t + \frac{l - 180^\circ - \odot}{(\odot' - \odot) - (l' - l)} (t' - t),$$

(118)

$$t_0 = t' + \frac{l' - 180^\circ - \odot'}{(\odot' - \odot) - (l' - l)} (t' - t),$$

or

the first of these equations being used when $l - 180^\circ - \odot$ is less

than $l' - 180^\circ - \odot'$. If the time t_0 differs considerably from t or t' , it may be necessary, in order to obtain an accurate result, to repeat the latter part of the calculation, using t_0 for t , and taking t' at a small interval from this, and so that the true time of opposition shall fall between t and t' . The longitudes of the planet and of the sun must be measured from the same equinox.

When the eccentricity is considerable, it will facilitate the calculation to use two terms of equation (53) in finding the equation of the centre, and, if e is expressed in seconds, this gives

$$v - M = 2e \sin M + \frac{5}{4} \cdot \frac{e^2}{s} \sin 2M,$$

s being the number of seconds corresponding to a length of arc equal to the radius, or $206264''.8$; and the value of $v - M$ will then be expressed in seconds of arc. In all cases in which circular arcs are involved in an equation, great care must be taken, in the numerical application, in reference to the homogeneity of the different terms. If the arcs are expressed by an abstract number, or by the length of arc expressed in parts of the radius taken as the unit, to express them in seconds we must multiply by the number 206264.8 ; but if the arcs are expressed in seconds, each term of the equation must contain only one concrete factor, the other concrete factors, if there be any, being reduced to abstract numbers by dividing each by s the number of seconds in an arc equal to the radius.

43. It is unnecessary to illustrate further the numerical application of the various formulæ which have been derived, since by reference to the formulæ themselves the course of procedure is obvious. It may be remarked, however, that in many cases in which auxiliary angles have been introduced so as to render the equations convenient for logarithmic calculation, by the use of tables which determine the logarithms of the sum or difference of two numbers when the logarithms of these numbers are given, the calculation is abbreviated, and is often even more accurately performed than by the aid of the auxiliary angles.

The logarithm of the sum of two numbers may be found by means of the tables of common logarithms. Thus, we have

$$\log(a + b) = \log a \left(1 + \frac{b}{a} \right) = \log b \left(1 + \frac{a}{b} \right).$$

If we put

$$\log \tan x = \frac{1}{2} (\log b - \log a),$$

we shall have

$$\log(a + b) = \log a - 2 \log \cos x,$$

or

$$\log(a + b) = \log b - 2 \log \sin x.$$

The first form is used when $\cos x$ is greater than $\sin x$, and the second form when $\cos x$ is less than $\sin x$.

It should also be observed that in the solution of equations of the form of (89), after $\tan(\lambda - \odot)$ —using the notation of this particular case—has been found by dividing the second equation by the first, the second members of these equations being divided by $\cos(\lambda - \odot)$ and $\sin(\lambda - \odot)$, respectively, give two values of $\Delta \cos \beta$, which should agree within the limits of the unavoidable errors of the logarithmic tables; but, in order that the errors of these tables shall have the least influence, the value derived from the first equation is to be preferred when $\cos(\lambda - \odot)$ is greater than $\sin(\lambda - \odot)$, and that derived from the second equation when $\cos(\lambda - \odot)$ is less than $\sin(\lambda - \odot)$. The value of Δ , if the greatest accuracy possible is required, should be derived from $\Delta \cos \beta$ when β is less than 45° , and from $\Delta \sin \beta$ when β is greater than 45° .

In the application of numbers to equations (109), when the values of the second members have been computed, we first, by division, find $\tan \frac{1}{2}(\Omega' + \omega_0)$ and $\tan \frac{1}{2}(\Omega' - \omega_0)$; then, if $\sin \frac{1}{2}(\Omega' + \omega_0)$ is greater than $\cos \frac{1}{2}(\Omega' + \omega_0)$, we find $\cos \frac{1}{2}i'$ from the first equation; but if $\sin \frac{1}{2}(\Omega' + \omega_0)$ is less than $\cos \frac{1}{2}(\Omega' + \omega_0)$, we find $\cos \frac{1}{2}i'$ from the second equation. The same principle is applied in finding $\sin \frac{1}{2}i'$ by means of the third and fourth equations. Finally, from $\sin \frac{1}{2}i'$ and $\cos \frac{1}{2}i'$ we get $\tan \frac{1}{2}i'$, and hence i' . The check obtained by the agreement of the values of $\sin \frac{1}{2}i'$ and $\cos \frac{1}{2}i'$, with those computed from the value of i' derived from $\tan \frac{1}{2}i'$, does not absolutely prove the calculation. This proof, however, may be obtained by means of the equation

$$\sin i' \sin \Omega' = \sin i \sin \Omega,$$

or by

$$\sin i' \sin \omega_0 = \sin \epsilon \sin \Omega.$$

In all cases, care should be taken in determining the quadrant in which the angles sought are situated, the criteria for which are fixed either by the nature of the problem directly, or by the relation of the algebraic signs of the trigonometrical functions involved.

CHAPTER II.

INVESTIGATION OF THE DIFFERENTIAL FORMULÆ WHICH EXPRESS THE RELATION BETWEEN THE GEOCENTRIC OR HELIOCENTRIC PLACES OF A HEAVENLY BODY AND THE VARIATION OF THE ELEMENTS OF ITS ORBIT.

44. IN many calculations relating to the motion of a heavenly body, it becomes necessary to determine the variations which small increments applied to the values of the elements of its orbit will produce in its geocentric or heliocentric place. The form, however, in which the problem most frequently presents itself is that in which approximate elements are to be corrected by means of the differences between the places derived from computation and those derived from observation. In this case it is required to find the variations of the elements such that they will cause the differences between calculation and observation to vanish; and, since there are six elements, it follows that six separate equations, involving the variations of the elements as the unknown quantities, must be formed. Each longitude or right ascension, and each latitude or declination, derived from observation, will furnish one equation; and hence at least three complete observations will be required for the solution of the problem. When more than three observations are employed, and the number of equations exceeds the number of unknown quantities, the equations of condition which are obtained must be reduced to six final equations, from which, by elimination, the corrections to be applied to the elements may be determined.

If we suppose the corrections which must be applied to the elements, in order to satisfy the data furnished by observation, to be so small that their squares and higher powers may be neglected, the variations of those elements which involve angular measure being expressed in parts of the radius as unity, the relations sought may be determined by differentiating the various formulæ which determine the position of the body. Thus, if we represent by θ any co-ordinate of the place of the body computed from the assumed elements of the orbit, we shall have, in the case of an elliptic orbit,

$$\theta = f(\pi, \Omega, i, \varphi, M_0, \mu),$$

M_0 being the mean anomaly at the epoch T . Let θ' denote the value of this co-ordinate as derived directly or indirectly from observation; then, if we represent the variations of the elements by $\Delta\pi$, $\Delta\Omega$, Δi , &c., and if we suppose these variations to be so small that their squares and higher powers may be neglected, we shall have

$$\begin{aligned} \theta' - \theta = \Delta\theta = \frac{d\theta}{d\pi} \Delta\pi + \frac{d\theta}{d\Omega} \Delta\Omega + \frac{d\theta}{di} \Delta i + \frac{d\theta}{d\varphi} \Delta\varphi \\ + \frac{d\theta}{dM_0} \Delta M_0 + \frac{d\theta}{d\mu} \Delta\mu. \end{aligned} \quad (1)$$

The differential coefficients $\frac{d\theta}{d\pi}$, $\frac{d\theta}{d\Omega}$, &c. must now be derived from the equations which determine the place of the body when the elements are known.

We shall first take the equator as the plane to which the positions of the body are referred, and find the differential coefficients of the geocentric right ascension and declination with respect to the elements of the orbit, these elements being referred to the ecliptic as the fundamental plane. Let x, y, z be the heliocentric co-ordinates of the body in reference to the equator, and we have

$$\theta = f(x, y, z),$$

or

$$d\theta = \frac{d\theta}{dx} dx + \frac{d\theta}{dy} dy + \frac{d\theta}{dz} dz.$$

Hence we obtain

$$\frac{d\theta}{d\pi} = \frac{d\theta}{dx} \cdot \frac{dx}{d\pi} + \frac{d\theta}{dy} \cdot \frac{dy}{d\pi} + \frac{d\theta}{dz} \cdot \frac{dz}{d\pi}; \quad (2)$$

and similarly for the differential coefficients of θ with respect to the other elements. We must, therefore, find the partial differential coefficients of θ with respect to x, y , and z , and then the partial differential coefficients of these co-ordinates with respect to the elements. In the case of the right ascension we put $\theta = \alpha$, and in the case of the declination we put $\theta = \delta$.

45. If we differentiate the equations

$$\begin{aligned} x + X &= \Delta \cos \delta \cos \alpha, \\ y + Y &= \Delta \cos \delta \sin \alpha, \\ z + Z &= \Delta \sin \delta, \end{aligned}$$

regarding X, Y , and Z as constant, we find

$$\begin{aligned} dx &= \cos \alpha \cos \delta d\Delta - \Delta \sin \alpha \cos \delta d\alpha - \Delta \cos \alpha \sin \delta d\delta, \\ dy &= \sin \alpha \cos \delta d\Delta + \Delta \cos \alpha \cos \delta d\alpha - \Delta \sin \alpha \sin \delta d\delta, \\ dz &= \sin \delta d\Delta + \Delta \cos \delta d\delta. \end{aligned}$$

From these equations, by elimination, we obtain

$$\begin{aligned} \cos \delta d\alpha &= -\frac{\sin \alpha}{\Delta} dx + \frac{\cos \alpha}{\Delta} dy, \\ d\delta &= -\frac{\cos \alpha \sin \delta}{\Delta} dx - \frac{\sin \alpha \sin \delta}{\Delta} dy + \frac{\cos \delta}{\Delta} dz. \end{aligned} \quad (3)$$

Therefore, the partial differential coefficients of α and δ with respect to the heliocentric co-ordinates are

$$\begin{aligned} \cos \delta \frac{d\alpha}{dx} &= -\frac{\sin \alpha}{\Delta}, & \frac{d\delta}{dx} &= -\frac{\cos \alpha \sin \delta}{\Delta}, \\ \cos \delta \frac{d\alpha}{dy} &= \frac{\cos \alpha}{\Delta}, & \frac{d\delta}{dy} &= -\frac{\sin \alpha \sin \delta}{\Delta}, \\ \cos \delta \frac{d\alpha}{dz} &= 0, & \frac{d\delta}{dz} &= \frac{\cos \delta}{\Delta}. \end{aligned} \quad (4)$$

Next, to find the partial differential coefficients of the co-ordinates x, y, z , with respect to the elements, if we differentiate the equations (100)₁, observing that $\sin a, \sin b, \sin c, A, B, C$, are functions of Ω and i , we get

$$\begin{aligned} dx &= \frac{x}{r} dr + x \cot(A + u) du + \frac{dx}{d\Omega} d\Omega + \frac{dx}{di} di, \\ dy &= \frac{y}{r} dr + y \cot(B + u) du + \frac{dy}{d\Omega} d\Omega + \frac{dy}{di} di, \\ dz &= \frac{z}{r} dr + z \cot(C + u) du + \frac{dz}{d\Omega} d\Omega + \frac{dz}{di} di. \end{aligned}$$

To find the expressions for $\frac{dx}{d\Omega}, \frac{dx}{di}$, &c., we have the equations

$$\begin{aligned} x &= r \cos u \cos \Omega - r \sin u \sin \Omega \cos i, \\ y &= r \cos u \sin \Omega \cos \epsilon + r \sin u \cos \Omega \cos i \cos \epsilon - r \sin u \sin i \sin \epsilon, \\ z &= r \cos u \sin \Omega \sin \epsilon + r \sin u \cos \Omega \cos i \sin \epsilon + r \sin u \sin i \cos \epsilon. \end{aligned}$$

which give, by differentiation,

$$\begin{aligned} \frac{dx}{d\Omega} &= -r \cos u \sin \Omega - r \sin u \cos \Omega \cos i, \\ \frac{dy}{d\Omega} &= r \cos u \cos \Omega \cos \epsilon - r \sin u \sin \Omega \cos i \cos \epsilon, \end{aligned}$$

$$\begin{aligned}\frac{dz}{d\Omega} &= r \cos u \cos \Omega \sin \varepsilon - r \sin u \sin \Omega \cos i \sin \varepsilon, \\ \frac{dx}{di} &= r \sin u \sin \Omega \sin i, \\ \frac{dy}{di} &= -r \sin u \cos \Omega \sin i \cos \varepsilon - r \sin u \cos i \sin \varepsilon, \\ \frac{dz}{di} &= -r \sin u \cos \Omega \sin i \sin \varepsilon + r \sin u \cos i \cos \varepsilon.\end{aligned}$$

The first three of these equations immediately reduce to

$$\frac{dx}{d\Omega} = -y \cos \varepsilon - z \sin \varepsilon, \quad \frac{dy}{d\Omega} = x \cos \varepsilon, \quad \frac{dz}{d\Omega} = x \sin \varepsilon; \quad (b)$$

and since

$$\begin{aligned}\cos a &= \sin \Omega \sin i, \\ \cos b &= -\cos \Omega \sin i \cos \varepsilon - \cos i \sin \varepsilon, \\ \cos c &= -\cos \Omega \sin i \sin \varepsilon + \cos i \cos \varepsilon,\end{aligned}$$

we have, also,

$$\frac{dx}{di} = r \sin u \cos a, \quad \frac{dy}{di} = r \sin u \cos b, \quad \frac{dz}{di} = r \sin u \cos c.$$

Further, we have

$$du = dv + d\pi - d\Omega,$$

and hence, finally,

$$\begin{aligned}dx &= \frac{x}{r} dr + x \cot(A + u) dv + x \cot(A + u) d\pi \\ &+ (-x \cot(A + u) - y \cos \varepsilon - z \sin \varepsilon) d\Omega + r \sin u \cos a di, \\ dy &= \frac{y}{r} dr + y \cot(B + u) dv + y \cot(B + u) d\pi \\ &+ (-y \cot(B + u) + x \cos \varepsilon) d\Omega + r \sin u \cos b di, \\ dz &= \frac{z}{r} dr + z \cot(C + u) dv + z \cot(C + u) d\pi \\ &+ (-z \cot(C + u) + x \sin \varepsilon) d\Omega + r \sin u \cos c di.\end{aligned} \quad (6)$$

These equations give, for the partial differential coefficients of the heliocentric co-ordinates with respect to the elements,

$$\begin{aligned}\frac{dx}{d\pi} = \frac{dx}{dv} &= x \cot(A + u), & \frac{dy}{d\pi} = \frac{dy}{dv} &= y \cot(B + u), \\ \frac{dz}{d\pi} = \frac{dz}{dv} &= z \cot(C + u);\end{aligned}$$

$$\frac{dx}{d\Omega} = -x \cot(A+u) - y \cos \epsilon - z \sin \epsilon, \quad \frac{dy}{d\Omega} = -y \cot(B+u) + x \cos \epsilon, \\ \frac{dz}{d\Omega} = -z \cot(C+u) + x \sin \epsilon;$$

$$\frac{dx}{di} = r \sin u \cos a, \quad \frac{dy}{di} = r \sin u \cos b, \quad \frac{dz}{di} = r \sin u \cos c; \quad (7) \\ \frac{dx}{dr} = \frac{x}{r}, \quad \frac{dy}{dr} = \frac{y}{r}, \quad \frac{dz}{dr} = \frac{z}{r}.$$

When the direct inclination is greater than 90° , if we introduce the distinction of retrograde motion, we have

$$du = dv - d\pi + d\Omega,$$

and hence

$$\frac{dx}{d\pi} = -\frac{dx}{dv} = -x \cot(A+u), \quad \frac{dy}{d\pi} = -\frac{dy}{dv} = -y \cot(B+u), \\ \frac{dz}{d\pi} = -\frac{dz}{dv} = -z \cot(C+u); \quad (8)$$

$$\frac{dx}{d\Omega} = \frac{dx}{dv} - y \cos \epsilon - z \sin \epsilon, \quad \frac{dy}{d\Omega} = \frac{dy}{dv} + x \cos \epsilon, \quad \frac{dz}{d\Omega} = \frac{dz}{dv} + x \sin \epsilon.$$

The expressions for $\frac{dx}{dr}$, $\frac{dy}{dr}$, and $\frac{dz}{dr}$ remain unchanged; and we have, also,

$$\frac{dx}{di} = -r \sin u \cos a, \quad \frac{dy}{di} = -r \sin u \cos b, \quad \frac{dz}{di} = -r \sin u \cos c. \quad (9)$$

It is advisable, in order to avoid the use of two sets of formulæ, in part, to regard the motion as direct and the inclination as susceptible of any value from 0° to 180° . If the elements which are given are for retrograde motion, we take the supplement of i instead of i ; and if we designate the longitude of the perihelion, when the motion is considered as being retrograde, by (π) , we shall have

$$\pi = 2\Omega - (\pi).$$

If we introduce, as one of the elements of the orbit, the distance of the perihelion from the ascending node, we have

$$du = dv + d\omega,$$

and, hence,

$$\frac{dx}{d\omega} = \frac{dx}{dv} = x \cot(A+u), \quad \frac{dy}{d\omega} = \frac{dy}{dv} = y \cot(B+u), \\ \frac{dz}{d\omega} = \frac{dz}{dv} = z \cot(C+u). \quad (10)$$

The values of $\frac{dx}{d\Omega}$, $\frac{dy}{d\Omega}$, and $\frac{dz}{d\Omega}$ must, in this case, be found by means of the equations (5).

By means of these expressions for the differential coefficients of the co-ordinates x, y, z , with respect to the various elements, and those given by (4), we may derive the differential coefficients of the geocentric right ascension and declination with respect to the elements Ω, i , and π or ω , and also with respect to r and v , by writing successively α and δ in place of θ , and Ω, i , &c., in place of π in the equation (2). The quantities r and v , however, are functions of the remaining elements φ, M_0 , and μ ; and we have

$$\begin{aligned} dr &= \frac{dr}{d\varphi} d\varphi + \frac{dr}{dM_0} dM_0 + \frac{dr}{d\mu} d\mu, \\ dv &= \frac{dv}{d\varphi} d\varphi + \frac{dv}{dM_0} dM_0 + \frac{dv}{d\mu} d\mu. \end{aligned}$$

Therefore, the partial differential coefficients of x , with respect to the elements φ, M_0 , and μ , are

$$\begin{aligned} \frac{dx}{d\varphi} &= \frac{dx}{dr} \cdot \frac{dr}{d\varphi} + \frac{dx}{dv} \cdot \frac{dv}{d\varphi}, \\ \frac{dx}{dM_0} &= \frac{dx}{dr} \cdot \frac{dr}{dM_0} + \frac{dx}{dv} \cdot \frac{dv}{dM_0}, \\ \frac{dx}{d\mu} &= \frac{dx}{dr} \cdot \frac{dr}{d\mu} + \frac{dx}{dv} \cdot \frac{dv}{d\mu}. \end{aligned} \tag{11}$$

The expressions for the partial differential coefficients in the case of the co-ordinates y and z are of precisely the same form, and are obtained by writing, successively, y and z in place of x . The values of $\frac{dx}{dr}, \frac{dx}{dv}, \frac{dy}{dr}, \frac{dy}{dv}, \frac{dz}{dr},$ and $\frac{dz}{dv}$ are given by the equations (7), and when the expressions for $\frac{dr}{d\varphi}, \frac{dv}{d\varphi}, \frac{dr}{dM_0}, \frac{dv}{dM_0}, \frac{dr}{d\mu},$ and $\frac{dv}{d\mu}$ have been found, the partial differential coefficients of the heliocentric co-ordinates with respect to the elements φ, M_0 , and μ will be completely determined, and hence, by means of (2), making the necessary changes, the differential coefficients of α and δ with respect to these elements.

46. If we differentiate the equation

$$M = E - e \sin E,$$

we shall have

$$dM = dE(1 - e \cos E) - \cos \varphi \sin E d\varphi.$$

But, since $1 - e \cos E = \frac{r}{a}$, and $\cos \varphi \sin E = \frac{r}{a} \sin v$, this reduces to

$$dM = \frac{r}{a} dE - \frac{r}{a} \sin v d\varphi,$$

or

$$dE = \frac{a}{r} dM + \sin v d\varphi.$$

If we take the logarithms of both members of the equation

$$\tan \frac{1}{2}v = \tan \frac{1}{2}E \tan (45^\circ + \frac{1}{2}\varphi),$$

and differentiate, we find

$$\frac{dv}{2 \sin \frac{1}{2}v \cos \frac{1}{2}v} = \frac{dE}{2 \sin \frac{1}{2}E \cos \frac{1}{2}E} + \frac{d\varphi}{2 \sin (45^\circ + \frac{1}{2}\varphi) \cos (45^\circ + \frac{1}{2}\varphi)},$$

which reduces to

$$dv = \frac{\sin v}{\sin E} dE + \frac{\sin v}{\cos \varphi} d\varphi.$$

Introducing into this equation the value of dE , already found, and replacing $\sin E$ by $\frac{r \sin v}{a \cos \varphi}$, we get

$$dv = \frac{a^2 \cos \varphi}{r^2} dM + \frac{\sin v}{\cos \varphi} \left(\frac{a \cos^2 \varphi}{r} + 1 \right) d\varphi.$$

But since $a \cos^2 \varphi = p$, and $\frac{p}{r} = 1 + \sin \varphi \cos v$, this becomes

$$dv = \frac{a^2 \cos \varphi}{r^2} dM + \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) \sin v d\varphi. \quad (12)$$

If we differentiate the equation

$$r = a(1 - e \cos E),$$

we shall have

$$dr = \frac{r}{a} da + ae \sin E dE - a \cos \varphi \cos E d\varphi;$$

and substituting for dE its value in terms of dM and $d\varphi$, the result is

$$dr = \frac{r}{a} da + a \tan \varphi \sin v dM + (ae \sin E \sin v - a \cos \varphi \cos E) d\varphi. \quad (13)$$

Now, since $\sin E = \frac{\sin v \cos \varphi}{1 + e \cos v}$, and $\cos E = \frac{\cos v + e}{1 + e \cos v}$, we shall have

$$ae \sin E \sin v - a \cos \varphi \cos E = \frac{ae \cos \varphi \sin^3 v}{1 + e \cos v} - \frac{a \cos \varphi (\cos v + e)}{1 + e \cos v},$$

which reduces to

$$ae \sin E \sin v - a \cos \varphi \cos E = -a \cos \varphi \cos v$$

Hence, the expression for dr becomes

$$dr = \frac{r}{a} da + a \tan \varphi \sin v dM - a \cos \varphi \cos v d\varphi. \quad (14)$$

Further, we have

$$M = M_0 + \mu(t - T),$$

T being the epoch for which the mean anomaly is M_0 , and

$$\mu = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}.$$

Differentiating these expressions, we get

$$\begin{aligned} dM &= dM_0 + (t - T) d\mu, \\ \frac{da}{a} &= -\frac{2}{3} \cdot \frac{d\mu}{\mu}; \end{aligned}$$

and substituting these values in the expressions for dr and dv , we have, finally,

$$\begin{aligned} dr &= a \tan \varphi \sin v dM_0 + \left(a \tan \varphi \sin v (t - T) - \frac{2r}{3\mu} \right) d\mu \\ &\quad - a \cos \varphi \cos v d\varphi, \end{aligned} \quad (15)$$

$$dv = \frac{a^2 \cos \varphi}{r^2} dM_0 + \frac{a^2 \cos \varphi}{r^2} (t - T) d\mu + \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) \sin v d\varphi.$$

From these equations for dr and dv we obtain the following values of the partial differential coefficients:—

$$\begin{aligned} \frac{dr}{d\varphi} &= -a \cos \varphi \cos v, & \frac{dv}{d\varphi} &= \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) \sin v, \\ \frac{dr}{dM_0} &= a \tan \varphi \sin v, & \frac{dv}{dM_0} &= \frac{a^2 \cos \varphi}{r^2}, \\ \frac{dr}{d\mu} &= a \tan \varphi \sin v (t - T) - \frac{2r}{3\mu} 206264.8, & \frac{dv}{d\mu} &= \frac{a^2 \cos \varphi}{r^2} (t - T). \end{aligned} \quad (16)$$

It will be observed that in the last term of the expression for $\frac{dr}{d\mu}$ we have supposed μ to be expressed in seconds of arc, and hence the factor 206264.8 is introduced in order to render the equation homogeneous.

47. The formulæ already derived are sufficient to find the variations of the right ascension and declination corresponding to the variations of the elements in the case of the elliptic orbit of a planet; but in the case of ellipses of great eccentricity, and also in the cases of parabolic and hyperbolic motion, these formulæ for the differential coefficients require some modification, which we now proceed to develop.

First, then, in the case of parabolic motion, $\sin \varphi = 1$, and instead of M_0 and μ we shall introduce the elements T and q , the differential coefficients relating to π , Ω , and i remaining unchanged from their form as already derived.

If we differentiate the equation

$$\frac{k(t-T)}{\sqrt{2}} = q^{\frac{3}{2}} (\tan \frac{1}{2}v + \frac{1}{3} \tan^3 \frac{1}{2}v),$$

regarding T , q , and v as variable, we shall have

$$-\frac{k dT}{\sqrt{2}} = \frac{3}{2} \frac{k(t-T)}{q\sqrt{2}} dq + \frac{1}{2} q^{\frac{3}{2}} \sec^4 \frac{1}{2}v dv,$$

or, since $r^2 = q^2 \sec^4 \frac{1}{2}v$,

$$-\frac{k dT}{\sqrt{2}} = \frac{3}{2} \frac{k(t-T)}{q\sqrt{2}} dq + \frac{1}{2} \frac{r^2}{q^{\frac{3}{2}}} dv.$$

Multiplying through by $\frac{2q^{\frac{1}{2}}}{r^2}$, and reducing, we get

$$dv = -\frac{k\sqrt{2q}}{r^2} dT - \frac{3k(t-T)}{r^2\sqrt{2q}} dq. \quad (17)$$

Instead of q , we may use $\log q$, and the equation will, therefore, become

$$dv = -\frac{k\sqrt{2q}}{r^2} dT - \frac{3k(t-T)\sqrt{2q}}{2r^2\lambda_0} d \log q, \quad (18)$$

in which λ_0 is the modulus of the system of logarithms.

If we take the logarithms of both members of the equation

$$r = \frac{q}{\cos^2 \frac{1}{2}v},$$

and differentiate, we find

$$dr = \frac{r}{q} dq + r \tan \frac{1}{2}v dv.$$

Introducing into this equation the value of dv from (17), we get

$$dr = r \left(\frac{1}{q} - \frac{3k(t-T) \tan \frac{1}{2}v}{r^2 \sqrt{2q}} \right) dq - \frac{k \sqrt{2q} \tan \frac{1}{2}v}{r} dT. \quad (19)$$

Now, since $\frac{k(t-T)}{\sqrt{2q}} = q(\tan \frac{1}{2}v + \frac{1}{3}\tan^3 \frac{1}{2}v)$, and $q = r \cos^2 \frac{1}{2}v$, we have

$$\begin{aligned} \frac{1}{q} - \frac{3k(t-T) \tan \frac{1}{2}v}{r^2 \sqrt{2q}} &= \frac{1}{r} (1 + \tan^2 \frac{1}{2}v - 3 \sin^2 \frac{1}{2}v - \sin^2 \frac{1}{2}v \tan^2 \frac{1}{2}v) \\ &= \frac{\cos v}{r}. \end{aligned}$$

We also have

$$\frac{k \sqrt{2q}}{r} \tan \frac{1}{2}v = \frac{k \sqrt{2q} \cos^2 \frac{1}{2}v \tan \frac{1}{2}v}{q} = \frac{k \sin v}{\sqrt{2q}}.$$

Therefore, equation (19) reduces to

$$dr = \cos v dq - \frac{k \sin v}{\sqrt{2q}} dT. \quad (20)$$

If we introduce $d \log q$ instead of dq , this equation becomes

$$dr = \frac{q \cos v}{\lambda_0} d \log q - \frac{k \sin v}{\sqrt{2q}} dT. \quad (21)$$

From the equations (17), (18), (20), and (21), we derive

$$\begin{aligned} \frac{dr}{dT} &= -\frac{k \sin v}{\sqrt{2q}}, & \frac{dv}{dT} &= -\frac{k \sqrt{2q}}{r^2}, \\ \frac{dr}{dq} &= \cos v, & \frac{dv}{dq} &= -\frac{3k(t-T)}{r^2 \sqrt{2q}}, \\ \frac{dr}{d \log q} &= \frac{q \cos v}{\lambda_0}, & \frac{dv}{d \log q} &= -\frac{3k(t-T) \sqrt{2q}}{2\lambda_0 r^2}, \end{aligned} \quad (22)$$

and then we have, for the differential coefficients of x with respect to T and q or $\log q$,

$$\begin{aligned}\frac{dx}{dT} &= \frac{dx}{dr} \cdot \frac{dr}{dT} + \frac{dx}{dv} \cdot \frac{dv}{dT} & \frac{dx}{dq} &= \frac{dx}{dr} \cdot \frac{dr}{dq} + \frac{dx}{dv} \cdot \frac{dv}{dq}, \\ \frac{dx}{d \log q} &= \frac{dx}{dr} \cdot \frac{dr}{d \log q} + \frac{dx}{dv} \cdot \frac{dv}{d \log q},\end{aligned}$$

and similarly for the differential coefficients of y and z with respect to these elements. The expressions for the partial differential coefficients of x , y , and z , respectively, with respect to r and v are the same as already found in the case of elliptic motion. We shall thus obtain the equations which express the relation between the variations of the geocentric places of a comet and the variation of the parabolic elements of its orbit, and which may be employed either to correct the approximate elements by means of equations of condition furnished by comparison of the computed place with the observed place, or to determine the change in the geocentric right ascension and declination corresponding to given increments assigned to the elements.

48. We may also, in the case of an elliptic orbit, introduce T , q , and e instead of the elements φ , M_0 , and μ . If we differentiate the expression

$$q = a(1 - e),$$

we shall have

$$da = \frac{a}{q} dq + \frac{a^2}{q} de.$$

We have, also,

$$M = k\sqrt{1+m} a^{-\frac{3}{2}} (t - T),$$

in which T is the time of perihelion passage, and

$$dM = -k\sqrt{1+m} a^{-\frac{3}{2}} dT - \frac{3}{2} k\sqrt{1+m} a^{-\frac{5}{2}} (t - T) da.$$

Hence we derive

$$\begin{aligned}dM &= -k\sqrt{1+m} a^{-\frac{3}{2}} dT - \frac{3}{2} \frac{k\sqrt{1+m} a^{-\frac{3}{2}}}{q} (t - T) dq \\ &\quad - \frac{3}{2} \frac{k\sqrt{1+m} a^{-\frac{1}{2}}}{q} (t - T) de.\end{aligned}$$

Substituting this value of dM in equation (12), replacing $\sin \varphi$ by e , and reducing, we get

$$\begin{aligned}dv &= -\frac{k\sqrt{p(1+m)}}{r^2} dT - \frac{3}{2} \frac{k\sqrt{p(1+m)}}{qr^2} (t - T) dq \\ &\quad - \left(p \frac{\frac{3}{2} k\sqrt{p(1+m)}}{qr^2} (t - T) - \left(\frac{p}{r} + 1 \right) \sin v \right) \frac{1}{1 - e^2} de. \quad (23)\end{aligned}$$

In a similar manner, by substituting the values of da and dM in equation (14), and reducing, we find

$$\begin{aligned} dr = & -\frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v \, dT \\ & + \left(\frac{r}{q} - \frac{3}{2} \frac{k\sqrt{1+m}(t-T)}{\sqrt{2}q^{\frac{3}{2}}} \sqrt{\frac{2}{1+e}} e \sin v \right) dq \\ & + \left(p \left(\frac{r}{q} - \cos v \right) - \frac{3}{2} k\sqrt{p(1+m)}(t-T) \frac{e \sin v}{q} \right) \frac{1}{1-e^2} de. \quad (24) \end{aligned}$$

These equations, (23) and (24), will furnish the expressions for the partial differential coefficients $\frac{dv}{dT}$, $\frac{dv}{dq}$, $\frac{dv}{de}$, $\frac{dr}{dT}$, $\frac{dr}{dq}$, and $\frac{dr}{de}$, which are required in finding the differential coefficients of the heliocentric co-ordinates with respect to the elements T , q , and e , these quantities being substituted for M_0 , μ , and φ , respectively, in the equations (11).

49. When the orbit is a hyperbola, we introduce, in place of M_0 , μ , and φ , the elements T , q , and ψ .

If we differentiate the equation

$$N_0 = e \tan F - \log_e \tan (45^\circ + \tfrac{1}{2}F),$$

we shall have

$$dN_0 = \left(\frac{e}{\cos F} - 1 \right) \frac{dF}{\cos F} + \tan F \, d\epsilon,$$

which is easily transformed into

$$dN_0 = \frac{r}{a} \cdot \frac{dF}{\cos F} + \tan F \frac{\tan \psi}{\cos \psi} d\psi,$$

or

$$\frac{dF}{\sin F} = \frac{a}{r \tan F} dN_0 - \frac{a}{r} \cdot \frac{\tan \psi}{\cos \psi} d\psi.$$

Let us now take the logarithms of both members of the equation

$$\tan \tfrac{1}{2}F = \tan \tfrac{1}{2}v \tan \tfrac{1}{2}\psi,$$

and differentiate, and we shall have

$$dv = \sin v \frac{dF}{\sin F} - \frac{\sin v}{\sin \psi} d\psi.$$

Introducing into this equation the value of $\frac{dF}{\sin F}$ already found, we get

$$dv = \frac{a \sin v}{r \tan F} dN_0 - \left(\frac{a \sin v}{r} \cdot \frac{\tan \psi}{\cos \psi} + \frac{\sin v}{\sin \psi} \right) d\psi.$$

But, since $r \sin v = a \tan \psi \tan F$, and $p = a \tan^2 \psi$, this reduces to

$$dv = \frac{a^{\frac{3}{2}}}{r^2} \sqrt{p} dN_0 - \left(\frac{p}{r} + 1 \right) \frac{\sin v}{\sin \psi} d\psi. \quad (25)$$

If we differentiate the equation

$$r = a \left(\frac{e}{\cos F} - 1 \right),$$

we get

$$dr = \frac{r}{a} da + ae \tan^2 F \frac{dF}{\sin F} + \frac{a}{\cos F} \cdot \frac{\tan \psi}{\cos \psi} d\psi.$$

Substituting in this equation the value of $\frac{dF}{\sin F}$, we obtain

$$dr = \frac{r}{a} da + \frac{a^2 e \tan F}{r} dN_0 - \left(\frac{a^2 e \tan^2 F}{r} - \frac{a}{\cos F} \right) \frac{\tan \psi}{\cos \psi} d\psi,$$

which is easily reduced to

$$dr = \frac{r}{a} da + a \frac{\sin v}{\sin \psi} dN_0 + \frac{p}{r} \left(\frac{r}{\cos F} - \frac{ae}{\cos^2 F} + ae \right) \frac{d\psi}{\sin \psi}.$$

But, since

$$\frac{r}{\cos F} = \frac{ae}{\cos^2 F} - \frac{a}{\cos F},$$

this reduces to

$$dr = \frac{r}{a} da + \frac{a \sin v}{\sin \psi} dN_0 + \frac{pa}{r} \left(e - \frac{1}{\cos F} \right) \frac{d\psi}{\sin \psi},$$

or

$$dr = \frac{r}{a} da + a \frac{\sin v}{\sin \psi} dN_0 + p \frac{\cos v}{\sin \psi} d\psi. \quad (26)$$

Now, since $q = a(e - 1)$, we have

$$dq = \frac{q}{a} da + \frac{a \tan \psi}{\cos \psi} d\psi,$$

or

$$da = \frac{a}{q} dq - \frac{a^{\frac{3}{2}} \sqrt{p}}{q \cos \psi} d\psi.$$

We have, also,

$$N_0 = ka^{-\frac{3}{2}}(t - T),$$

and hence

$$dN_0 = -ka^{-\frac{3}{2}}dT - \frac{3}{2}ka^{-\frac{3}{2}}(t - T) da.$$

By substituting the value of da , this becomes

$$dN_0 = -ka^{-\frac{3}{2}}dT - \frac{\frac{3}{2}ka^{-\frac{3}{2}}(t - T)}{q} dq + \frac{\frac{3}{2}k(t - T)\sqrt{p}}{aq \cos \psi} a\psi.$$

Substituting this value of dN_0 in equation (25), and reducing, we obtain

$$dv = -\frac{k\sqrt{p}}{r^2}dT - \frac{\frac{3}{2}k\sqrt{p}(t-T)}{qr^2}dq + \left(\frac{\frac{3}{2}kp^{\frac{3}{2}}(t-T)}{qr^2} - \left(\frac{p}{r} + 1 \right) \sin v \right) \frac{d\psi}{\sin \psi}. \quad (27)$$

In a similar manner, substituting in equation (26) the values of da and dN_0 , and reducing, we get

$$dr = -\frac{k}{\sqrt{p}} \cdot \frac{\sin v}{\cos \psi} dT + \left(\frac{r}{q} - \frac{\frac{3}{2}k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} \cdot \frac{\sin v}{\cos \frac{1}{2}\psi \sqrt{\cos \psi}} \right) dq + \left(\frac{\frac{3}{2}k\sqrt{p}(t-T)}{q} \cdot \frac{\sin v}{\cos \psi} - \left(\frac{r}{q} - \cos v \right) p \right) \frac{d\psi}{\sin \psi}. \quad (28)$$

The equations (27) and (28) will furnish the expressions for the partial differential coefficients of r and v with respect to the elements T , q , and ψ , required in forming the equations for $\cos \delta \, d\alpha$ and $d\delta$. It will be observed that these equations are analogous to the equations (23) and (24), and that by introducing the relation between e and ψ , and neglecting the mass, they become identical with them. We might, indeed, have derived the equations (27) and (28) directly from (23) and (24) by substituting for e its value in terms of ψ ; but the differential formulæ which have resulted in deriving them directly from the equations for hyperbolic motion, will not be superfluous.

50. It is evident, from an inspection of the terms of equations (23), (24), (27), and (28) which contain de and $d\psi$, that when the value of e is very nearly equal to unity, the coefficients for these differentials become indeterminate. It becomes necessary, therefore, to develop the corresponding expressions for the case in which these equations are insufficient. For this purpose, let us resume the equation

$$\frac{k(t-T)(1+e)^{\frac{1}{2}}}{2q^{\frac{3}{2}}} = u + \frac{1}{3}u^3 - 2i(\frac{1}{3}u^3 + \frac{1}{5}u^5) + 3i^2(\frac{1}{5}u^5 + \frac{1}{7}u^7) - \&c.,$$

in which $u = \tan \frac{1}{2}\psi$, and $i = \frac{1-e}{1+e}$. Then, since

$$i = \frac{1}{2}(1-e) + \frac{1}{4}(1-e)^2 + \&c.,$$

$$\sqrt{\frac{2}{1+e}} = \sqrt{\frac{1}{1-\frac{1}{2}(1-e)}} = 1 + \frac{1}{4}(1-e) + \frac{3}{32}(1-e)^2 + \&c.,$$

we shall have

$$\frac{k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} = u + \frac{1}{3}u^3 + (\frac{1}{4}u - \frac{1}{4}u^3 - \frac{1}{5}u^5)(1-e) \\ + (\frac{3}{2}u - \frac{7}{2}u^3 + \frac{3}{2}u^5)(1-e)^2 + \&c. \quad (29)$$

If it is required to find the expression for $\frac{dv}{de}$ in the case of the variation of the elements of parabolic motion, or when $1-e$ is very small, we may regard the coefficient of $1-e$ as constant, and neglect terms multiplied by the square and higher powers of $1-e$. By differentiating the equation (29) according to these conditions, and regarding u and e as variable, we get

$$0 = (1+u^2) du - (\frac{1}{4}u - \frac{1}{4}u^3 - \frac{1}{5}u^5) de;$$

and, since $du = \frac{1}{2}(1+u^2) dv$, this gives

$$\frac{dv}{de} = \frac{\frac{1}{2}u - \frac{1}{2}u^3 - \frac{2}{5}u^5}{(1+u^2)^2}. \quad (30)$$

The values of the second member, corresponding to different values of v , may be tabulated with the argument v ; but a table of this kind is by no means indispensable, since the expression for $\frac{dv}{de}$ may be changed to another form which furnishes a direct solution with the same facility. Thus, by division, we have

$$\frac{dv}{de} = -\frac{2}{5}u + \frac{9}{10} \frac{u + \frac{1}{3}u^3}{(1+u^2)^2}$$

and since, in the case of parabolic motion,

$$\frac{k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} = u + \frac{1}{3}u^3, \quad r^2 = q^2(1+u^2)^3,$$

this becomes

$$\frac{dv}{de} = \frac{9}{20} \frac{k(t-T)}{r^2} \sqrt{2}q - \frac{2}{5} \tan \frac{1}{2}v. \quad (31)$$

If we differentiate the equation

$$r = \frac{q(1+e)}{1+e \cos v},$$

regarding r , v , and e as variables, we shall have

$$\frac{dr}{de} = \frac{2r^2 \sin^2 \frac{1}{2}v}{q(1+e)^2} + \frac{r^2 e \sin v}{q(1+e)} \cdot \frac{dv}{de}. \quad (32)$$

In the case of parabolic motion, $e = 1$, and this equation is easily transformed into

$$\frac{dr}{de} = \frac{1}{2}r \tan \frac{1}{2}v \left(\tan \frac{1}{2}v + 2 \frac{dv}{de} \right). \quad (33)$$

Substituting for $\frac{dv}{de}$ its value from (31), and reducing, we get

$$\frac{dr}{de} = \frac{9}{2n} \frac{k(t-T)}{\sqrt{2q}} \sin v + \frac{1}{10}r \tan^2 \frac{1}{2}v. \quad (34)$$

The equations (31) and (34) furnish the values of $\frac{dv}{de}$ and $\frac{dr}{de}$ to be used in forming the expressions for the variation of the place of the body when the parabolic eccentricity is changed to the value $1 + de$. When the eccentricity to which the increment is assigned differs but little from unity, we may compute the value of $\frac{dv}{de}$ directly from equation (30). A still closer approximation would be obtained by using an additional term of (29) in finding the expression for $\frac{dv}{de}$; but a more convenient formula may be derived, of which the numerical application is facilitated by the use of Table IX. Thus, if we differentiate the equation

$$v = V + A(100i) + B(100i)^2 + C(100i)^3,$$

regarding the coefficients A , B , and C as constant, and introducing the value of i in terms of e , we have

$$\frac{dv}{de} = \frac{dV}{de} - \frac{200A}{s(1+e)^2} - \frac{400B}{s(1+e)^2}(100i) - \frac{600C}{s(1+e)^2}(100i)^2,$$

in which $s = 206264.8$, the values of A , B , and C , as derived from the table, being expressed in seconds. To find $\frac{dV}{de}$, we have

$$\frac{k(t-T)\sqrt{1+e}}{2q^{\frac{3}{2}}} = \tan \frac{1}{2}V + \frac{1}{3} \tan^3 \frac{1}{2}V,$$

which gives, by differentiation,

$$\frac{k(t-T)}{2q^{\frac{3}{2}}} \cdot \frac{de}{\sqrt{1+e}} = \frac{dV}{\cos^4 \frac{1}{2}V};$$

and if we introduce the expression for the value of M used as the argument in finding V by means of Table VI., the result is

$$\frac{dV}{de} = \frac{M \cos^4 \frac{1}{2} V}{75(1+e)}.$$

Hence we have

$$\frac{dv}{de} = \frac{M \cos^4 \frac{1}{2} V}{75(1+e)} - \frac{200A}{s(1+e)^2} - \frac{400B}{s(1+e)^3} (100i) - \frac{600C}{s(1+e)^4} (100i)^2, \quad (35)$$

by means of which the value of $\frac{dv}{de}$ is readily found.

When the eccentricity differs so much from that of the parabola that the terms of the last equation are not sufficiently convergent, the expression for $\frac{dv}{de}$, which will furnish the required accuracy, may be derived from the equations (75)₁ and (76)₁. If we differentiate the first of these equations with respect to e , since B may evidently be regarded as constant, we get

$$\frac{dw}{de} = \frac{9}{10} \frac{k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} \cdot \frac{\cos^4 \frac{1}{2} w}{B \sqrt{\frac{1}{10}(1+9e)}}. \quad (36)$$

If we take the logarithms of both members of equation (76)₁, and differentiate, we get

$$\frac{dv}{\sin v} = \frac{dC}{C} + \frac{dw}{\sin w} - \frac{4de}{(1+e)(1+9e)}. \quad (37)$$

To find the differential coefficient of C with respect to e , it will be sufficient to take

$$\frac{1}{C^2} = 1 - \frac{4}{5}A,$$

which gives

$$\frac{dC}{C} = \frac{2}{5} C^2 dA.$$

The equation

$$A = \frac{5(1-e)}{(1+9e)} \tan^2 \frac{1}{2} w$$

gives

$$dA = -\frac{50}{(1+9e)^2} \tan^2 \frac{1}{2} w de + \frac{A dw}{\tan \frac{1}{2} w \cos^2 \frac{1}{2} w};$$

and hence we obtain

$$\frac{dC}{C} = -\frac{20C^2}{(1+9e)^2} \tan^2 \frac{1}{2} w de + \frac{4}{5} \frac{A C^2}{\sin w} dw.$$

Substituting this value in equation (37), we get

$$\frac{dv}{de} = -\frac{20C^2}{(1+9e)^2} \sin v \tan^2 \frac{1}{2} w + \frac{C^2 \sin v}{\sin w} \cdot \frac{dw}{de} - \frac{4 \sin v}{(1+e)(1+9e)};$$

and substituting, finally, the value of $\frac{dw}{de}$, we obtain

$$\frac{dv}{de} = \frac{9}{2^0} \cdot \frac{k(t-T)}{\sqrt{2} q^{\frac{3}{2}}} \cdot \frac{C^2 \sin v}{B \sqrt{\frac{1}{10}(1+9e)}} \cdot \frac{\cos^2 \frac{1}{2} w}{\tan \frac{1}{2} w} - \frac{20 C^2}{(1+9e)^2} \sin v \tan^2 \frac{1}{2} w - \frac{4 \sin v}{(1+e)(1+9e)},$$

which, by means of (76)₁, reduces to

$$\frac{dv}{de} = \frac{9}{2^0} \cdot \frac{k(t-T)}{\sqrt{2} q^{\frac{3}{2}}} \cdot \frac{C^2 \sin v}{B \sqrt{\frac{1}{10}(1+9e)}} \cdot \frac{\cos^2 \frac{1}{2} w}{\tan \frac{1}{2} w} - \frac{8 \tan \frac{1}{2} v}{(1+e)(1+9e)}. \quad (38)$$

If we introduce the quantity M which is used as the argument in finding w by means of Table VI., this equation becomes

$$\frac{dv}{de} = \frac{9}{2(1+9e)} \cdot \frac{M \cos^2 \frac{1}{2} w}{75 \tan \frac{1}{2} w} C^2 \sin v - \frac{8 \tan \frac{1}{2} v}{(1+e)(1+9e)}. \quad (39)$$

This equation remains unchanged in the case of hyperbolic motion, the value of C being taken from the column of the table which corresponds to this case; and it will furnish the correct value of $\frac{dv}{de}$ in all cases in which the last term of equation (23) is not conveniently applicable. The value of $\frac{dr}{de}$ is then given by the equation (32).

When the eccentricity differs very little from unity, we may put $B = 1$, and

$$\begin{aligned} \tan \frac{1}{2} w &= \tan \frac{1}{2} v \sqrt{\frac{1}{10}(1+9e)}, \\ \cos^2 \frac{1}{2} w &= C^2 \cos^2 \frac{1}{2} v. \end{aligned}$$

Then we shall have

$$\frac{M \cos^2 \frac{1}{2} w}{75 \tan \frac{1}{2} w} C^2 \sin v = \frac{2k(t-T)}{\sqrt{2} q^{\frac{3}{2}}} \cos^4 \frac{1}{2} w.$$

The equation

$$\frac{q}{r} = (1 + AC^2) \cos^2 \frac{1}{2} v = (1 + \frac{1}{5}A) \cos^2 \frac{1}{2} w,$$

gives

$$\frac{q^2}{r^2} = (1 + \frac{2}{5}A) \cos^4 \frac{1}{2} w = C \cos^4 \frac{1}{2} w.$$

Hence we derive

$$\frac{M \cos^2 \frac{1}{2} w}{75 \tan \frac{1}{2} w} C^2 \sin v = \frac{k(t-T) \sqrt{p}}{r^2} \cdot \sqrt{\frac{2}{C^2(1+e)}}.$$

If we substitute this value in equation (39), and put $C^2(1+e)=2$, we get

$$\frac{dv}{de} = \frac{9}{2(1+9e)} \cdot \frac{k\sqrt{p}}{r^2} (t-T) - \frac{8 \tan \frac{1}{2}v}{(1+e)(1+9e)}, \quad (40)$$

and when $e=1$, this becomes identical with equation (31).

51. EXAMPLES.—We will now illustrate, by numerical examples, the formulæ for the calculation of the variations of the geocentric right ascension and declination arising from small increments assigned to the elements. Let it be required to find for the date 1865 February 24.5 mean time at Washington, the differential coefficients of the right ascension and declination of the planet *Eurynome* ⑨ with respect to the elements of its orbit, using the data and results given in Art. 41. Thus we have

$$\begin{aligned} \alpha &= 181^\circ 8' 29''.29, & \delta &= -4^\circ 42' 21''.56, & \log A &= 0.2450054, \\ \log r &= 0.428285, & v &= 129^\circ 3' 50''.5, & u &= 326^\circ 41' 40''.1, \\ A &= 296^\circ 39' 5''.0, & B &= 205^\circ 55' 27''.1, & C &= 212^\circ 32' 17''.7, \\ \log \sin \alpha &= 9.999716, & \log \sin b &= 9.974825, & \log \sin c &= 9.522219, \\ \log x &= 0.425066_n, & \log y &= 9.511920, & \log z &= 8.077315, \\ \epsilon &= 23^\circ 27' 24''.0, & t-T &= 420.714018. \end{aligned}$$

First, by means of the equations (4), we compute the following values:—

$$\begin{aligned} \log \cos \delta \frac{d\alpha}{dx} &= 8.054308, & \log \frac{d\delta}{dx} &= 8.668959_n, \\ \log \cos \delta \frac{d\alpha}{dy} &= 9.754919_n, & \log \frac{d\delta}{dy} &= 6.968348_n, \\ & & \log \frac{d\delta}{dz} &= 9.753529. \end{aligned}$$

Then we find the differential coefficients of the heliocentric co-ordinates, with respect to π , Ω , i , v , and r , from the formulæ (7), which give

$$\begin{aligned} \log \frac{dx}{d\pi} &= \log \frac{dx}{dv} = 9.491991_n, & \log \frac{dy}{d\pi} &= \log \frac{dy}{dv} = 0.399496_n, \\ & \log \frac{dz}{d\pi} &= \log \frac{dz}{dv} &= 9.950466_n, \\ \log \frac{dx}{d\Omega} &= 7.876553, & \log \frac{dy}{d\Omega} &= 8.830941, & \log \frac{dz}{d\Omega} &= 9.222898_n, \\ \log \frac{dx}{di} &= 8.726364, & \log \frac{dy}{di} &= 9.687577, & \log \frac{dz}{di} &= 0.142443_n, \\ \log \frac{dx}{dr} &= 9.996780_n, & \log \frac{dy}{dr} &= 9.083635, & \log \frac{dz}{dr} &= 7.649030. \end{aligned}$$

In computing the values of $\frac{dx}{di}$, $\frac{dy}{di}$, and $\frac{dz}{di}$, those of $\cos a$, $\cos b$, and $\cos c$ may generally be obtained with sufficient accuracy from $\sin a$, $\sin b$, and $\sin c$. Their algebraic signs, however, must be strictly attended to. The quantities $\sin a$, $\sin b$, and $\sin c$ are always positive; and the algebraic signs of $\cos a$, $\cos b$, and $\cos c$ are indicated at once by the equations (101)₁, from which, also, their numerical values may be derived. In the case of the example proposed, it will be observed that $\cos a$ and $\cos b$ are negative, and that $\cos c$ is positive.

To find the values of $\cos \delta \frac{d\alpha}{d\pi}$ and $\frac{d\delta}{d\pi}$, we have, according to equation (2),

$$\begin{aligned}\cos \delta \frac{d\alpha}{d\pi} &= \cos \delta \frac{d\alpha}{dx} \cdot \frac{dx}{d\pi} + \cos \delta \frac{d\alpha}{dy} \cdot \frac{dy}{d\pi}, \\ \frac{d\delta}{d\pi} &= \frac{d\delta}{dx} \cdot \frac{dx}{d\pi} + \frac{d\delta}{dy} \cdot \frac{dy}{d\pi} + \frac{d\delta}{dz} \cdot \frac{dz}{d\pi},\end{aligned}\quad (41)$$

which give

$$\cos \delta \frac{d\alpha}{d\pi} = \cos \delta \frac{d\alpha}{dv} = +1.42345, \quad \frac{d\delta}{d\pi} = \frac{d\delta}{dv} = -0.48900.$$

In the case of Ω , i , and r , we write these quantities successively in place of π in the equations (41), and hence we derive

$$\begin{aligned}\cos \delta \frac{d\alpha}{d\Omega} &= -0.03845, & \frac{d\delta}{d\Omega} &= -0.09533, \\ \cos \delta \frac{d\alpha}{di} &= -0.27641, & \frac{d\delta}{di} &= -0.78993, \\ \cos \delta \frac{d\alpha}{dr} &= -0.08020, & \frac{d\delta}{dr} &= +0.04873\end{aligned}$$

Next, from (16), we compute the following values:—

$$\begin{aligned}\log \frac{dr}{d\varphi} &= 0.179155, & \log \frac{dr}{dM_0} &= 9.577453, & \log \frac{dr}{d\mu} &= 2.376581, \\ \log \frac{dv}{d\varphi} &= 0.171999, & \log \frac{dv}{dM_0} &= 9.911247, & \log \frac{dv}{d\mu} &= 2.535234.\end{aligned}$$

We may now find $\frac{dx}{d\varphi}$, $\frac{dx}{dM_0}$, &c. by means of the equations (11), and thence the values of $\cos \delta \frac{d\alpha}{d\varphi}$, $\frac{d\delta}{d\varphi}$, &c.; but it is most convenient to derive these values directly from $\cos \delta \frac{d\alpha}{dr}$, $\cos \delta \frac{d\alpha}{dv}$, $\frac{d\delta}{dr}$, and $\frac{d\delta}{dv}$, in connection with the numerical values last found, according to the

equations which result from the analytical substitution of the expressions for $\frac{dx}{d\varphi}$, $\frac{dy}{d\varphi}$, $\frac{dz}{d\varphi}$, &c., in equation (2), writing successively φ , M_0 , and μ in place of π . Thus, we have

$$\begin{aligned}\cos \delta \frac{d\alpha}{d\varphi} &= \cos \delta \frac{d\alpha}{dr} \cdot \frac{dr}{d\varphi} + \cos \delta \frac{d\alpha}{dv} \cdot \frac{dv}{d\varphi}, \\ \frac{d\delta}{d\varphi} &= \frac{d\delta}{dr} \cdot \frac{dr}{d\varphi} + \frac{d\delta}{dv} \cdot \frac{dv}{d\varphi},\end{aligned}\quad (42)$$

and similarly for M_0 and μ , which give

$$\begin{aligned}\cos \delta \frac{d\alpha}{d\varphi} &= +1.99400, & \frac{d\delta}{d\varphi} &= -0.65307, \\ \cos \delta \frac{d\alpha}{dM_0} &= +1.13004, & \frac{d\delta}{dM_0} &= -0.38023, \\ \cos \delta \frac{d\alpha}{d\mu} &= +507.264, & \frac{d\delta}{d\mu} &= -179.315.\end{aligned}$$

Therefore, according to (1), we shall have

$$\begin{aligned}\cos \delta \Delta \alpha &= +1.42345 \Delta \pi - 0.03845 \Delta \Omega - 0.27641 \Delta i + 1.99400 \Delta \varphi \\ &\quad + 1.13004 \Delta M_0 + 507.264 \Delta \mu, \\ \Delta \delta &= -0.48900 \Delta \pi - 0.09533 \Delta \Omega - 0.78993 \Delta i - 0.65307 \Delta \varphi \\ &\quad - 0.38023 \Delta M_0 - 179.315 \Delta \mu.\end{aligned}$$

To prove the calculation of the coefficients in these equations, we assign to the elements the increments

$$\begin{aligned}\Delta M_0 &= +10'', & \Delta \pi &= -20'', & \Delta \Omega &= -10'', & \Delta i &= +10'', \\ \Delta \varphi &= +10'', & \Delta \mu &= +0''.01,\end{aligned}$$

so that they become

Epoch = 1864 Jan. 1.0 Greenwich mean time.

$$M_0 = 1^\circ 29' 50''.21$$

$$\pi = 44 \quad 20 \quad 13.09$$

$$\Omega = 206 \quad 42 \quad 30.13$$

$$i = 4 \quad 37 \quad 0.51$$

$$\varphi = 11 \quad 16 \quad 1.02$$

$$\log a = 0.3881288$$

$$\mu = 928''.56745$$

} Mean Equinox 1864.0

With these elements we compute the geocentric place for 1865 February 24.5 mean time at Washington; and the result is

$$\alpha = 181^\circ 8' 34''.81, \quad \delta = -4^\circ 42' 30''.58, \quad \log A = 0.2450284,$$

which are referred to the mean equinox and equator of 1865.0. The difference between these values of α and δ and those already given, as derived from the unchanged elements, gives

$$\Delta\alpha = + 5''.52, \quad \cos \delta \Delta\alpha = + 5''.50, \quad \Delta\delta = - 9''.02,$$

and the direct substitution of the assumed values of $\Delta\pi$, $\Delta\Omega$, Δi , &c. in the equations for $\cos \delta \Delta\alpha$ and $\Delta\delta$, gives

$$\cos \delta \Delta\alpha = + 5''.46, \quad \Delta\delta = - 9''.29.$$

The agreement of these results is sufficiently close to show that the computation of the differential coefficients has been correctly performed, the difference being due chiefly to terms of the second order.

When the differential coefficients are required for several dates, if we compute their values for successive dates at equal intervals, the use of differences will serve to check the accuracy of the calculation; but, to provide against the possibility of a systematic error, it may be advisable to calculate at least one place directly from the changed elements. Throughout the calculation of the various differential coefficients, great care must be taken in regard to the algebraic signs involved in the successive numerical substitutions. In the example given, we have employed logarithms of six decimal places; but it would have been sufficient if logarithms of five decimals had been used; and such is generally the case.

It will be observed that the calculation of the coefficients of $\Delta\pi$, $\Delta\Omega$, and Δi is independent of the form of the orbit, depending simply on the position of the plane of the orbit and on the position of the orbit in this plane. Hence, in the case of parabolic and hyperbolic orbits, the only deviation from the process already illustrated is in the computation of the coefficients of the variations of the elements which determine the magnitude and form of the orbit and the position of the body in its orbit at a given epoch. In all cases, the values of $\cos \delta \frac{d\alpha}{dv}$, $\cos \delta \frac{d\alpha}{dr} \cdot \frac{dr}{dv}$, and $\frac{d\delta}{dr}$ are determined as already exemplified. If we introduce the elements T , q , and e , we shall have

$$\begin{aligned} \cos \delta \frac{d\alpha}{dT} &= \cos \delta \frac{d\alpha}{dr} \cdot \frac{dr}{dT} + \cos \delta \frac{d\alpha}{dv} \cdot \frac{dv}{dT}, \\ \frac{d\delta}{dT} &= \frac{d\delta}{dr} \cdot \frac{dr}{dT} + \frac{d\delta}{dv} \cdot \frac{dv}{dT}, \end{aligned} \quad (43)$$

and similarly for the differential coefficients with respect to q and e .

The mode of calculating the values of $\frac{dr}{dT}$, $\frac{dv}{dT}$, $\frac{dr}{dq}$, $\frac{dv}{dq}$, $\frac{dr}{de}$, and $\frac{dv}{de}$ depends on the nature of the orbit.

In the case of passing from one system of parabolic elements to another system of parabolic elements, the coefficients of Δe vanish. To illustrate the calculation of $\frac{dr}{dT}$, $\frac{dv}{dT}$, &c. in the case of parabolic motion, let us resume the values $t - T = 75.364$ days, and $\log q = 9.9650486$, from which we have found

$$\log r = 0.1961120, \quad v = 79^\circ 55' 57''.26.$$

Then, by means of the equations (22), we find

$$\begin{aligned} \log \frac{dr}{dT} &= 8.095802_n, & \log \frac{dr}{dq} &= 9.242547, \\ \log \frac{dv}{dT} &= 7.976397_n, & \log \frac{dv}{dq} &= 0.064602_n. \end{aligned}$$

If, instead of dq , we introduce $d \log q$, we shall have

$$\log \frac{dr}{d \log q} = 9.569812, \quad \log \frac{dv}{d \log q} = 0.391867_n.$$

From these, by means of (43), we obtain the differential coefficients of α and δ with respect to T and q or $\log q$. The same values are also used when the variation of the parabolic eccentricity is taken into account. But in this case we compute also $\frac{dv}{de}$ from equation (31) and $\frac{dr}{de}$ from (33) or (34), which give, for $v = 79^\circ 55' 57''.3$,

$$\log \frac{dv}{de} = 8.147367_n, \quad \log \frac{dr}{de} = 9.726869.$$

In the case of very eccentric orbits, the values of $\frac{dr}{dT}$, $\frac{dv}{dT}$, &c. are found from

$$\begin{aligned} \frac{dv}{dT} &= -\frac{k\sqrt{p}}{r^2}, & \frac{dr}{dT} &= -\frac{k}{\sqrt{p}} e \sin v, & (44) \\ \frac{dv}{dq} &= -\frac{3}{2} \frac{k\sqrt{p}}{qr^2} (t - T), & \frac{dr}{dq} &= \frac{r}{q} - \frac{3}{2} \frac{k(t - T)}{q\sqrt{p}} e \sin v \\ & & \frac{dr}{dq} &= \frac{r}{q} + \frac{r^2 e \sin v}{p} \cdot \frac{dv}{dq}, \end{aligned}$$

the mass being neglected.

To illustrate the application of these formulæ, let us resume the values, $t - T = 68.25$ days, $e = 0.9675212$, and $\log q = 9.7668134$, from which we have found (Art. 41)

$$v = 102^\circ 20' 52''.20, \quad \log r = 0.1614052.$$

Hence we derive

$$\log p = 0.0607328,$$

and

$$\log \frac{dv}{dT} = 7.943137, \quad \log \frac{dr}{dT} = 8.180711,$$

$$\log \frac{dv}{dq} = 0.186517, \quad \log \frac{dr}{dq} = 0.186517.$$

If we wish to obtain the differential coefficients of v and r with respect to $\log q$ instead of q , we have

$$\frac{dv}{d \log q} = \frac{q}{\lambda_0} \cdot \frac{dv}{dq}, \quad \frac{dr}{d \log q} = \frac{q}{\lambda_0} \cdot \frac{dr}{dq},$$

in which λ_0 is the modulus of the system of logarithms.

Then we compute the value of $\frac{dv}{de}$ by means of the equation (30). (35), (39), or (40). The correct value as derived from (39) is

$$\frac{dv}{de} = -0.24289.$$

The values derived from (35), omitting the last term, from (40) and from (30), are, respectively, -0.24440 , -0.24291 , and -0.23531 . The close agreement of the value derived from (40) with the correct value is accidental, and arises from the particular value of v , which is here such as to make the assumptions, according to which equation (40) is derived from (39), almost exact.

Finally, the value of $\frac{dr}{de}$ may be found by means of (32), which gives

$$\frac{dr}{de} = +0.70855.$$

When, in addition to the differential coefficients which depend on the elements T , q , and e , those which depend on the position of the orbit in space have been found, the expressions for the variation of the geocentric right ascension and declination become

$$\begin{aligned}\cos \delta \Delta \alpha &= \cos \delta \frac{da}{d\pi} \Delta \pi + \cos \delta \frac{da}{d\Omega} \Delta \Omega + \cos \delta \frac{da}{di} \Delta i + \cos \delta \frac{da}{dT} \Delta T \\ &\quad + \cos \delta \frac{da}{dq} \Delta q + \cos \delta \frac{da}{de} \Delta e, \\ \Delta \delta &= \frac{d\delta}{d\pi} \Delta \pi + \frac{d\delta}{d\Omega} \Delta \Omega + \frac{d\delta}{di} \Delta i + \frac{d\delta}{dT} \Delta T + \frac{d\delta}{dq} \Delta q + \frac{d\delta}{de} \Delta e.\end{aligned}$$

If we introduce $\log q$ instead of q , the terms containing q become respectively $\cos \delta \frac{da}{d \log q} \Delta \log q$ and $\frac{d\delta}{d \log q} \Delta \log q$. It should be observed that if $\Delta \pi$, $\Delta \Omega$, and Δi are expressed in seconds, in order that these equations may be homogeneous, the terms containing ΔT , Δq , and Δe must be multiplied by 206264.8; but if $\Delta \pi$, $\Delta \Omega$, and Δi are expressed in parts of the radius as unity, the resulting values of $\cos \delta \Delta \alpha$ and $\Delta \delta$ must be multiplied by 206264.8 in order to express them in seconds of arc.

The most general application of the equations for $\cos \delta \Delta \alpha$ and $\Delta \delta$ in terms of the variations of the elements is for the cases in which the values of $\cos \delta \Delta \alpha$ and of $\Delta \delta$ are already known by comparison of the computed place of the body with the observed place, and in which it is required to find the values of $\Delta \pi$, $\Delta \Omega$, Δi , &c., which, being applied to the elements, will make the computed and the observed places agree. When the variations of all the elements of the orbit are taken into account, at least six equations thus derived are necessary, and, if more than six equations are employed, they must first be reduced to six final equations, from which, by elimination, the values of the unknown quantities $\Delta \pi$, $\Delta \Omega$, &c. may be found. In all such cases, the values of $\Delta \alpha$ and $\Delta \delta$, as derived from the comparison of the computed with the observed place, are expressed in seconds of arc; and if the elements involved are expressed in seconds of arc, the coefficients of the several terms of the equations must be abstract numbers. But if some of the elements are not expressed in seconds, as in the case of T , q , and e , the equations formed must be rendered homogeneous. For this purpose we multiply the coefficients of the variations of those elements which are not expressed in seconds of arc by 206264.8. Further, it is generally inconvenient to express the variations ΔT , Δq , and Δe in parts of the units of T , q , and e , respectively; and, to avoid this inconvenience, we may express these variations in terms of certain parts of the actual units. Thus, in the case of T , we may adopt as the unit of ΔT the n th part of a mean solar day, and the coefficients of the terms of the equations for $\cos \delta \Delta \alpha$ and $\Delta \delta$ which involve ΔT

must evidently be divided by n . In the same manner, it appears that if we adopt as the unit of Δq the unit of the m th decimal place of its value expressed in parts of the unit of q , we must divide its coefficient by 10^m , and similarly in the case of Δe , so that the equations become

$$\begin{aligned}\cos \delta \Delta \alpha &= \cos \delta \frac{d\alpha}{d\pi} \Delta \pi + \cos \delta \frac{d\alpha}{d\Omega} \Delta \Omega + \cos \delta \frac{d\alpha}{di} \Delta i + \frac{s}{n} \cos \delta \frac{d\alpha}{dT} \Delta T \\ &\quad + \frac{s}{10^m} \cos \delta \frac{d\alpha}{dq} \Delta q + \frac{s}{10^m} \cos \delta \frac{d\alpha}{de} \Delta e, \\ \Delta \delta &= \frac{d\delta}{d\pi} \Delta \pi + \frac{d\delta}{d\Omega} \Delta \Omega + \frac{d\delta}{di} \Delta i + \frac{s}{n} \frac{d\delta}{dT} \Delta T + \frac{s}{10^m} \frac{d\delta}{dq} \Delta q \\ &\quad + \frac{s}{10^m} \frac{d\delta}{de} \Delta e,\end{aligned}\tag{45}$$

in which $s = 206264.8$. When $\log q$ is introduced in place of q , the coefficients of $\Delta \log q$ are multiplied by the same factor as in the case of Δq , the unit of $\Delta \log q$ being the unit of the m th decimal place of the logarithms. The equations are thus rendered homogeneous, and also convenient for the numerical solution in finding the values of the unknown quantities $\Delta \pi$, $\Delta \Omega$, Δi , ΔT , &c. When ΔT , Δq , and Δe have been found by means of the equations thus formed, the corrections to be applied to the corresponding elements are $\frac{\Delta T}{n}$, $\frac{\Delta q}{10^m}$,

and $\frac{\Delta e}{10^m}$. In the same manner, we may adopt as the unknown quantity, instead of the actual variation of any one of the elements of the orbit, n times that variation, in which case its coefficient in the equations must be divided by n .

The value of $\Delta \alpha$, derived by taking the difference between the computed and the observed place, is affected by the uncertainty necessarily incident to the determination of α by observation. The unavoidable error of observation being supposed the same in the case of α as in the case of δ , when expressed in parts of the same unit, it is evident that an error of a given magnitude will produce a greater apparent error in α than in δ , since in the case of α it is measured on a small circle, of which the radius is $\cos \delta$; and hence, in order that the difference between computation and observation in α and δ may have the same influence in the determination of the corrections to be applied to the elements, we introduce $\cos \delta \Delta \alpha$ instead of $\Delta \alpha$. The same principle is applied in the case of the longitude and of all corresponding spherical co-ordinates.

52. The formulæ already given will determine also the variations of the geocentric longitude and latitude corresponding to small increments assigned to the elements of the orbit of a heavenly body. In this case we put $\varepsilon = 0$, and compute the values of A , B , $\sin a$, and $\sin b$ by means of the equations (94)₁. We have also $C = 0$, $\sin c = \sin i$, and, in place of α and δ , respectively, we write λ and β . But when the elements are referred to the same fundamental plane as the geocentric places of the body, the formulæ which depend on the position of the plane of the orbit may be put in a form which is more convenient for numerical application.

If we differentiate the equations

$$\begin{aligned}x' &= r \cos u \cos \Omega - r \sin u \sin \Omega \cos i, \\y' &= r \cos u \sin \Omega + r \sin u \cos \Omega \cos i, \\z' &= r \sin u \sin i,\end{aligned}$$

we obtain

$$\begin{aligned}dx' &= \frac{x'}{r} dr - r(\sin u \cos \Omega + \cos u \sin \Omega \cos i) du \\&\quad - r(\cos u \sin \Omega + \sin u \cos \Omega \cos i) d\Omega + r \sin u \sin \Omega \sin i di, \\dy' &= \frac{y'}{r} dr - r(\sin u \sin \Omega - \cos u \cos \Omega \cos i) du \\&\quad + r(\cos u \cos \Omega - \sin u \sin \Omega \cos i) d\Omega - r \sin u \cos \Omega \sin i di, \quad (46) \\dz' &= \frac{z'}{r} dr + r \cos u \sin i du + r \sin u \cos i di,\end{aligned}$$

in which x' , y' , z' are the heliocentric co-ordinates of the body in reference to the ecliptic, the positive axis of x being directed to the vernal equinox. Let us now suppose the place of the body to be referred to a system of co-ordinates in which the ecliptic remains as the plane of xy , but in which the positive axis of x is directed to the point whose longitude is Ω ; then we shall have

$$\begin{aligned}dx &= dx' \cos \Omega + dy' \sin \Omega, \\dy &= -dx' \sin \Omega + dy' \cos \Omega, \\dz &= dz',\end{aligned}$$

and the preceding equations give

$$\begin{aligned}dx &= \frac{x}{r} dr - r \sin u du - r \sin u \cos i d\Omega, \\dy &= \frac{y}{r} dr + r \cos u \cos i du + r \cos u d\Omega - r \sin u \sin i di, \quad (47) \\dz &= \frac{z}{r} dr + r \cos u \sin i du + r \sin u \cos i di.\end{aligned}$$

This transformation, it will be observed, is equivalent to diminishing the longitudes in the equations (46) by the angle Ω through which the axis of x has been moved.

Let X, Y, Z , denote the heliocentric co-ordinates of the earth referred to the same system of co-ordinates, and we have

$$\begin{aligned}x + X &= \Delta \cos \beta \cos (\lambda - \Omega), \\y + Y &= \Delta \cos \beta \sin (\lambda - \Omega), \\z + Z &= \Delta \sin \beta,\end{aligned}$$

in which λ is the geocentric longitude and β the geocentric latitude. In differentiating these equations so as to find the relation between the variations of the heliocentric co-ordinates and the geocentric longitude and latitude, we must regard Ω as constant, since it indicates here the position of the axis of x in reference to the vernal equinox, and this position is supposed to be fixed. Therefore, we shall have

$$\begin{aligned}dx &= \cos \beta \cos (\lambda - \Omega) d\Delta - \Delta \sin \beta \cos (\lambda - \Omega) d\beta - \Delta \cos \beta \sin (\lambda - \Omega) d\lambda, \\dy &= \cos \beta \sin (\lambda - \Omega) d\Delta - \Delta \sin \beta \sin (\lambda - \Omega) d\beta + \Delta \cos \beta \cos (\lambda - \Omega) d\lambda, \\dz &= \sin \beta d\Delta + \Delta \cos \beta d\beta,\end{aligned}$$

from which, by elimination, we find

$$\begin{aligned}\cos \beta d\lambda &= -\frac{\sin (\lambda - \Omega)}{\Delta} dx + \frac{\cos (\lambda - \Omega)}{\Delta} dy, \\d\beta &= -\frac{\sin \beta \cos (\lambda - \Omega)}{\Delta} dx - \frac{\sin \beta \sin (\lambda - \Omega)}{\Delta} dy + \frac{\cos \beta}{\Delta} dz.\end{aligned}$$

These equations give

$$\begin{aligned}\cos \beta \frac{d\lambda}{dx} &= -\frac{\sin (\lambda - \Omega)}{\Delta}, & \frac{d\beta}{dx} &= -\frac{\sin \beta \cos (\lambda - \Omega)}{\Delta}, \\ \cos \beta \frac{d\lambda}{dy} &= \frac{\cos (\lambda - \Omega)}{\Delta}, & \frac{d\beta}{dy} &= -\frac{\sin \beta \sin (\lambda - \Omega)}{\Delta}, \\ \cos \beta \frac{d\lambda}{dz} &= 0, & \frac{d\beta}{dz} &= \frac{\cos \beta}{\Delta}.\end{aligned} \quad (48)$$

If we introduce the distance ω between the ascending node and the place of the perihelion as one of the elements of the orbit, we have

$$du = dv + d\omega,$$

and the equations (47) give

$$\begin{aligned}\frac{dx}{dr} &= \frac{x}{r} = \cos u, & \frac{dy}{dr} &= \frac{y}{r} = \sin u \cos i, & \frac{dz}{dr} &= \frac{z}{r} = \sin u \sin i; \\ \frac{dx}{dv} &= \frac{dx}{d\omega} = -r \sin u, & \frac{dy}{dv} &= \frac{dy}{d\omega} = r \cos u \cos i, & \frac{dz}{dv} &= \frac{dz}{d\omega} = r \cos u \sin i,\end{aligned}$$

$$\begin{aligned} \frac{dx}{d\Omega} &= -r \sin u \cos i, & \frac{dy}{d\Omega} &= r \cos u, & \frac{dz}{d\Omega} &= 0; \\ \frac{dx}{di} &= 0, & \frac{dy}{di} &= -r \sin u \sin i, & \frac{dz}{di} &= r \sin u \cos i. \end{aligned} \quad (49)$$

If we introduce π , the longitude of the perihelion, we have

$$du = dv + d\pi - d\Omega,$$

and hence the expressions for the partial differential coefficients of the heliocentric co-ordinates with respect to π and Ω become

$$\begin{aligned} \frac{dx}{d\pi} &= -r \sin u, & \frac{dy}{d\pi} &= r \cos u \cos i, & \frac{dz}{d\pi} &= r \cos u \sin i; \\ \frac{dx}{d\Omega} &= 2r \sin u \sin^2 \frac{1}{2}i, & \frac{dy}{d\Omega} &= 2r \cos u \sin^2 \frac{1}{2}i, & \frac{dz}{d\Omega} &= -r \cos u \sin i. \end{aligned} \quad (50)$$

When the direct inclination exceeds 90° and the motion is regarded as being retrograde, we find, by making the necessary distinctions in regard to the algebraic signs in the general equations,

$$\frac{dx}{di} = 0, \quad \frac{dy}{di} = r \sin u \sin i, \quad \frac{dz}{di} = -r \sin u \cos i; \quad (51)$$

and the expressions for $\frac{dx}{dr}$, $\frac{dx}{dv}$, $\frac{dx}{d\Omega}$, $\frac{dy}{dr}$, &c. are derived directly from (49) by writing $180^\circ - i$ in place of i . If we introduce the longitude of the perihelion, we have, in this case,

$$du = dv - d\pi + d\Omega,$$

and hence

$$\begin{aligned} \frac{dx}{d\pi} &= r \sin u, & \frac{dy}{d\pi} &= r \cos u \cos i, & \frac{dz}{d\pi} &= -r \cos u \sin i; \\ \frac{dx}{d\Omega} &= -2r \sin u \sin^2 \frac{1}{2}i, & \frac{dy}{d\Omega} &= 2r \cos u \sin^2 \frac{1}{2}i, & \frac{dz}{d\Omega} &= r \cos u \sin i. \end{aligned} \quad (52)$$

But, to prevent confusion and the necessity of using so many formulæ, it is best to regard i as admitting any value from 0° to 180° , and to transform the elements which are given with the distinction of retrograde motion into those of the general case by taking $180^\circ - i$ instead of i , and $2\Omega - \pi$ instead of π , the other elements remaining the same in both cases.

53. The equations already derived enable us to form those for the differential coefficients of λ and β with respect to r , v , Ω , i , and ω or π , by writing successively λ and β in place of θ , and Ω , i , &c. in

place of π in equation (2). The expressions for the differential coefficients of r and v , with respect to the elements which determine the form of the orbit and the position of the body in its orbit, being independent of the position of the plane of the orbit, are the same as those already given; and hence, according to (42) and (43), we may derive the values of the partial differential coefficients of λ and β with respect to these elements. The numerical application, however, is facilitated by the introduction of certain auxiliary quantities. Thus, if we substitute the values given by (48) and (49) in the equations

$$\begin{aligned}\cos \beta \frac{d\lambda}{dv} &= \cos \beta \frac{d\lambda}{dx} \cdot \frac{dx}{dv} + \cos \beta \frac{d\lambda}{dy} \cdot \frac{dy}{dv}, \\ \frac{d\beta}{dv} &= \frac{d\beta}{dx} \cdot \frac{dx}{dv} + \frac{d\beta}{dy} \cdot \frac{dy}{dv} + \frac{d\beta}{dz} \cdot \frac{dz}{dv},\end{aligned}$$

and put

$$\begin{aligned}\cos i \cos (\lambda - \Omega) &= A_0 \sin A, \\ \sin (\lambda - \Omega) &= A_0 \cos A, \\ \sin i &= n \sin N, \\ -\sin (\lambda - \Omega) \cos i &= n \cos N,\end{aligned}\tag{53}$$

in which A_0 and n are always positive, they become

$$\begin{aligned}\cos \beta \frac{d\lambda}{dv} &= \cos \beta \frac{d\lambda}{d\omega} = \frac{r}{\Delta} A_0 \sin (A + u), \\ \frac{d\beta}{dv} &= \frac{d\beta}{d\omega} = \frac{r}{\Delta} (\sin \beta \cos (\lambda - \Omega) \sin u + n \cos u \sin (N + \beta)).\end{aligned}$$

Let us also put

$$\begin{aligned}n \sin (N + \beta) &= B_0 \sin B, \\ \sin \beta \cos (\lambda - \Omega) &= B_0 \cos B,\end{aligned}\tag{54}$$

and we have

$$\begin{aligned}\cos \beta \frac{d\lambda}{dv} &= \cos \beta \frac{d\lambda}{d\omega} = \frac{r}{\Delta} A_0 \sin (A + u), \\ \frac{d\beta}{dv} &= \frac{d\beta}{d\omega} = \frac{r}{\Delta} B_0 \sin (B + u).\end{aligned}\tag{55}$$

The expressions for $\cos \beta \frac{d\lambda}{dr}$ and $\frac{d\beta}{dr}$ give, by means of the same auxiliary quantities,

$$\begin{aligned}\cos \beta \frac{d\lambda}{dr} &= -\frac{A_0}{\Delta} \cos (A + u), \\ \frac{d\beta}{dr} &= -\frac{B_0}{\Delta} \cos (B + u).\end{aligned}\tag{56}$$

In the same manner, if we put

$$\begin{aligned}
 \cos (\lambda - \Omega) &= C_0 \sin C, \\
 \cos i \sin (\lambda - \Omega) &= C_0 \cos C; \\
 \cos i &= D_0 \sin D, \\
 \sin (\lambda - \Omega) \sin i &= D_0 \cos D;
 \end{aligned}
 \tag{57}$$

we obtain

$$\begin{aligned}
 \cos \beta \frac{d\lambda}{d\Omega} &= \frac{r}{A} C_0 \sin (C + u), \\
 \frac{d\beta}{d\Omega} &= -\frac{r}{A} A_0 \sin \beta \cos (A + u); \\
 \cos \beta \frac{d\lambda}{di} &= -\frac{r}{A} \sin i \sin u \cos (\lambda - \Omega), \\
 \frac{d\beta}{di} &= \frac{A}{r} D_0 \sin u \sin (D + \beta).
 \end{aligned}
 \tag{58}$$

If we substitute the expressions (55) and (56) in the equations

$$\begin{aligned}
 \cos \beta \frac{d\lambda}{d\varphi} &= \cos \beta \frac{d\lambda}{dr} \cdot \frac{dr}{d\varphi} + \cos \beta \frac{d\lambda}{dv} \cdot \frac{dv}{d\varphi}, \\
 \frac{d\beta}{d\varphi} &= \frac{d\beta}{dr} \cdot \frac{dr}{d\varphi} + \frac{d\beta}{dv} \cdot \frac{dv}{d\varphi},
 \end{aligned}$$

and put

$$\begin{aligned}
 -\frac{dr}{d\varphi} &= f \sin F = a \cos \varphi \cos v, \\
 r \frac{dv}{d\varphi} &= f \cos F = \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) r \sin v,
 \end{aligned}
 \tag{59}$$

we get

$$\begin{aligned}
 \cos \beta \frac{d\lambda}{d\varphi} &= \frac{f}{A} A_0 \sin (A + F + u), \\
 \frac{d\beta}{d\varphi} &= \frac{f}{A} B_0 \sin (B + F + u).
 \end{aligned}
 \tag{60}$$

In a similar manner, if we put

$$\begin{aligned}
 -\frac{dr}{dM_0} &= g \sin G = -a \tan \varphi \sin v, \\
 r \frac{dv}{dM_0} &= g \cos G = \frac{a^2 \cos \varphi}{r}, \\
 -\frac{dr}{d\mu} &= h \sin H = -\left(a \tan \varphi \sin v (t - T) - \frac{2r}{3\mu} 206264.8 \right), \\
 r \frac{dv}{d\mu} &= h \cos H = \frac{a^2 \cos \varphi}{r} (t - T),
 \end{aligned}
 \tag{61}$$

we obtain

$$\begin{aligned}
 \cos \beta \frac{d\lambda}{dM_0} &= \frac{g}{\Delta} A_0 \sin (A + G + u), \\
 \frac{d\beta}{dM_0} &= \frac{g}{\Delta} B_0 \sin (B + G + u); \\
 \cos \beta \frac{d\lambda}{d\mu} &= \frac{h}{\Delta} A_0 \sin (A + H + u), \\
 \frac{d\beta}{d\mu} &= \frac{h}{\Delta} B_0 \sin (B + H + u).
 \end{aligned} \tag{62}$$

The quadrants in which the auxiliary angles must be taken are determined by the condition that A_0, B_0, C_0, f, g , and h are always positive.

54. If the elements T, q , and e are introduced in place of M_0, μ , and φ , we must put

$$\begin{aligned}
 f \sin F &= -\frac{dr}{de}, & f \cos F &= r \frac{dv}{de}, \\
 g \sin G &= -\frac{dr}{dT}, & g \cos G &= r \frac{dv}{dT}, \\
 h \sin H &= -\frac{dr}{dq}, & h \cos H &= r \frac{dv}{dq},
 \end{aligned} \tag{63}$$

and the equations become

$$\begin{aligned}
 \cos \beta \frac{d\lambda}{de} &= \frac{f}{\Delta} A_0 \sin (A + F + u), \\
 \frac{d\beta}{de} &= \frac{f}{\Delta} B_0 \sin (B + F + u); \\
 \cos \beta \frac{d\lambda}{dT} &= \frac{g}{\Delta} A_0 \sin (A + G + u), \\
 \frac{d\beta}{dT} &= \frac{g}{\Delta} B_0 \sin (B + G + u); \\
 \cos \beta \frac{d\lambda}{dq} &= \frac{h}{\Delta} A_0 \sin (A + H + u), \\
 \frac{d\beta}{dq} &= \frac{h}{\Delta} B_0 \sin (B + H + u).
 \end{aligned} \tag{64}$$

In the numerical application of these formulæ, the values of the second members of the equations (63) are found as already exemplified for the cases of parabolic orbits and of elliptic and hyperbolic orbits in which the eccentricity differs but little from unity. In the same manner, the differential coefficients of λ and β with respect to any other elements which determine the form of the orbit may be computed.

In the case of a parabolic orbit, if the parabolic eccentricity is supposed to be invariable, the terms involving e vanish. Further, in the case of parabolic elements, we have

$$\begin{aligned} g \sin G &= -\frac{dr}{dT} = \frac{k \sin v}{\sqrt{2q}} = -r \tan \frac{1}{2}v \frac{dv}{dT}, \\ g \cos G &= r \frac{dv}{dT}, \end{aligned}$$

which give

$$\tan G = -\tan \frac{1}{2}v.$$

Hence there results $G = 180^\circ - \frac{1}{2}v$, and $g = k\sqrt{\frac{2}{r}}$, which is the expression for the linear velocity of a comet moving in a parabola. Therefore,

$$\begin{aligned} \cos \beta \frac{d\lambda}{dT} &= -\frac{k\sqrt{2}}{\Delta\sqrt{r}} A_0 \sin(A + u - \tfrac{1}{2}v), \\ \frac{d\beta}{dT} &= -\frac{k\sqrt{2}}{\Delta\sqrt{r}} B_0 \sin(B + u - \tfrac{1}{2}v). \end{aligned} \quad (65)$$

For the case in which the motion is considered as being retrograde, $180^\circ - i$ must be used instead of i in computing the values of A_0 , A , n , N , C_0 , and C , and the equations (55), (56), and the first two of (58), remain unchanged. But, for the differential coefficients with respect to i , the values of D_0 and D must be found from the last two of equations (57), using the given value of i directly; and then we shall have

$$\begin{aligned} \cos \beta \frac{d\lambda}{di} &= \frac{r}{\Delta} \sin i \sin u \cos(\lambda - \Omega), \\ \frac{d\beta}{di} &= -\frac{r}{\Delta} D_0 \sin u \sin(D + \beta). \end{aligned} \quad (66)$$

55. EXAMPLES.—The equations thus derived for the differential coefficients of λ and β with respect to the elements of the orbit, referred to the ecliptic as the fundamental plane, are applicable when any other plane is taken as the fundamental plane, if we consider λ and β as having the same signification in reference to the new plane that they have in reference to the ecliptic, the longitudes, however, being measured from the place of the descending node of this plane on the ecliptic. To illustrate their numerical application, let it be required to find the differential coefficients of the geocentric right ascension and declination of *Eurynome* [Ⓔ] with respect to the elements of its orbit referred to the equator, for the date 1865 February 24.5 mean time at Washington, using the data given in Art. 41.

In the first place, the elements which are referred to the ecliptic must be referred to the equator as the fundamental plane; and, by means of the equations (109)₁, we obtain

$$\Omega' = 353^\circ 45' 35''.87, \quad i' = 19^\circ 26' 25''.76, \quad \omega_0 = 212^\circ 32' 17''.71,$$

and

$$\omega' = \omega + \omega_0 = 50^\circ 10' 7''.29,$$

which are the elements which determine the position of the orbit in space when the equator is taken as the fundamental plane. These elements are referred to the mean equinox and equator of 1865.0. Writing α and δ in place of λ and β , and Ω' , i' , ω' in place of Ω , i , and ω , respectively, we have

$$\begin{aligned} A_0 \sin A &= \cos(\alpha - \Omega') \cos i', & A_0 \cos A &= \sin(\alpha - \Omega'); \\ n \sin N &= \sin i', & n \cos N &= -\cos i' \sin(\alpha - \Omega'); \\ B_0 \sin B &= n \sin(N + \delta), & B_0 \cos B &= \sin \delta \cos(\alpha - \Omega'); \\ C_0 \sin C &= \cos(\alpha - \Omega'), & C_0 \cos C &= \sin(\alpha - \Omega') \cos i'; \\ D_0 \sin D &= \cos i', & D_0 \cos D &= \sin i' \sin(\alpha - \Omega'); \\ f \sin F &= a \cos \varphi \cos v, \\ f \cos F &= \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) r \sin v; \\ g \sin G &= -a \tan \varphi \sin v, \\ g \cos G &= \frac{a^2 \cos \varphi}{r}; \\ h \sin H &= -\left(a \tan \varphi \sin v (t - T) - \frac{2r}{3\mu} 206264.8 \right), \\ h \cos H &= \frac{a^2 \cos \varphi}{r} (t - T). \end{aligned}$$

The values of A_0 , n , B_0 , C_0 , D_0 , f , g , and h must always be positive, thus determining the quadrants in which the angles A , B , &c. must be taken; and these equations give

$$\begin{aligned} \log A_0 &= 9.97497, & A &= 262^\circ 10' 40'', \\ \log B_0 &= 9.52100, & B &= 75 \quad 48 \quad 35, \\ \log C_0 &= 9.99961, & C &= 263 \quad 2 \quad 6, \\ \log D_0 &= 9.97497, & D &= 92 \quad 35 \quad 47, \\ \log f &= 0.62946, & F &= 339 \quad 14 \quad 0, \\ \log g &= 0.34593, & G &= 350 \quad 11 \quad 16, \\ \log h &= 2.97759, & H &= 14 \quad 30 \quad 48, \\ u' &= v + \omega' = 179^\circ 13' 58''. \end{aligned}$$

Substituting these values in the equations (55), (58), (60), and (62), and writing α and δ instead of λ and β , and u' in place of u , we find

$$\begin{aligned}\cos \delta \frac{d\alpha}{d\omega'} &= + 1.4235, & \frac{d\delta}{d\omega'} &= - 0.4890, \\ \cos \delta \frac{d\alpha}{d\Omega'} &= + 1.5098, & \frac{d\delta}{d\Omega'} &= + 0.0176, \\ \cos \delta \frac{d\alpha}{di'} &= + 0.0067, & \frac{d\delta}{di'} &= + 0.0193, \\ \cos \delta \frac{d\alpha}{d\varphi} &= + 1.9940, & \frac{d\delta}{d\varphi} &= - 0.6530, \\ \cos \delta \frac{d\alpha}{dM_0} &= + 1.1300, & \frac{d\delta}{dM_0} &= - 0.3802, \\ \cos \delta \frac{d\alpha}{d\mu} &= + 507.25, & \frac{d\delta}{d\mu} &= - 179.34;\end{aligned}$$

and hence

$$\begin{aligned}\cos \delta \Delta \alpha &= + 1.4235 \Delta \omega' + 1.5098 \Delta \Omega' + 0.0067 \Delta i' + 1.9940 \Delta \varphi \\ &\quad + 1.1300 \Delta M_0 + 507.25 \Delta \mu, \\ \Delta \delta &= - 0.4890 \Delta \omega' + 0.0176 \Delta \Omega' + 0.0193 \Delta i' - 0.6530 \Delta \varphi \\ &\quad - 0.3802 \Delta M_0 - 179.34 \Delta \mu.\end{aligned}$$

If we put

$$\begin{aligned}\Delta \omega' &= - 6''.64, & \Delta \Omega' &= - 14''.12, & \Delta i' &= - 8''.86, \\ \Delta \varphi &= + 10'', & \Delta M_0 &= + 10'', & \Delta \mu &= + 0''.01,\end{aligned}$$

we get

$$\cos \delta \Delta \alpha = + 5''.47, \quad \Delta \delta = - 9''.29;$$

and the values calculated directly from the elements corresponding to the increments thus assigned, are

$$\cos \delta \Delta \alpha = + 5''.50, \quad \Delta \delta = - 9''.02.$$

The agreement of these results is sufficiently close to prove the calculation of the coefficients in the equations for $\cos \delta \Delta \alpha$ and $\Delta \delta$.

When the values of $\Delta \omega'$, $\Delta \Omega'$, and $\Delta i'$ are small, the corresponding values of $\Delta \omega$, $\Delta \Omega$, and Δi may be determined by means of differential formulæ. From the spherical triangle formed by the intersection of the planes of the orbit, ecliptic, and equator with the celestial vault, we have

$$\begin{aligned}\cos i &= \cos i' \cos \varepsilon + \sin i' \sin \varepsilon \cos \Omega', \\ \sin i \cos \Omega &= - \cos i' \sin \varepsilon + \sin i' \cos \varepsilon \cos \Omega', \\ \sin i \sin \Omega &= \sin i' \sin \Omega', \\ \sin i \sin \omega_0 &= \sin \Omega' \sin \varepsilon, \\ \sin i \cos \omega_0 &= \cos \varepsilon \sin i' - \sin \varepsilon \cos i' \cos \Omega',\end{aligned}\tag{67}$$

from which the values of Ω , i , and ω_0 may be found from those of Ω' and i' . If we differentiate the first of these equations, regarding ϵ as constant, and reduce by means of the other given relations, we get

$$di = \cos \omega_0 di' + \sin \omega_0 \sin i' d\Omega'. \quad (68)$$

Interchanging i and $180^\circ - i'$, and also Ω and Ω' , we obtain

$$di' = \cos \omega_0 di - \sin \omega_0 \sin i d\Omega.$$

Eliminating di from these equations, and introducing the value

$$\frac{\sin i'}{\sin i} = \frac{\sin \Omega}{\sin \Omega'},$$

the result is

$$d\Omega = \frac{\sin \Omega}{\sin \Omega'} \cos \omega_0 d\Omega' - \frac{\sin \omega_0}{\sin i} di'. \quad (69)$$

If we differentiate the expression for $\cos \omega_0$ derived from the same spherical triangle, and reduce, we find

$$d\omega_0 = \cos i d\Omega - \cos i' d\Omega'.$$

Substituting for $d\Omega$ its value given by the preceding equation, and reducing by means of

$$\sin \Omega' \cos i' = \sin \Omega \cos \omega_0 \cos i - \cos \Omega \sin \omega_0,$$

we get

$$d\omega_0 = \frac{\sin \omega_0}{\sin \Omega'} \cos \Omega d\Omega' - \frac{\sin \omega_0}{\sin i} \cos i di'. \quad (70)$$

The equations (68), (69), and (70) give the partial differential coefficients of Ω , i , and ω_0 with respect to Ω' and i' , and if we suppose the variations of the elements, expressed in parts of the radius as unity, to be so small that their squares may be neglected, we shall have

$$\begin{aligned} \Delta \omega_0 &= \frac{\sin \omega_0}{\sin \Omega'} \cos \Omega \Delta \Omega' - \frac{\sin \omega_0}{\sin i} \cos i \Delta i', \\ \Delta \Omega &= \frac{\sin \Omega}{\sin \Omega'} \cos \omega_0 \Delta \Omega' - \frac{\sin \omega_0}{\sin i} \Delta i', \\ \Delta i &= \sin \omega_0 \sin i' \Delta \Omega' + \cos \omega_0 \Delta i', \\ \Delta \omega &= \Delta \omega' - \Delta \omega_0. \end{aligned} \quad (71)$$

If we apply these formulæ to the case of *Eurynome*, the result is

$$\begin{aligned} \Delta \omega_0 &= -4.420 \Delta \Omega' + 6.665 \Delta i', \\ \Delta \Omega &= -3.488 \Delta \Omega' + 6.686 \Delta i', \\ \Delta i &= -0.179 \Delta \Omega' - 0.843 \Delta i'; \end{aligned}$$

and if we assign the values

$$\Delta\Omega' = -14''.12, \quad \Delta i' = -8''.86, \quad \Delta\omega' = -6''.64,$$

we get

$$\Delta\omega_0 = +3''.36, \quad \Delta\Omega = -10''.0, \quad \Delta i = +10''.0, \quad \Delta\omega = -10''.0,$$

and, hence, the elements which determine the position of the orbit in reference to the ecliptic.

The elements ω' , Ω' , and i' may also be changed into those for which the ecliptic is the fundamental plane, by means of equations which may be derived from (109)₁ by interchanging Ω and Ω' and $180^\circ - i'$ and i .

56. If we refer the geocentric places of the body to a plane whose inclination to the plane of the ecliptic is i , and the longitude of whose ascending node on the ecliptic is Ω ,—which is equivalent to taking the plane of the orbit corresponding to the unchanged elements as the fundamental plane,—the equations are still further simplified. Let x' , y' , z' be the heliocentric co-ordinates of the body referred to a system of co-ordinates for which the plane of the unchanged orbit is the plane of xy , the positive axis of x being directed to the ascending node of this plane on the ecliptic; and let x , y , z be the heliocentric co-ordinates referred to a system in which the plane of xy is the plane of the ecliptic, the positive axis of x being directed to the point whose longitude is Ω . Then we shall have

$$\begin{aligned} dx' &= dx, \\ dy' &= dy \cos i + dz \sin i, \\ dz' &= -dy \sin i + dz \cos i. \end{aligned}$$

Substituting for dx , dy , and dz their values given by the equations (47), we get

$$\begin{aligned} dx' &= \frac{x'}{r} dr - r \sin u \, du - r \sin u \cos i \, d\Omega, \\ dy' &= \frac{y'}{r} dr + r \cos u \, du + r \cos u \cos i \, d\Omega, \\ dz' &= \frac{z'}{r} dr - r \cos u \sin i \, d\Omega + r \sin u \, di. \end{aligned}$$

It will be observed that we have, so long as the elements remain unchanged,

$$x' = r \cos u, \quad y' = r \sin u, \quad z' = 0,$$

and hence, omitting the accents, so that x, y, z will refer to the plane of the unchanged orbit as the plane of xy , the preceding equations give

$$\begin{aligned} dx &= \cos u \, dr - r \sin u \, du - r \sin u \cos i \, d\Omega, \\ dy &= \sin u \, dr + r \cos u \, du + r \cos u \cos i \, d\Omega, \\ dz &= -r \cos u \sin i \, d\Omega + r \sin u \, di. \end{aligned}$$

The value of ω is subject to two distinct changes, the one arising from the variation of the position of the orbit in its own plane, and the other, from the variation of the position of the plane of the orbit. Let us take a fixed line in the plane of the orbit and directed from the centre of the sun to a point the angular distance of which, back from the place of the ascending node on the ecliptic, we shall designate by σ ; and let the angle between this fixed line and the semi-transverse axis be designated by χ . Then we have

$$\chi = \omega + \sigma.$$

The fixed line thus taken is supposed to be so situated that, so long as the position of the plane of the orbit remains unchanged, we have

$$\sigma = \Omega, \quad \chi = \pi.$$

But if the elements which fix the position of the plane of the orbit are supposed to vary, we have the relations

$$\begin{aligned} d\sigma &= \cos i \, d\Omega, \\ d\omega &= d\chi - \cos i \, d\Omega, \\ d\pi &= d\chi + (1 - \cos i) \, d\Omega = d\chi + 2 \sin^2 \frac{1}{2}i \, d\Omega. \end{aligned} \tag{72}$$

Now, since $u = v + \omega$, we have

$$u = v + \chi - \sigma,$$

and

$$du = dv + d\chi - d\sigma = dv + d\chi - \cos i \, d\Omega.$$

Substituting this value of du in the equations for dx, dy, dz , they reduce to

$$\begin{aligned} dx &= \cos u \, dr - r \sin u \, dv - r \sin u \, d\chi, \\ dy &= \sin u \, dr + r \cos u \, dv + r \cos u \, d\chi, \\ dz &= -r \cos u \sin i \, d\Omega + r \sin u \, di. \end{aligned} \tag{73}$$

The inclination is here supposed to be susceptible of any value from 0° to 180° , and if the elements are given with the distinction of retrograde motion we must use $180^\circ - i$ instead of i .

Let us now denote by θ the geocentric longitude of the body measured in the plane of the unchanged orbit (which is here taken as the

fundamental plane) from the ascending node of this plane on the ecliptic, and let the geocentric latitude in reference to the same plane be denoted by η . Then we shall have

$$\begin{aligned}x + X &= \Delta \cos \eta \cos \theta, \\y + Y &= \Delta \cos \eta \sin \theta, \\z + Z &= \Delta \sin \eta,\end{aligned}$$

in which X, Y, Z are the geocentric co-ordinates of the sun referred to the same system of co-ordinates as x, y , and z . These equations give, by differentiation,

$$\begin{aligned}dx &= \cos \eta \cos \theta d\Delta - \Delta \sin \eta \cos \theta d\eta - \Delta \cos \eta \sin \theta d\theta, \\dy &= \cos \eta \sin \theta d\Delta - \Delta \sin \eta \sin \theta d\eta + \Delta \cos \eta \cos \theta d\theta, \\dz &= \sin \eta d\Delta + \Delta \cos \eta d\eta;\end{aligned}$$

and hence we obtain

$$\begin{aligned}\cos \eta d\theta &= -\frac{\sin \theta}{\Delta} dx + \frac{\cos \theta}{\Delta} dy, \\d\eta &= -\frac{\sin \eta \cos \theta}{\Delta} dx - \frac{\sin \eta \sin \theta}{\Delta} dy + \frac{\cos \eta}{\Delta} dz.\end{aligned}$$

These give

$$\begin{aligned}\cos \eta \frac{d\theta}{dx} &= -\frac{\sin \theta}{\Delta}, & \cos \eta \frac{d\theta}{dy} &= \frac{\cos \theta}{\Delta}, & \cos \eta \frac{d\theta}{dz} &= 0; \\ \frac{d\eta}{dx} &= -\frac{\sin \eta \cos \theta}{\Delta}, & \frac{d\eta}{dy} &= -\frac{\sin \eta \sin \theta}{\Delta}, & \frac{d\eta}{dz} &= \frac{\cos \eta}{\Delta};\end{aligned}\tag{74}$$

and from (73) we get

$$\begin{aligned}\frac{dx}{dr} &= \cos u, & \frac{dy}{dr} &= \sin u, & \frac{dz}{dr} &= 0; \\ \frac{dx}{dv} &= \frac{dx}{d\chi} = -r \sin u, & \frac{dy}{dv} &= \frac{dy}{d\chi} = r \cos u, & \frac{dz}{dv} &= \frac{dz}{d\chi} = 0; \\ \frac{dx}{d\Omega} &= 0, & \frac{dy}{d\Omega} &= 0, & \frac{dz}{d\Omega} &= -r \cos u \sin u; \\ \frac{dx}{di} &= 0, & \frac{dy}{di} &= 0, & \frac{dz}{di} &= r \sin u.\end{aligned}\tag{75}$$

Substituting the values thus found, in the equations

$$\begin{aligned}\cos \eta \frac{d\theta}{dv} &= \cos \eta \frac{d\theta}{dx} \cdot \frac{dx}{dv} + \cos \eta \frac{d\theta}{dy} \cdot \frac{dy}{dv}, \\ \frac{d\eta}{dv} &= \frac{d\eta}{dx} \cdot \frac{dx}{dv} + \frac{d\eta}{dy} \cdot \frac{dy}{dv} + \frac{d\eta}{dz} \cdot \frac{dz}{dv},\end{aligned}$$

we get

$$\begin{aligned}\cos \eta \frac{d\theta}{dv} &= \cos \eta \frac{d\theta}{d\chi} = \frac{r}{\Delta} \cos (\theta - u), \\ \frac{d\eta}{dv} &= \frac{d\eta}{d\chi} = -\frac{r}{\Delta} \sin \eta \sin (\theta - u).\end{aligned}\quad (76)$$

In a similar manner, we derive

$$\begin{aligned}\cos \eta \frac{d\theta}{dr} &= -\frac{1}{\Delta} \sin (\theta - u), & \frac{d\eta}{dr} &= -\frac{1}{\Delta} \sin \eta \cos (\theta - u), \\ \cos \eta \frac{d\theta}{d\Omega} &= 0, & \frac{d\eta}{d\Omega} &= -\frac{r}{\Delta} \cos \eta \sin i \cos u, \\ \cos \eta \frac{d\theta}{di} &= 0, & \frac{d\eta}{di} &= +\frac{r}{\Delta} \cos \eta \sin u.\end{aligned}\quad (77)$$

If we introduce the elements φ , M_0 , and μ , which determine r and v , we have, from

$$\begin{aligned}\cos \eta \frac{d\theta}{d\varphi} &= \cos \eta \frac{d\theta}{dr} \cdot \frac{dr}{d\varphi} + \cos \eta \frac{d\theta}{dv} \cdot \frac{dv}{d\varphi}, \\ \frac{d\eta}{d\varphi} &= \frac{d\eta}{dr} \cdot \frac{dr}{d\varphi} + \frac{d\eta}{dv} \cdot \frac{dv}{d\varphi},\end{aligned}$$

if we introduce also the auxiliary quantities f and F , as determined by means of the equations (59),

$$\cos \eta \frac{d\theta}{d\varphi} = \frac{f}{\Delta} \cos (\theta - u - F), \quad \frac{d\eta}{d\varphi} = -\frac{f}{\Delta} \sin \eta \sin (\theta - u - F). \quad (78)$$

Finally, using the auxiliaries g , h , G , and H , according to the equations (61), we get

$$\begin{aligned}\cos \eta \frac{d\theta}{dM_0} &= \frac{g}{\Delta} \cos (\theta - u - G), & \frac{d\eta}{dM_0} &= -\frac{g}{\Delta} \sin \eta \sin (\theta - u - G), \\ \cos \eta \frac{d\theta}{d\mu} &= \frac{h}{\Delta} \cos (\theta - u - H), & \frac{d\eta}{d\mu} &= -\frac{h}{\Delta} \sin \eta \sin (\theta - u - H).\end{aligned}\quad (79)$$

If we express r and v in terms of the elements T , q , and e , the values of the auxiliaries f , g , h , F , &c. must be found by means of (64); and, in the same manner, any other elements which determine the form of the orbit and the position of the body in its orbit, may be introduced.

The partial differential coefficients with respect to the elements having been found, we have

$$\begin{aligned}\cos \eta \Delta \theta &= \cos \eta \frac{d\theta}{d\chi} \Delta \chi + \cos \eta \frac{d\theta}{d\varphi} \Delta \varphi + \cos \eta \frac{d\theta}{dM_0} \Delta M_0 + \cos \eta \frac{d\theta}{d\mu} \Delta \mu, \\ \Delta \eta &= \frac{d\eta}{d\Omega} \Delta \Omega + \frac{d\eta}{di} \Delta i + \frac{d\eta}{d\chi} \Delta \chi + \frac{d\eta}{d\varphi} \Delta \varphi + \frac{d\eta}{dM_0} \Delta M_0 + \frac{d\eta}{d\mu} \Delta \mu,\end{aligned}$$

from which it appears that, by the introduction of χ as one of the elements of the orbit, when the geocentric places are referred directly to the plane of the unchanged orbit as the fundamental plane, the variation of the geocentric longitude in reference to this plane depends on only four elements.

57. It remains now to derive the formulæ for finding the values of η and θ from those of λ and β . Let x_0, y_0, z_0 be the geocentric co-ordinates of the body referred to a system in which the ecliptic is the plane of xy , the positive axis of x being directed to the point whose longitude is Ω ; and let x'_0, y'_0, z'_0 be the geocentric co-ordinates of the body referred to a system in which the axis of x remains the same, but in which the plane of the unchanged orbit is the plane of xy ; then we shall have

$$\begin{aligned} x_0 &= \Delta \cos \beta \cos (\lambda - \Omega), & x'_0 &= \Delta \cos \eta \cos \theta, \\ y_0 &= \Delta \cos \beta \sin (\lambda - \Omega), & y'_0 &= \Delta \cos \eta \sin \theta, \\ z_0 &= \Delta \sin \beta, & z'_0 &= \Delta \sin \eta, \end{aligned}$$

and also

$$\begin{aligned} x'_0 &= x_0, \\ y'_0 &= y_0 \cos i + z_0 \sin i, \\ z'_0 &= -y_0 \sin i + z_0 \cos i. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \cos \eta \cos \theta &= \cos \beta \cos (\lambda - \Omega), \\ \cos \eta \sin \theta &= \cos \beta \sin (\lambda - \Omega) \cos i + \sin \beta \sin i, \\ \sin \eta &= -\cos \beta \sin (\lambda - \Omega) \sin i + \sin \beta \cos i. \end{aligned} \quad (80)$$

These equations correspond to the relations between the parts of a spherical triangle of which the sides are i , $90^\circ - \eta$, and $90^\circ - \beta$, the angles opposite to $90^\circ - \eta$ and $90^\circ - \beta$ being respectively $90^\circ + (\lambda - \Omega)$ and $90^\circ - \theta$. Let the other angle of the triangle be denoted by γ , and we have

$$\begin{aligned} \cos \eta \sin \gamma &= \sin i \cos (\lambda - \Omega), \\ \cos \eta \cos \gamma &= \sin i \sin (\lambda - \Omega) \sin \beta + \cos i \cos \beta. \end{aligned} \quad (81)$$

The equations thus obtained enable us to determine η , θ , and γ from λ and β . Their numerical application is facilitated by the introduction of auxiliary angles. Thus, if we put

$$\begin{aligned} n \sin N &= \sin \beta, \\ n \cos N &= \cos \beta \sin (\lambda - \Omega), \end{aligned} \quad (82)$$

in which n is always positive, we get

$$\begin{aligned}\cos \eta \cos \theta &= \cos \beta \cos (\lambda - \Omega), \\ \cos \eta \sin \theta &= n \cos (N - i), \\ \sin \eta &= n \sin (N - i),\end{aligned}\tag{83}$$

from which η and θ may be readily found. If we also put

$$\begin{aligned}n' \sin N' &= \cos i, \\ n' \cos N' &= \sin i \sin (\lambda - \Omega),\end{aligned}\tag{84}$$

we shall have

$$\begin{aligned}\cot N' &= \tan i \sin (\lambda - \Omega), \\ \tan \gamma &= \frac{\cos N'}{\sin (N' + \beta)} \cot (\lambda - \Omega).\end{aligned}\tag{85}$$

If γ is small, it may be found from the equation

$$\sin \gamma = \frac{\sin i \cos (\lambda - \Omega)}{\cos \eta}.\tag{86}$$

The quadrants in which the angles sought must be taken, are easily determined by the relations of the quantities involved ; and the accuracy of the numerical calculation may be checked as already illustrated for similar cases.

If we apply Gauss's analogies to the same spherical triangle, we get

$$\begin{aligned}\sin (45^\circ - \tfrac{1}{2}\eta) \sin (45^\circ - \tfrac{1}{2}(\theta + \gamma)) &= \\ &\cos (45^\circ + \tfrac{1}{2}(\lambda - \Omega)) \sin (45^\circ - \tfrac{1}{2}(\beta + i)), \\ \sin (45^\circ - \tfrac{1}{2}\eta) \cos (45^\circ - \tfrac{1}{2}(\theta + \gamma)) &= \\ &\sin (45^\circ + \tfrac{1}{2}(\lambda - \Omega)) \sin (45^\circ - \tfrac{1}{2}(\beta - i)), \\ \cos (45^\circ - \tfrac{1}{2}\eta) \sin (45^\circ - \tfrac{1}{2}(\theta - \gamma)) &= \\ &\cos (45^\circ + \tfrac{1}{2}(\lambda - \Omega)) \cos (45^\circ - \tfrac{1}{2}(\beta + i)), \\ \cos (45^\circ - \tfrac{1}{2}\eta) \cos (45^\circ - \tfrac{1}{2}(\theta - \gamma)) &= \\ &\sin (45^\circ + \tfrac{1}{2}(\lambda - \Omega)) \cos (45^\circ - \tfrac{1}{2}(\beta - i)),\end{aligned}\tag{87}$$

from which we may derive η , θ , and γ .

When the problem is to determine the corrections to be applied to the elements of the orbit of a heavenly body, in order to satisfy given observed places, it is necessary to find the expressions for $\cos \eta \Delta \theta$ and $\Delta \eta$ in terms of $\cos \beta \Delta \lambda$ and $\Delta \beta$. If we differentiate the first and second of equations (80), regarding Ω and i (which here determine the position of the fundamental plane adopted) as constant, eliminate the terms containing $d\eta$ from the resulting equations, and reduce by means of the relations of the parts of the spherical triangle, we get

$$\cos \eta \, d\theta = \cos \gamma \cos \beta \, d\lambda + \sin \gamma \, d\beta.$$

Differentiating the last of equations (80), and reducing, we find

$$d\eta = -\sin \gamma \cos \beta \, d\lambda + \cos \gamma \, d\beta.$$

The equations thus derived give the values of the differential coefficients of θ and η with respect to λ and β ; and if the differences $\Delta\lambda$ and $\Delta\beta$ are small, we shall have

$$\begin{aligned} \cos \eta \, \Delta\theta &= \cos \gamma \cos \beta \, \Delta\lambda + \sin \gamma \, \Delta\beta, \\ \Delta\eta &= -\sin \gamma \cos \beta \, \Delta\lambda + \cos \gamma \, \Delta\beta. \end{aligned} \quad (88)$$

The value of γ required in the application of numbers to these equations may generally be derived with sufficient accuracy from (86), the algebraic sign of $\cos \gamma$ being indicated by the second of equations (81); and the values of η and θ required in the calculation of the differential coefficients of these quantities with respect to the elements of the orbit, need not be determined with extreme accuracy.

58. EXAMPLE.—Since the spherical co-ordinates which are furnished directly by observation are the right ascension and declination, the formulæ will be most frequently required in the form for finding η and θ from α and δ . For this purpose, it is only necessary to write α and δ in place of λ and β , respectively, and also Ω' , i' , ω' , χ' , and u' in place of Ω , i , ω , χ , and u , in the equations which have been derived for the determination of η and θ , and for the differential coefficients of these quantities with respect to the elements of the orbit.

To illustrate this clearly, let it be required to find the expressions for $\cos \eta \, \Delta\theta$ and $\Delta\eta$ in terms of the variations of the elements in the case of the example already given; for which we have

$$\omega' = 50^\circ 10' 7''.29, \quad \Omega' = 353^\circ 45' 35''.87, \quad i' = 19^\circ 26' 25''.76.$$

These are the elements which determine the position of the orbit of *Eurynome* $\textcircled{79}$, referred to the mean equinox and equator of 1865.0. We have, further,

$$\begin{aligned} \log f &= 0.62946, & \log g &= 0.34593, & \log h &= 2.97759, \\ F &= 339^\circ 14' 0'', & G &= 350^\circ 11' 16'', & H &= 14^\circ 30' 48'', \\ u' &= 179^\circ 13' 58''. \end{aligned}$$

In the first place, we compute η , θ , and γ by means of the formulæ

(83) and (85), or by means of (87), writing α, δ, Ω' , and i' instead of λ, β, Ω , and i , respectively. Hence we obtain

$$\theta = 188^\circ 31' 9'', \quad \eta = -1^\circ 59' 28'', \quad \gamma = -19^\circ 17' 7''.$$

Since the equator is here considered as the fundamental plane, the longitude θ is measured on the equator from the place of the ascending node of the orbit on this plane. The values of the differential coefficients are then found by means of the formulæ

$$\begin{aligned} \cos \eta \frac{d\theta}{d\Omega'} &= 0, & \frac{d\eta}{d\Omega'} &= -\frac{r}{\Delta} \cos \eta \sin i' \cos u', \\ \cos \eta \frac{d\theta}{di'} &= 0, & \frac{d\eta}{di'} &= +\frac{r}{\Delta} \cos \eta \sin u', \\ \cos \eta \frac{d\theta}{d\chi'} &= \frac{r}{\Delta} \cos (\theta - u'), & \frac{d\eta}{d\chi'} &= -\frac{r}{\Delta} \sin \eta \sin (\theta - u'), \\ \cos \eta \frac{d\theta}{d\varphi} &= \frac{f}{\Delta} \cos (\theta - u' - F), & \frac{d\eta}{d\varphi} &= -\frac{f}{\Delta} \sin \eta \sin (\theta - u' - F), \\ \cos \eta \frac{d\theta}{dM_0} &= \frac{g}{\Delta} \cos (\theta - u' - G), & \frac{d\eta}{dM_0} &= -\frac{g}{\Delta} \sin \eta \sin (\theta - u' - G), \\ \cos \eta \frac{d\theta}{d\mu} &= \frac{h}{\Delta} \cos (\theta - u' - H), & \frac{d\eta}{d\mu} &= -\frac{h}{\Delta} \sin \eta \sin (\theta - u' - H), \end{aligned}$$

which give

$$\begin{aligned} \cos \eta \frac{d\theta}{d\Omega'} &= 0, & \frac{d\eta}{d\Omega'} &= + 0.5072, \\ \cos \eta \frac{d\theta}{di'} &= 0, & \frac{d\eta}{di'} &= + 0.0204, \\ \cos \eta \frac{d\theta}{d\chi'} &= + 1.5051, & \frac{d\eta}{d\chi'} &= + 0.0086, \\ \cos \eta \frac{d\theta}{d\varphi} &= + 2.0978, & \frac{d\eta}{d\varphi} &= + 0.0422. \\ \cos \eta \frac{d\theta}{dM_0} &= + 1.1922, & \frac{d\eta}{dM_0} &= + 0.0143, \\ \cos \eta \frac{d\theta}{d\mu} &= + 538.00, & \frac{d\eta}{d\mu} &= - 1.71. \end{aligned}$$

Therefore, the equations for $\cos \eta \Delta\theta$ and $\Delta\eta$ become

$$\begin{aligned} \cos \eta \Delta\theta &= + 1.5051 \Delta\chi' + 2.0978 \Delta\varphi + 1.1922 \Delta M_0 + 538.00 \Delta\mu, \\ \Delta\eta &= + 0.0086 \Delta\chi' + 0.0422 \Delta\varphi + 0.0143 \Delta M_0 - 1.71 \Delta\mu \\ &\quad + 0.5072 \Delta\Omega' + 0.0204 \Delta i'. \end{aligned}$$

If we assign to the elements of the orbit the variations

$$\begin{array}{lll} \Delta\omega' = -6''.64, & \Delta\Omega' = -14''.12, & \Delta i' = -8''.86, \\ \Delta\varphi = +10'', & \Delta M_0 = +10'', & \Delta\mu = +0''.01, \end{array}$$

we have

$$\Delta\chi' = \Delta\omega' + \cos i' \Delta\Omega' = -19''.96;$$

and the preceding equations give

$$\cos \eta \Delta\theta = +8''.24, \quad \Delta\eta = -6''.96.$$

With the same values of $\Delta\omega'$, $\Delta\Omega'$, &c., we have already found

$$\cos \delta \Delta\alpha = +5''.47, \quad \Delta\delta = -9''.29,$$

which, by means of the equations (88), writing α and δ in place of λ and β , give

$$\cos \eta \Delta\theta = +8''.23, \quad \Delta\eta = -6''.96.$$

59. In special cases, in which the differences between the calculated and the observed values of two spherical co-ordinates are given, and the corrections to be applied to the assumed elements are sought, it may become necessary, on account of difficulties to be encountered in the solution of the equations of condition, to introduce other elements of the orbit of the body. The relation of the elements chosen to those commonly used will serve, without presenting any difficulty, for the transformation of the equations into a form adapted to the special case. Thus, in the case of the elements which determine the form of the orbit, we may use a or $\log a$ instead of μ , and the equation

$$\mu = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}$$

gives

$$d\mu = -\frac{3}{2} \frac{\mu}{a} da = -\frac{3}{2} \frac{\mu}{\lambda_0} d \log a, \quad (89)$$

in which λ_0 is the modulus of the system of logarithms. Therefore, the coefficient of $\Delta\mu$ is transformed into that of $\Delta \log a$ by multiplying it by $-\frac{3}{2} \frac{\mu}{\lambda_0}$; and if the unit of the m th decimal place of the logarithms is taken as the unit of $\Delta \log a$, the coefficient must be also multiplied by 10^{-m} . The homogeneity of the equation is not disturbed, since μ is here supposed to be expressed in seconds.

If we introduce $\log p$ as one of the elements, from the equation

$$p = a \cos^2 \varphi$$

we get

$$d \log p = -\frac{3}{2} \frac{\lambda_0}{\mu} d\mu - 2\lambda_0 \tan \varphi d\varphi,$$

or

$$d\mu = -\frac{3}{2} \frac{\mu}{\lambda_0} d \log p - 3\mu \tan \varphi d\varphi. \quad (90)$$

Hence it appears that the coefficients of $\Delta \log p$ are the same as those of $\Delta \log a$, but since p is also a function of φ , the coefficients of $\Delta \varphi$ are changed; and if we denote by $\cos \delta \left(\frac{d\alpha}{d\varphi} \right)$ and $\left(\frac{d\delta}{d\varphi} \right)$ the values of the partial differential coefficients when the element μ is used in connection with φ , we shall have, for the case under consideration,

$$\begin{aligned} \cos \delta \frac{d\alpha}{d\varphi} &= \cos \delta \left(\frac{d\alpha}{d\varphi} \right) - 3 \frac{\mu}{s} \tan \varphi \cos \delta \frac{d\alpha}{d\mu}, \\ \frac{d\delta}{d\varphi} &= \left(\frac{d\delta}{d\varphi} \right) - 3 \frac{\mu}{s} \tan \varphi \frac{d\delta}{d\mu}, \end{aligned}$$

in which $s = 206264'' \cdot 8$. If the values of the differential coefficients with respect to μ and φ have not already been found, it will be advantageous to compute the values of $\frac{dr}{d\varphi}$, $\frac{dv}{d\varphi}$, $\frac{dr}{d \log p}$, and $\frac{dv}{d \log p}$ by means of the expressions which may be derived by substituting in the equations (15) the value of $d\mu$ given by (90), and then we may compute directly the values of $\cos \delta \frac{d\alpha}{d\varphi}$, $\cos \delta \frac{d\alpha}{d \log p}$, $\frac{d\delta}{d\varphi}$, and $\frac{d\delta}{d \log p}$.

In place of M_0 , it is often convenient to introduce L_0 , the mean longitude for the epoch; and since

$$L_0 = M_0 + \pi,$$

we have

$$dL_0 = dM_0 + d\pi = dM_0 + d\omega + d\Omega,$$

and, when χ is used,

$$dL_0 = dM_0 + d\chi + (1 - \cos i) d\Omega.$$

Instead of the elements Ω and i which indicate the position of the plane of the orbit, we may use

$$b = \sin i \sin \Omega, \quad c = \sin i \cos \Omega,$$

and the expressions for the relations between the differentials of b and c and those of i and Ω are easily derived. The cosines of the angles which the line of apsides or any other line in the orbit makes with the three co-ordinate axes, may also be taken as elements of the

orbit in the formation of the equations for the variation of the geocentric place.

60. The equations (48), by writing l and b in place of λ and β , respectively, will give the values of the differential coefficients of the heliocentric longitude and latitude with respect to x , y , and z . Combining these with the expressions for the differential coefficients of the heliocentric co-ordinates with respect to the elements of the orbit, we obtain the values of $\cos b \, \Delta l$ and Δb in terms of the variations of the elements.

The equations for dx , dy , and dz in terms of du , $d\Omega$, and di , may also be used to determine the corrections to be applied to the co-ordinates in order to reduce them from the ecliptic and mean equinox of one epoch to those of another, or to the apparent equinox of the date. In this case, we have

$$du = d\pi - d\Omega.$$

When the auxiliary constants A , B , a , b , &c. are introduced, to find the variations of these arising from the variations assigned to the elements, we have, from the equations (99)₁,

$$\begin{aligned}\cot A &= -\tan \Omega \cos i, \\ \cot B &= \cot \Omega \cos i - \sin i \operatorname{cosec} \Omega \tan \varepsilon, \\ \cot C &= \cot \Omega \cos i + \sin i \operatorname{cosec} \Omega \cot \varepsilon,\end{aligned}$$

in which i may have any value from 0° to 180° . If we differentiate these, regarding all the quantities involved as variable, and reduce by means of the values of $\sin a$, $\sin b$, and $\sin c$, we get

$$\begin{aligned}dA &= \frac{\cos i}{\sin^2 a} d\Omega - \frac{\sin A}{\sin a} \sin \Omega \sin i \, di, \\ dB &= \frac{\cos \varepsilon}{\sin^2 b} (\cos i \cos \varepsilon - \sin i \sin \varepsilon \cos \Omega) d\Omega \\ &\quad + \frac{\sin B}{\sin b} (\cos \Omega \sin i \cos \varepsilon + \cos i \sin \varepsilon) di + \frac{\sin i \sin \Omega}{\sin^2 b} d\varepsilon, \\ dC &= \frac{\sin \varepsilon}{\sin^2 c} (\cos i \sin \varepsilon + \sin i \cos \varepsilon \cos \Omega) d\Omega \\ &\quad + \frac{\sin C}{\sin c} (\cos \Omega \sin i \sin \varepsilon - \cos i \cos \varepsilon) di + \frac{\sin i \sin \Omega}{\sin^2 c} d\varepsilon;\end{aligned}$$

and these, by means of (101)₁, reduce to

$$\begin{aligned}
dA &= \frac{\cos i}{\sin^2 a} d\Omega - \sin A \cot a di, \\
dB &= \frac{\cos \varepsilon \cos c}{\sin^2 b} d\Omega - \sin B \cot b di + \frac{\cos a}{\sin^2 b} d\varepsilon, \\
dC &= -\frac{\sin \varepsilon \cos b}{\sin^2 c} d\Omega - \sin C \cot c di + \frac{\cos a}{\sin^2 c} d\varepsilon.
\end{aligned} \tag{91}$$

Let us now differentiate the equations (101)₁, using only the upper sign, and the result is

$$\begin{aligned}
da &= -\sin i \sin A d\Omega + \cos A di, \\
db &= -\sin i \sin B d\Omega + \cos B di + \cos c \operatorname{cosec} b d\varepsilon, \\
dc &= -\sin i \sin C d\Omega + \cos C di - \cos b \operatorname{cosec} c d\varepsilon.
\end{aligned}$$

If we multiply the first of these equations by $\cot a$, the second by $\cot b$, and the third by $\cot c$, and denote by λ_0 the modulus of the system of logarithms, we get

$$\begin{aligned}
d \log \sin a &= -\lambda_0 \sin i \cot a \sin A d\Omega + \lambda_0 \cot a \cos A di, \\
d \log \sin b &= -\lambda_0 \sin i \cot b \sin B d\Omega + \lambda_0 \cot b \cos B di + \lambda_0 \frac{\cos b \cos c}{\sin^2 b} d\varepsilon, \\
d \log \sin c &= -\lambda_0 \sin i \cot c \sin C d\Omega + \lambda_0 \cot c \cos C di - \lambda_0 \frac{\cos b \cos c}{\sin^2 c} d\varepsilon.
\end{aligned} \tag{92}$$

The equations (91) and (92) furnish the differential coefficients of A , B , C , $\log \sin a$, &c. with respect to Ω , i , and ε ; and if the variations assigned to Ω , i , and ε are so small that their squares may be neglected, the same equations, writing ΔA , $\Delta \Omega$, Δi , &c. instead of the differentials, give the variations of the auxiliary constants. In the case of equations (92), if the variations of Ω , i , and ε are expressed in seconds, each term of the second member must be divided by 206264.8, and if the variations of $\log \sin a$, $\log \sin b$, and $\log \sin c$ are required in units of the m th decimal place of the logarithms, each term of the second member must also be divided by 10^m .

If we differentiate the equations (81)₁, and reduce by means of the same equations, we easily find

$$\begin{aligned}
\cos b dl &= \cos i \sec b du + \cos b d\Omega - \sin b \cos(l - \Omega) di, \\
db &= \sin i \cos(l - \Omega) du + \sin(l - \Omega) di,
\end{aligned} \tag{93}$$

which determine the relations between the variations of the elements of the orbit and those of the heliocentric longitude and latitude.

By differentiating the equations (88)₁, neglecting the latitude of

the sun, and considering λ , β , Δ , and \odot as variables, we derive, after reduction,

$$\begin{aligned}\cos \beta \, d\lambda &= \frac{R}{\Delta} \cos (\lambda - \odot) \, d\odot, \\ d\beta &= -\frac{R}{\Delta} \sin \beta \sin (\lambda - \odot) \, d\odot,\end{aligned}\tag{94}$$

which determine the variation of the geocentric latitude and longitude arising from an increment assigned to the longitude of the sun. It appears, therefore, that an error in the longitude of the sun will produce the greatest error in the computed geocentric longitude of a heavenly body when the body is in opposition.

CHAPTER III.

INVESTIGATION OF FORMULÆ FOR COMPUTING THE ORBIT OF A COMET MOVING IN A PARABOLA, AND FOR CORRECTING APPROXIMATE ELEMENTS BY THE VARIATION OF THE GEOCENTRIC DISTANCE.

61. THE observed spherical co-ordinates of the place of a heavenly body furnish each one equation of condition for the correction of the elements of its orbit approximately known, and similarly for the determination of the elements in the case of an orbit wholly unknown; and since there are six *elements*, neglecting the mass,—which must always be done in the first approximation, the perturbations not being considered,—three complete observations will furnish the six equations necessary for finding these unknown quantities. Hence, the data required for the determination of the orbit of a heavenly body are three complete observations, namely, three observed longitudes and the corresponding latitudes, or any other spherical co-ordinates which completely determine three places of the body as seen from the earth. Since these observations are given as made at some point or at different points on the earth's surface, it becomes necessary in the first place to apply the corrections for parallax. In the case of a body whose orbit is wholly unknown, it is impossible to apply the correction for parallax directly to the place of the body; but an equivalent correction may be applied to the places of the earth, according to the formulæ which will be given in the next chapter. However, in the first determination of approximate elements of the orbit of a comet, it will be sufficient to neglect entirely the correction for parallax. The uncertainty of the observed places of these bodies is so much greater than in the case of well-defined objects like the planets, and the intervals between the observations which will be generally employed in the first determination of the orbit will be so small, that an attempt to represent the observed places with extreme accuracy will be superfluous.

When approximate elements have been derived, we may find the distances of the comet from the earth corresponding to the three observed places, and hence determine the parallax in right ascension

and in declination for each observation by means of the usual formulæ. Thus, we have

$$\begin{aligned}\Delta \alpha &= -\frac{\pi \rho \cos \varphi'}{\Delta} \cdot \frac{\sin (\alpha - \Theta)}{\cos \delta}, \\ \tan \gamma &= \frac{\tan \varphi'}{\cos (\alpha - \Theta)}, \\ \Delta \delta &= \frac{\pi \rho \sin \varphi'}{\Delta} \cdot \frac{\sin (\gamma - \delta)}{\sin \gamma},\end{aligned}$$

in which α is the right ascension, δ the declination, Δ the distance of the comet from the earth, φ' the geocentric latitude of the place of observation, Θ the sidereal time corresponding to the time of observation, ρ the radius of the earth expressed in parts of the equatorial radius, and π the equatorial horizontal parallax of the sun.

In order to obtain the most accurate representation of the observed place by means of the elements computed, the correction for aberration must also be applied. When the distance Δ is known, the time of observation may be corrected for the time of aberration; but if Δ is not approximately known, this correction may be neglected in the first approximation.

The transformation of the observed right ascension and declination into latitude and longitude is effected by means of the equations which may be derived from (92)₁ by interchanging α and λ , δ and β , and writing $-\varepsilon$ instead of ε . Thus, we have

$$\begin{aligned}\tan N &= \frac{\tan \delta}{\sin \alpha}, \\ \tan \lambda &= \frac{\cos (N - \varepsilon)}{\cos N} \tan \alpha, \\ \tan \beta &= \tan (N - \varepsilon) \sin \lambda,\end{aligned}\tag{1}$$

and also

$$\frac{\cos (N - \varepsilon)}{\cos N} = \frac{\cos \beta \sin \lambda}{\cos \delta \sin \alpha},$$

which will serve to check the numerical calculation of λ and β . Since $\cos \beta$ and $\cos \delta$ are always positive, $\cos \lambda$ and $\cos \alpha$ must have the same sign, thus determining the quadrant in which λ is to be taken.

62. As soon as these preliminary corrections and transformations have been effected, and the times of observation have been reduced to the same meridian, the longitudes having been reduced to the

same equinox, we are prepared to proceed with the determination of the elements of the orbit. For this purpose, let t, t', t'' be the times of observation, r, r', r'' the radii-vectores of the body, and u, u', u'' the corresponding arguments of the latitude, R, R', R'' the distances of the earth from the sun, and \odot, \odot', \odot'' the longitudes of the sun corresponding to these times.

Let $[rr']$ denote double the area of the triangle formed between the radii-vectores r, r' and the chord of the orbit between the corresponding places of the body, and similarly for the other triangles thus formed. The angle at the sun in this triangle is the difference between the corresponding arguments of the latitude, and we shall have

$$\begin{aligned} [rr'] &= rr' \sin(u' - u), \\ [rr''] &= rr'' \sin(u'' - u), \\ [r'r''] &= r'r'' \sin(u'' - u'), \end{aligned} \quad (2)$$

If we designate by $x, y, z, x', y', z', x'', y'', z''$ the heliocentric co-ordinates of the body at the times $t, t',$ and t'' , we shall have

$$\begin{aligned} x &= r \sin a \sin(A + u), \\ x' &= r' \sin a \sin(A + u'), \\ x'' &= r'' \sin a \sin(A + u''), \end{aligned}$$

in which a and A are auxiliary constants which are functions of the elements Ω and i , and these elements may refer to any fundamental plane whatever. If we multiply the first of these equations by $\sin(u'' - u')$, the second by $-\sin(u'' - u)$, and the third by $\sin(u' - u)$, and add the products, we find, after reduction,

$$\frac{x}{r} \sin(u'' - u') - \frac{x'}{r'} \sin(u'' - u) + \frac{x''}{r''} \sin(u' - u) = 0,$$

which, by introducing the values of $[rr']$, $[rr'']$, and $[r'r'']$, becomes

$$[r'r'']x - [rr'']x' + [rr']x'' = 0.$$

If we put

$$n = \frac{[r'r'']}{[rr'']}, \quad n'' = \frac{[rr']}{[rr'']}, \quad (3)$$

we get

$$nx - x' + n''x'' = 0. \quad (4)$$

In precisely the same manner, we find

$$\begin{aligned} ny - y' + n''y'' &= 0, \\ nz - z' + n''z'' &= 0. \end{aligned} \quad (5)$$

Since the coefficients in these equations are independent of the positions of the co-ordinate planes, except that the origin is at the centre of the sun, it is evident that the three equations are identical, and express simply the condition that the plane of the orbit passes through the centre of the sun; and the last two might have been derived from the first by writing successively y and z in place of x .

Let $\lambda, \lambda', \lambda''$ be the three observed longitudes, β, β', β'' the corresponding latitudes, and $\Delta, \Delta', \Delta''$ the distances of the body from the earth; and let

$$\Delta \cos \beta = \rho, \quad \Delta' \cos \beta' = \rho', \quad \Delta'' \cos \beta'' = \rho'',$$

which are called *curtate* distances. Then we shall have

$$\begin{aligned} x &= \rho \cos \lambda - R \cos \odot, & x' &= \rho' \cos \lambda' - R' \cos \odot', \\ y &= \rho \sin \lambda - R \sin \odot, & y' &= \rho' \sin \lambda' - R' \sin \odot', \\ z &= \rho \tan \beta, & z' &= \rho' \tan \beta', \\ & & x'' &= \rho'' \cos \lambda'' - R'' \cos \odot'', \\ & & y'' &= \rho'' \sin \lambda'' - R'' \sin \odot'', \\ & & z'' &= \rho'' \tan \beta'', \end{aligned}$$

in which the latitude of the sun is neglected. The data may be so transformed that the latitude of the sun becomes 0, as will be explained in the next chapter; but in the computation of the orbit of a comet, in which this preliminary reduction has not been made, it will be unnecessary to consider this latitude which never exceeds $1''$, while its introduction into the formulæ would unnecessarily complicate some of those which will be derived. If we substitute these values of $x, x', \&c.$ in the equations (4) and (5), they become

$$\begin{aligned} 0 &= n(\rho \cos \lambda - R \cos \odot) - (\rho' \cos \lambda' - R' \cos \odot') \\ &\quad + n''(\rho'' \cos \lambda'' - R'' \cos \odot''), \\ 0 &= n(\rho \sin \lambda - R \sin \odot) - (\rho' \sin \lambda' - R' \sin \odot') \\ &\quad + n''(\rho'' \sin \lambda'' - R'' \sin \odot''), \\ 0 &= n\rho \tan \beta - \rho' \tan \beta' + n''\rho'' \tan \beta''. \end{aligned} \tag{6}$$

These equations simply satisfy the condition that the plane of the orbit passes through the centre of the sun, and they only become distinct or independent of each other when n and n'' are expressed in functions of the time, so as to satisfy the conditions of undisturbed motion in accordance with the law of gravitation. Further, they involve five unknown quantities in the case of an orbit wholly unknown, namely, $n, n'', \rho, \rho',$ and ρ'' ; and if the values of n and n'' are first found, they will be sufficient to determine $\rho, \rho',$ and ρ'' .

The determination, however, of n and n'' to a sufficient degree of accuracy, by means of the intervals of time between the observations, requires that ρ' should be approximately known, and hence, in general, it will become necessary to derive first the values of n , n'' , and ρ' ; after which those of ρ and ρ'' may be found from equations (6) by elimination. But since the number of equations will then exceed the number of unknown quantities, we may combine them in such a manner as will diminish, in the greatest degree possible, the effect of the errors of the observations. In special cases in which the conditions of the problem are such that when the ratio of two curtate distances is known, the distances themselves may be determined, the elimination must be so performed as to give this ratio with the greatest accuracy practicable.

63. If, in the first and second of equations (6), we change the direction of the axis of x from the vernal equinox to the place of the sun at the time t' , and again in the second, from the equinox to the second place of the body, we must diminish the longitudes in these equations by the angle through which the axis of x has been moved, and we shall have

$$\begin{aligned}
 0 &= n(\rho \cos(\lambda - \odot') - R \cos(\odot' - \odot)) - (\rho' \cos(\lambda' - \odot') - R') \\
 &\quad + n''(\rho'' \cos(\lambda'' - \odot') - R'' \cos(\odot'' - \odot')), \\
 0 &= n(\rho \sin(\lambda - \odot') + R \sin(\odot' - \odot)) - \rho' \sin(\lambda' - \odot') \\
 &\quad + n''(\rho'' \sin(\lambda'' - \odot') - R'' \sin(\odot'' - \odot')), \\
 0 &= n(\rho \sin(\lambda' - \lambda) + R \sin(\odot - \lambda')) - R' \sin(\odot' - \lambda') \\
 &\quad - n''(\rho'' \sin(\lambda'' - \lambda') - R'' \sin(\odot'' - \lambda')), \\
 0 &= n\rho \tan \beta - \rho' \tan \beta' + n''\rho'' \tan \beta''.
 \end{aligned} \tag{7}$$

If we multiply the second of these equations by $\tan \beta'$, and the fourth by $-\sin(\lambda' - \odot')$, and add the products, we get

$$\begin{aligned}
 0 &= n''\rho''(\tan \beta' \sin(\lambda'' - \odot') - \tan \beta'' \sin(\lambda' - \odot')) \\
 -n''R'' \sin(\odot'' - \odot') \tan \beta' &+ n\rho(\tan \beta' \sin(\lambda - \odot') - \tan \beta \sin(\lambda' - \odot')) \\
 &+ nR \sin(\odot' - \odot) \tan \beta'.
 \end{aligned} \tag{8}$$

Let us now denote double the area of the triangle formed by the sun and two places of the earth corresponding to R and R' by $[RR']$, and we shall have

$$[RR'] = RR' \sin(\odot' - \odot),$$

and similarly

$$\begin{aligned}
 [RR''] &= RR'' \sin(\odot'' - \odot), \\
 [R'R''] &= R'R'' \sin(\odot'' - \odot').
 \end{aligned}$$

Then, if we put

$$N = \frac{[R'R'']}{[RR'']}, \quad N'' = \frac{[RR']}{[RR'']}, \quad (9)$$

we obtain

$$R'' \sin(\odot'' - \odot') = R \sin(\odot' - \odot) \frac{N}{N''}.$$

Substituting this in the equation (8), and dividing by the coefficient of ρ'' , the result is

$$\rho'' = \rho \frac{n}{n''} \cdot \frac{\tan \beta' \sin(\lambda - \odot') - \tan \beta \sin(\lambda' - \odot')}{\tan \beta'' \sin(\lambda' - \odot') - \tan \beta' \sin(\lambda'' - \odot')} \\ + \left(\frac{n}{n''} - \frac{N}{N''} \right) \frac{R \sin(\odot' - \odot) \tan \beta'}{\tan \beta'' \sin(\lambda' - \odot') - \tan \beta' \sin(\lambda'' - \odot')}.$$

Let us also put

$$M' = \frac{\tan \beta' \sin(\lambda - \odot') - \tan \beta \sin(\lambda' - \odot')}{\tan \beta'' \sin(\lambda' - \odot') - \tan \beta' \sin(\lambda'' - \odot')}, \quad (10) \\ M'' = \frac{\sin(\odot' - \odot) \tan \beta'}{\tan \beta'' \sin(\lambda' - \odot') - \tan \beta' \sin(\lambda'' - \odot')}.$$

and the preceding equation reduces to

$$\rho'' = \frac{n}{n''} M' \rho + \left(\frac{n}{n''} - \frac{N}{N''} \right) M'' R. \quad (11)$$

We may transform the values of M' and M'' so as to be better adapted to logarithmic calculation with the ordinary tables. Thus, if w' denotes the inclination to the ecliptic of a great circle passing through the second place of the comet and the second place of the sun, the longitude of its ascending node will be \odot' , and we shall have

$$\sin(\lambda' - \odot') \tan w' = \tan \beta'. \quad (12)$$

Let β_0, β_0'' be the latitudes of the points of this circle corresponding to the longitudes λ and λ'' , and we have, also,

$$\tan \beta_0 = \sin(\lambda - \odot') \tan w', \quad (13) \\ \tan \beta_0'' = \sin(\lambda'' - \odot') \tan w'.$$

Substituting these values for $\tan \beta'$, $\sin(\lambda - \odot')$ and $\sin(\lambda'' - \odot')$ in the expressions for M' and M'' , and reducing, they become

$$M' = \frac{\sin(\beta_0 - \beta)}{\sin(\beta'' - \beta_0'')} \cdot \frac{\cos \beta'' \cos \beta_0''}{\cos \beta_0 \cos \beta}, \quad (14) \\ M'' = \tan w' \sin(\odot' - \odot) \frac{\cos \beta'' \cos \beta_0''}{\sin(\beta'' - \beta_0'')}.$$

When the value of $\frac{n}{n''}$ has been found, equation (11) will give the relation between ρ and ρ'' in terms of known quantities. It is evident, however, from equations (14), that when the apparent path of the comet is in a plane passing through the second place of the sun, since, in this case, $\beta = \beta_0$ and $\beta'' = \beta_0''$, we shall have $M' = \frac{0}{0}$ and $M'' = \infty$. In this case, therefore, and also when $\beta_0 - \beta$ and $\beta'' - \beta_0''$ are very nearly 0, we must have recourse to some other equation which may be derived from the equations (7), and which does not involve this indetermination.

It will be observed, also, that if, at the time of the middle observation, the comet is in opposition or conjunction with the sun, the values of M' and M'' as given by equation (14) will be indeterminate in form, but that the original equations (10) will give the values of these quantities provided that the apparent path of the comet is not in a great circle passing through the second place of the sun. These values are

$$M' = -\frac{\sin(\lambda - \odot')}{\sin(\lambda'' - \odot')}, \quad M'' = -\frac{\sin(\odot' - \odot)}{\sin(\lambda'' - \odot')}.$$

Hence it appears that whenever the apparent path of the body is nearly in a plane passing through the place of the sun at the time of the middle observation, the errors of observation will have great influence in vitiating the resulting values of M' and M'' ; and to obviate the difficulties thus encountered, we obtain from the third of equations (7) the following value of ρ'' :—

$$\begin{aligned} \rho'' = \rho \frac{n}{n''} \cdot \frac{\sin(\lambda' - \lambda)}{\sin(\lambda'' - \lambda')} \\ + \frac{\frac{n}{n''} R \sin(\odot - \lambda') - \frac{1}{n''} R' \sin(\odot' - \lambda') + R'' \sin(\odot'' - \lambda')}{\sin(\lambda'' - \lambda')} \end{aligned} \quad (15)$$

We may also eliminate ρ between the first and fourth of equations (7). If we multiply the first by $\tan \beta'$, and the second by $-\cos(\lambda' - \odot')$, and add the products, we obtain

$$\begin{aligned} 0 = n'' \rho'' (\tan \beta' \cos(\lambda'' - \odot') - \tan \beta'' \cos(\lambda' - \odot')) \\ - n'' R'' \tan \beta' \cos(\odot'' - \odot') + n \rho (\tan \beta' \cos(\lambda - \odot') - \tan \beta \cos(\lambda' - \odot')) \\ - n R \tan \beta' \cos(\odot' - \odot) + R' \tan \beta', \end{aligned}$$

from which we derive

$$\rho'' = \rho \frac{n}{n''} \cdot \frac{\tan \beta' \cos(\lambda - \odot') - \tan \beta \cos(\lambda' - \odot')}{\tan \beta'' \cos(\lambda' - \odot') - \tan \beta' \cos(\lambda'' - \odot')} \quad (16)$$

$$- \frac{R'' \tan \beta' \cos(\odot'' - \odot') + \frac{n}{n''} R \tan \beta' \cos(\odot' - \odot) - \frac{1}{n''} R' \tan \beta'}{\tan \beta'' \cos(\lambda' - \odot') - \tan \beta' \cos(\lambda'' - \odot')}.$$

Let us now denote by I' the inclination to the ecliptic of a great circle passing through the second place of the comet and that point of the ecliptic whose longitude is $\odot' - 90^\circ$, which will therefore be the longitude of its ascending node, and we shall have

$$\cos(\lambda' - \odot') \tan I' = \tan \beta'; \quad (17)$$

and, if we designate by β , and $\beta_{''}$, the latitudes of the points of this circle corresponding to the longitudes λ and λ'' , we shall also have

$$\begin{aligned} \tan \beta &= \cos(\lambda - \odot') \tan I', \\ \tan \beta_{''} &= \cos(\lambda'' - \odot') \tan I'. \end{aligned} \quad (18)$$

Introducing these values into equation (16), it reduces to

$$\rho'' = \rho \frac{n}{n''} \cdot \frac{\sin(\beta - \beta_{''})}{\sin(\beta'' - \beta_{''})} \cdot \frac{\cos \beta'' \cos \beta_{''}}{\cos \beta \cos \beta_{''}} \quad (19)$$

$$- \frac{\tan I' \cos \beta'' \cos \beta_{''}}{\sin(\beta'' - \beta_{''})} \left(R'' \cos(\odot'' - \odot') + \frac{n}{n''} R \cos(\odot' - \odot) - \frac{R'}{n''} \right),$$

from which it appears that this equation becomes indeterminate when the apparent path of the body is in a plane passing through that point of the ecliptic whose longitude is equal to the longitude of the second place of the sun diminished by 90° . In this case we may use equation (11) provided that the path of the comet is not nearly in the ecliptic. When the comet, at the time of the second observation, is in quadrature with the sun, equation (19) becomes indeterminate in form, and we must have recourse to the original equation (16), which does not necessarily fail in this case.

When both equations (11) and (16) are simultaneously nearly indeterminate, so as to be greatly affected by errors of observation, the relation between ρ and ρ'' must be determined by means of equation (15), which fails only when the motion of the comet in longitude is very small. It will rarely happen that all three equations, (14), (15), and (16), are inapplicable, and when such a case does occur it will indicate that the data are not sufficient for the determination of the elements of the orbit. In general, equation (16) or (19) is to be used when the motion of the comet in latitude is considerable, and equation (15) when the motion in longitude is greater than in latitude.

64. The formulæ already derived are sufficient to determine the relation between ρ'' and ρ when the values of n and n'' are known, and it remains, therefore, to derive the expressions for these quantities.

If we put

$$\begin{aligned} k(t' - t) &= \tau'', \\ k(t'' - t') &= \tau, \\ k(t'' - t) &= \tau', \end{aligned} \quad (20)$$

and express the values of x, y, z, x'', y'', z'' in terms of x', y', z' by expansion into series, we have

$$\begin{aligned} x &= x' - \frac{dx'}{dt} \cdot \frac{\tau''}{k} + \frac{1}{1.2} \cdot \frac{d^2x'}{dt^2} \cdot \frac{\tau''^2}{k^2} - \frac{1}{1.2.3} \cdot \frac{d^3x'}{dt^3} \cdot \frac{\tau''^3}{k^3} + \&c., \\ x'' &= x' + \frac{dx'}{dt} \cdot \frac{\tau}{k} + \frac{1}{1.2} \cdot \frac{d^2x'}{dt^2} \cdot \frac{\tau^2}{k^2} + \frac{1}{1.2.3} \cdot \frac{d^3x'}{dt^3} \cdot \frac{\tau^3}{k^3} + \&c., \end{aligned} \quad (21)$$

and similar expressions for y, y'', z , and z'' . We shall, however, take the plane of the orbit as the fundamental plane, in which case z, z' , and z'' vanish.

The fundamental equations for the motion of a heavenly body relative to the sun are, if we neglect its mass in comparison with that of the sun,

$$\begin{aligned} \frac{d^2x'}{dt^2} + \frac{k^2x'}{r'^3} &= 0, \\ \frac{d^2y'}{dt^2} + \frac{k^2y'}{r'^3} &= 0. \end{aligned}$$

If we differentiate the first of these equations, we get

$$\frac{d^3x'}{dt^3} = \frac{3k^2x'}{r'^4} \cdot \frac{dr'}{dt} - \frac{k^2}{r'^3} \cdot \frac{dx'}{dt}.$$

Differentiating again, we find

$$\frac{d^4x'}{dt^4} = \left(\frac{k^4}{r'^6} - \frac{12k^2}{r'^5} \left(\frac{dr'}{dt} \right)^2 + \frac{3k^2}{r'^4} \cdot \frac{d^2r'}{dt^2} \right) x' + \frac{6k^2}{r'^4} \cdot \frac{dr'}{dt} \cdot \frac{dx'}{dt}.$$

Writing y instead of x , we shall have the expressions for $\frac{d^3y'}{dt^3}$ and $\frac{d^4y'}{dt^4}$. Substituting these values of the differential coefficients in equations (21), and the corresponding expressions for y and y'' , and putting

$$\begin{aligned}
a &= 1 - \frac{1}{2} \frac{\tau'^2}{r'^3} - \frac{1}{2} \frac{\tau'^3}{kr'^4} \cdot \frac{dr'}{dt} + \frac{1}{24} \left(\frac{1}{r'^6} - \frac{12}{k^2 r'^5} \left(\frac{dr'}{dt} \right)^2 + \frac{3}{k^2 r'^4} \cdot \frac{d^2 r'}{dt^2} \right) \tau'^4 \dots, \\
b &= \frac{\tau''}{k} - \frac{1}{6} \frac{\tau''^3}{kr'^3} - \frac{1}{4} \frac{\tau''^4}{k^2 r'^4} \cdot \frac{dr'}{dt} \dots, \\
a'' &= 1 - \frac{1}{2} \frac{\tau^2}{r'^3} + \frac{1}{2} \frac{\tau^3}{kr'^4} \cdot \frac{dr'}{dt} + \frac{1}{24} \left(\frac{1}{r'^6} - \frac{12}{k^2 r'^5} \left(\frac{dr'}{dt} \right)^2 + \frac{3}{k^2 r'^4} \cdot \frac{d^2 r'}{dt^2} \right) \tau^4 \dots, \\
b'' &= \frac{\tau}{k} - \frac{1}{6} \frac{\tau^3}{kr'^3} + \frac{1}{4} \frac{\tau^4}{k^2 r'^4} \cdot \frac{dr'}{dt} \dots,
\end{aligned} \tag{22}$$

we obtain

$$\begin{aligned}
x &= ax' - b \frac{dx'}{dt}, & x'' &= a''x' + b'' \frac{dx'}{dt}, \\
y &= ay' - b \frac{dy'}{dt}, & y'' &= a''y' + b'' \frac{dy'}{dt}.
\end{aligned}$$

From these equations we easily derive

$$\begin{aligned}
y'x - x'y &= b \frac{x'dy' - y'dx'}{dt}, \\
y''x' - x''y' &= b'' \frac{x'dy' - y'dx'}{dt}, \\
y''x - x'y &= (ab'' + a''b) \frac{x'dy' - y'dx'}{dt}.
\end{aligned} \tag{23}$$

The first members of these equations are double the areas of the triangles formed by the radii-vectores and the chords of the orbit between the places of the comet or planet. Thus,

$$y'x - x'y = [rr'], \quad y''x' - x''y' = [r'r''], \quad y''x - x'y = [rr''], \tag{24}$$

and $x'dy' - y'dx'$ is double the area described by the radius-vector during the element of time dt , and, consequently, $\frac{x'dy' - y'dx'}{dt}$ is double the areal velocity. Therefore we shall have, neglecting the mass of the body,

$$\frac{x'dy' - y'dx'}{dt} = 2f = k\sqrt{p},$$

in which p is the semi-parameter of the orbit. The equations (23), therefore, become

$$[rr'] = bk\sqrt{p}, \quad [r'r''] = b''k\sqrt{p}, \quad [rr''] = (ab'' + a''b)k\sqrt{p}.$$

Substituting for a, b, a'', b'' their values from (22), we find, since $\tau' = \tau + \tau''$,

$$\begin{aligned}
[r'r'] &= \tau'' \sqrt{p} \left(1 - \frac{1}{6} \frac{\tau'^2}{r'^3} - \frac{1}{4} \frac{\tau''^3}{kr'^4} \cdot \frac{dr'}{dt} \dots \right), \\
[r'r''] &= \tau \sqrt{p} \left(1 - \frac{1}{6} \frac{\tau^2}{r'^3} + \frac{1}{4} \frac{\tau^3}{kr'^4} \cdot \frac{dr'}{dt} \dots \right), \\
[r'r''] &= \tau' \sqrt{p} \left(1 - \frac{1}{6} \frac{\tau'^2}{r'^3} + \frac{1}{4} \frac{\tau'^2(\tau - \tau'')}{kr'^4} \cdot \frac{dr'}{dt} \dots \right).
\end{aligned} \tag{25}$$

From these equations the values of $n = \frac{[r'r'']}{[r'r']}$ and $n'' = \frac{[r'r']}{[r'r']}$ may be derived; and the results are

$$\begin{aligned}
n &= \frac{\tau}{\tau'} \left(1 + \frac{1}{6} \frac{\tau''(\tau' + \tau)}{r'^3} + \frac{1}{4} \frac{\tau''(\tau'^2 + \tau\tau'' - \tau^2)}{kr'^4} \cdot \frac{dr'}{dt} \dots \right), \\
n'' &= \frac{\tau''}{\tau'} \left(1 + \frac{1}{6} \frac{\tau(\tau' + \tau'')}{r'^3} - \frac{1}{4} \frac{\tau(\tau^2 + \tau\tau'' - \tau'^2)}{kr'^4} \cdot \frac{dr'}{dt} \dots \right),
\end{aligned} \tag{26}$$

which values are exact to the third powers of the time, inclusive.

In the case of the orbit of the earth, the term of the third order, being multiplied by the very small quantity $\frac{dR'}{dt}$, is reduced to a superior order, and, therefore, it may be neglected, so that in this case we shall have, to the same degree of approximation as in (26),

$$\begin{aligned}
N &= \frac{\tau}{\tau'} \left(1 + \frac{1}{6} \frac{\tau''(\tau' + \tau)}{R'^3} \dots \right), \\
N'' &= \frac{\tau''}{\tau'} \left(1 + \frac{1}{6} \frac{\tau(\tau' + \tau'')}{R'^3} \dots \right).
\end{aligned} \tag{27}$$

From the equations (26) or from (25), since $\frac{n}{n''} = \frac{[r'r'']}{[r'r']}$, we find

$$\frac{n}{n''} = \frac{\tau}{\tau''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau''^2}{r'^3} + \frac{1}{4} \frac{\tau^3 + \tau''^3}{kr'^4} \cdot \frac{dr'}{dt} \dots \right). \tag{28}$$

Since this equation involves r' and $\frac{dr'}{dt}$, it is evident that the value of $\frac{n}{n''}$, in the case of an orbit wholly unknown, can be determined only by successive approximations. In the first approximation to the elements of the orbit of a heavenly body, the intervals between the observations will usually be small, and the series of terms of (28) will converge rapidly, so that we may take

$$\frac{n}{n''} = \frac{\tau}{\tau''}$$

and similarly

$$\frac{N}{N''} = \frac{\tau}{\tau''}.$$

Hence the equation (11) reduces to

$$\rho'' = \frac{\tau}{\tau''} M' \rho. \quad (29)$$

It will be observed, further, that if the intervals between the observations are equal, the term of the second order in equation (28) vanishes, and the supposition that $\frac{n}{n''} = \frac{\tau}{\tau''}$ is correct to terms of the third order. It will be advantageous, therefore, to select observations whose intervals approach nearest to equality. But if the observations available do not admit of the selection of those which give nearly equal intervals, and these intervals are necessarily very unequal, it will be more accurate to assume

$$\frac{n}{n''} = \frac{N}{N''}$$

and compute the values of N and N'' by means of equations (9), since, according to (27) and (28), if r' does not differ much from R' , the error of this assumption will only involve terms of the third order, even when the values of τ and τ'' differ very much.

Whenever the values of ρ and ρ'' can be found when that of their ratio is given, we may at once derive the corresponding values of r and r'' , as will be subsequently explained.

The values of r and r'' may also be expressed in terms of r' by means of series, and we have

$$\begin{aligned} r &= r' - \frac{dr'}{dt} \cdot \frac{\tau''}{k} + \frac{1}{2} \frac{d^2 r'}{dt^2} \cdot \frac{\tau''^2}{k^2} - \&c., \\ r'' &= r' + \frac{dr'}{dt} \cdot \frac{\tau}{k} + \frac{1}{2} \frac{d^2 r'}{dt^2} \cdot \frac{\tau^2}{k^2} + \&c., \end{aligned}$$

from which we derive

$$r'' - r = \frac{\tau + \tau''}{k} \cdot \frac{dr'}{dt},$$

neglecting terms of the third order. Therefore

$$\frac{dr'}{dt} = \frac{k(r'' - r)}{\tau + \tau''}; \quad (30)$$

and when the intervals are equal, this value is exact to terms of the fourth order. We have, also,

$$r + r'' = 2r' + \frac{\tau - \tau''}{k} \cdot \frac{dr'}{dt},$$

which gives

$$r' = \frac{1}{2} (r + r'') - \frac{1}{2} (r'' - r) \frac{\tau - \tau''}{\tau}. \quad (31)$$

Therefore, when r and r'' have been determined by a first approximation, the approximate values of r' and $\frac{dr'}{dt}$ are obtained from these equations, by means of which the value of $\frac{n}{n''}$ may be recomputed from equation (28). We also compute

$$\frac{N}{N''} = \frac{R'R'' \sin(\odot'' - \odot')}{RR' \sin(\odot' - \odot)}, \quad (32)$$

and substitute in equation (11) the values of $\frac{n}{n''}$ and $\frac{N}{N''}$ thus found.

If we designate by M the ratio of the curtate distances ρ and ρ'' , we have

$$M = \frac{\rho''}{\rho} = M' \frac{n}{n''} + M'' \left(\frac{n}{n''} - \frac{N}{N''} \right) \frac{R}{\rho}. \quad (33)$$

In the numerical application of this, the approximate value of ρ will be used in computing the last term of the second member.

In the case of the determination of an orbit when the approximate elements are already known, the value of $\frac{n}{n''}$ may be computed from

$$\frac{n}{n''} = \frac{r'r'' \sin(v'' - v')}{rr' \sin(v' - v)}, \quad (34)$$

and that of $\frac{N}{N''}$ from (32); and the value of M derived by means of these from (33) will not require any further correction.

65. When the apparent path of the body is such that the value of M' , as derived from the first of equations (10), is either indeterminate or greatly affected by errors of observation, the equations (15) and (16) must be employed. The last terms of these equations may be changed to a form which is more convenient in the approximations to the value of the ratio of ρ'' to ρ .

Let Y, Y', Y'' be the ordinates of the sun when the axis of

abscissas is directed to that point in the ecliptic whose longitude is λ' , and we have

$$\begin{aligned} Y &= R \sin(\odot - \lambda'), \\ Y' &= R' \sin(\odot' - \lambda'), \\ Y'' &= R'' \sin(\odot'' - \lambda'). \end{aligned}$$

Now, in the last term of equation (15), it will be sufficient to put

$$\frac{n}{n''} = \frac{N}{N''},$$

and, introducing Y , Y' , Y'' , it becomes

$$\left(\frac{N}{N''} Y - \frac{1}{n''} Y' + Y'' \right) \operatorname{cosec}(\lambda'' - \lambda'). \quad (35)$$

It now remains to find the value of $\frac{1}{n''}$. From the second of equations (26) we find, to terms of the second order inclusive,

$$\frac{1}{n''} = \frac{\tau'}{\tau''} \left(1 - \frac{1}{6} \frac{\tau(\tau' + \tau'')}{r'^3} \right).$$

We have, also,

$$\frac{1}{N''} = \frac{\tau'}{\tau''} \left(1 - \frac{1}{6} \frac{\tau(\tau' + \tau'')}{R'^3} \right),$$

and hence

$$\frac{1}{n''} = \frac{1}{N''} - \frac{1}{6} \frac{\tau'}{\tau''} \tau(\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right).$$

Therefore, the expression (35) becomes

$$\frac{1}{N'' \sin(\lambda'' - \lambda')} \left(NY - Y' + N'' Y'' + \frac{1}{6} \frac{\tau'}{\tau''} \tau(\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) N'' Y' \right).$$

But, according to equations (5),

$$NY - Y' + N'' Y'' = 0,$$

and the foregoing expression reduces to

$$+ \frac{1}{6} \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) \frac{R' \sin(\odot' - \lambda')}{\sin(\lambda'' - \lambda')},$$

since $Y' = R' \sin(\odot' - \lambda')$. Hence the equation (15) becomes

$$\rho'' = \rho \frac{n}{n''} \cdot \frac{\sin(\lambda' - \lambda)}{\sin(\lambda'' - \lambda')} - \frac{1}{6} \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) \frac{R' \sin(\lambda' - \odot')}{\sin(\lambda'' - \lambda')} \quad (36)$$

If we put

$$M_0 = \frac{n}{n''} \cdot \frac{\sin(\lambda' - \lambda)}{\sin(\lambda'' - \lambda')},$$

$$F = 1 - \frac{1}{6} \frac{n''}{n} \cdot \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \frac{\sin(\lambda' - \odot')}{\sin(\lambda'' - \lambda)} \cdot \frac{R'}{\rho} \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right),$$

we have

$$\frac{\rho''}{\rho} = M = M_0 F. \quad (37)$$

Let us now consider the equation (16), and let us designate by X , X' , X'' the abscissas of the earth, the axis of abscissas being directed to that point of the ecliptic for which the longitude is \odot' , then

$$\begin{aligned} X &= R \cos(\odot - \odot'), \\ X' &= R', \\ X'' &= R'' \cos(\odot'' - \odot'). \end{aligned}$$

It will be sufficient, in the last term of (16), to put

$$\frac{n}{n''} = \frac{N}{N''},$$

and for $\frac{1}{n''}$ its value in terms of N'' as already found. Then, since

$$NX - X' + N''X'' = 0,$$

this term reduces to

$$- \frac{1}{6} \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) \frac{R' \tan \beta'}{\tan \beta'' \cos(\lambda' - \odot') - \tan \beta' \cos(\lambda'' - \odot')};$$

and if we put

$$M'_0 = \frac{n}{n''} \cdot \frac{\tan \beta' \cos(\lambda - \odot') - \tan \beta \cos(\lambda' - \odot')}{\tan \beta'' \cos(\lambda' - \odot') - \tan \beta' \cos(\lambda'' - \odot')}, \quad (38)$$

$$F' = 1 - \frac{1}{6} \frac{n''}{n} \cdot \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) \frac{\tan \beta'}{\tan \beta' \cos(\lambda - \odot') - \tan \beta \cos(\lambda' - \odot')} \cdot \frac{R'}{\rho},$$

the equation (16) becomes

$$M = \frac{\rho''}{\rho} = M'_0 F'. \quad (39)$$

In the numerical application of these formulæ, if the elements are not approximately known, we first assume

$$\frac{n}{n''} = \frac{\tau}{\tau''},$$

when the intervals are nearly equal, and

$$\frac{n}{n''} = \frac{N}{N''},$$

as given by (32), when the intervals are very unequal, and neglect the factors F and F' . The values of ρ and ρ'' which are thus obtained, enable us to find an approximate value of r' , and with this a more exact value of $\frac{n}{n''}$ may be found, and also the value of F or F' .

Whenever equation (11) is not materially affected by errors of observation, it will furnish the value of M with more accuracy than the equations (37) and (39), since the neglected terms will not be so great as in the case of these equations. In general, therefore, it is to be preferred, and, in the case in which it fails, the very circumstance that the geocentric path of the body is nearly in a great circle, makes the values of F and F' differ but little from unity, since, in order that the apparent path of the body may be nearly in a great circle, r' must differ very little from R' .

66. When the value of M has been found, we may proceed to determine, by means of other relations between ρ and ρ'' , the values of the quantities themselves.

The co-ordinates of the first place of the earth referred to the third, are

$$\begin{aligned} x, &= R'' \cos \odot'' - R \cos \odot, \\ y, &= R'' \sin \odot'' - R \sin \odot. \end{aligned} \quad (40)$$

If we represent by g the chord of the earth's orbit between the places corresponding to the first and third observations, and by G the longitude of the first place of the earth as seen from the third, we shall have

$$x, = g \cos G, \quad y, = g \sin G,$$

and, consequently,

$$\begin{aligned} R'' \cos (\odot'' - \odot) - R &= g \cos (G - \odot), \\ R'' \sin (\odot'' - \odot) &= g \sin (G - \odot). \end{aligned} \quad (41)$$

If ψ represents the angle at the earth between the sun and comet at the first observation, and if we designate by w the inclination to the ecliptic of a plane passing through the places of the earth, sun, and comet or planet for the first observation, the longitude of the ascending node of this plane on the ecliptic will be \odot , and we shall have, in accordance with equations (81)₁,

$$\begin{aligned} \cos \psi &= \cos \beta \cos (\lambda - \odot), \\ \sin \psi \cos w &= \cos \beta \sin (\lambda - \odot), \\ \sin \psi \sin w &= \sin \beta, \end{aligned}$$

from which

$$\begin{aligned}\tan w &= \frac{\tan \beta}{\sin(\lambda - \odot)}, \\ \tan \psi &= \frac{\tan(\lambda - \odot)}{\cos w}.\end{aligned}\tag{42}$$

Since $\cos \beta$ is always positive, $\cos \psi$ and $\cos(\lambda - \odot)$ must have the same sign; and, further, ψ cannot exceed 180° .

In the same manner, if w'' and ψ'' represent analogous quantities for the time of the third observation, we obtain

$$\begin{aligned}\tan w'' &= \frac{\tan \beta''}{\sin(\lambda'' - \odot'')}, \\ \tan \psi'' &= \frac{\tan(\lambda'' - \odot'')}{\cos w''}, \\ \cos \psi'' &= \cos \beta'' \cos(\lambda'' - \odot'').\end{aligned}\tag{43}$$

We also have

$$r^2 = \Delta^2 + R^2 - 2\Delta R \cos \psi,$$

which may be transformed into

$$r^2 = (\rho \sec \beta - R \cos \psi)^2 + R^2 \sin^2 \psi;\tag{44}$$

and in a similar manner we find

$$r''^2 = (\rho'' \sec \beta'' - R'' \cos \psi'')^2 + R''^2 \sin^2 \psi''.\tag{45}$$

Let x designate the chord of the orbit of the body between the first and third places, and we have

$$x^2 = (x'' - x)^2 + (y'' - y)^2 + (z'' - z)^2.$$

But

$$\begin{aligned}x &= \rho \cos \lambda - R \cos \odot, \\ y &= \rho \sin \lambda - R \sin \odot, \\ z &= \rho \tan \beta,\end{aligned}$$

and, since $\rho'' = M\rho$,

$$\begin{aligned}x'' &= M\rho \cos \lambda'' - R'' \cos \odot'', \\ y'' &= M\rho \sin \lambda'' - R'' \sin \odot'', \\ z'' &= M\rho \tan \beta''\end{aligned}$$

from which we derive, introducing g and G ,

$$\begin{aligned}x'' - x &= M\rho \cos \lambda'' - \rho \cos \lambda - g \cos G, \\ y'' - y &= M\rho \sin \lambda'' - \rho \sin \lambda - g \sin G, \\ z'' - z &= M\rho \tan \beta'' - \rho \tan \beta.\end{aligned}$$

Let us now put

$$\begin{aligned}
 M\rho \cos \lambda'' - \rho \cos \lambda &= \rho h \cos \zeta \cos H, \\
 M\rho \sin \lambda'' - \rho \sin \lambda &= \rho h \cos \zeta \sin H, \\
 M\rho \tan \beta'' - \rho \tan \beta &= \rho h \sin \zeta.
 \end{aligned}
 \tag{46}$$

Then we have

$$\begin{aligned}
 x'' - x &= \rho h \cos \zeta \cos H - g \cos G, \\
 y'' - y &= \rho h \cos \zeta \sin H - g \sin G, \\
 z'' - z &= \rho h \sin \zeta.
 \end{aligned}$$

Squaring these values, and adding, we get, by reduction,

$$x^2 = \rho^2 h^2 - 2g \rho h \cos \zeta \cos (G - H) + g^2; \tag{47}$$

and if we put

$$\cos \zeta \cos (G - H) = \cos \varphi, \tag{48}$$

we have

$$x^2 = (\rho h - g \cos \varphi)^2 + g^2 \sin^2 \varphi. \tag{49}$$

If we multiply the first of equations (46) by $\cos \lambda''$, and the second by $\sin \lambda''$, and add the products; then multiply the first by $\sin \lambda''$, and the second by $\cos \lambda''$, and subtract, we obtain

$$\begin{aligned}
 h \cos \zeta \cos (H - \lambda'') &= M - \cos (\lambda'' - \lambda), \\
 h \cos \zeta \sin (H - \lambda'') &= \sin (\lambda'' - \lambda), \\
 h \sin \zeta &= M \tan \beta'' - \tan \beta,
 \end{aligned}
 \tag{50}$$

by means of which we may determine h , ζ , and H .

Let us now put

$$\begin{aligned}
 g \sin \varphi &= A, \\
 R \sin \psi &= B, & h \cos \beta &= b, \\
 R'' \sin \psi'' &= B'', & \frac{h \cos \beta''}{M} &= b'', \\
 g \cos \varphi - bR \cos \psi &= c, & g \cos \varphi - b''R'' \cos \psi'' &= c'', \\
 \rho h - g \cos \varphi &= d,
 \end{aligned}
 \tag{51}$$

and the equations (44), (45), and (49) become

$$\begin{aligned}
 x &= \sqrt{d^2 + A^2}, \\
 r &= \sqrt{\left(\frac{d+c}{b}\right)^2 + B^2}, \\
 r'' &= \sqrt{\left(\frac{d+c''}{b''}\right)^2 + B'^2}.
 \end{aligned}
 \tag{52}$$

The equations thus derived are independent of the form of the orbit, and are applicable to the case of any heavenly body revolving around the sun. They will serve to determine r and r'' in all cases in which the unknown quantity d can be determined. If ρ is known,

d becomes known directly; but in the case of an unknown orbit, these equations are applicable only when ρ or d may be determined directly or indirectly from the data furnished by observation.

67. Since the equations (52) involve two radii-vectores r and r'' and the chord \varkappa joining their extremities, it is evident that an additional equation involving these and known quantities will enable us to derive d , if not directly, at least by successive approximations. There is, indeed, a remarkable relation existing between two radii-vectores, the chord joining their extremities, and the time of describing the part of the orbit included by these radii-vectores. In general, the equation which expresses this relation involves also the semi-transverse axis of the orbit; and hence, in the case of an unknown orbit, it will not be sufficient, in connection with the equations (52), for the determination of d , unless some assumption is made in regard to the value of the semi-transverse axis. For the special case of parabolic motion, the semi-transverse axis is infinite, and the resulting equation involves only the time, the two radii-vectores, and the chord of the part of the orbit included by these. It is, therefore, adapted to the determination of the elements when the orbit is supposed to be a parabola, and, though it is transcendental in form, it may be easily solved by trial. To determine this expression, let us resume the equations

$$\frac{k(t - T)}{\sqrt{2}q^{\frac{3}{2}}} = \tan \frac{1}{2}v + \frac{1}{3} \tan^3 \frac{1}{2}v$$

and, for the time t'' ,

$$\frac{k(t'' - T)}{\sqrt{2}q^{\frac{3}{2}}} = \tan \frac{1}{2}v'' + \frac{1}{3} \tan^3 \frac{1}{2}v''.$$

Subtracting the former from the latter, and reducing, we obtain

$$\frac{3k(t'' - t)}{\sqrt{2}q^{\frac{3}{2}}} = \frac{\sin \frac{1}{2}(v'' - v)}{\cos \frac{1}{2}v'' \cos \frac{1}{2}v} \left(\frac{r''}{q} + \frac{\cos \frac{1}{2}(v'' - v)}{\cos \frac{1}{2}v'' \cos \frac{1}{2}v} + \frac{r}{q} \right),$$

and, since $r = q \sec^2 \frac{1}{2}v$, this gives

$$\frac{3k(t'' - t)}{\sqrt{2}} = \frac{\sin \frac{1}{2}(v'' - v) \sqrt{rr''}}{\sqrt{q}} \left(r + r'' + \cos \frac{1}{2}(v'' - v) \sqrt{rr''} \right). \quad (53)$$

But we have, also, from the triangle formed by the chord \varkappa and the radii-vectores r and r'' ,

$$\begin{aligned} \varkappa^2 &= r^2 + r''^2 - 2rr'' \cos(v'' - v) \\ &= (r + r'')^2 - 4rr'' \cos^2 \frac{1}{2}(v'' - v). \end{aligned}$$

Therefore,

$$\cos \frac{1}{2}(v'' - v) = \pm \frac{\sqrt{(r + r'' + x)(r + r'' - x)}}{2\sqrt{rr''}}.$$

Let us now put

$$r + r'' + x = m^2, \quad r + r'' - x = n^2,$$

m and n being positive quantities. Then we shall have

$$\begin{aligned} r + r'' &= \frac{1}{2}(m^2 + n^2), \\ 2 \cos \frac{1}{2}(v'' - v) \sqrt{rr''} &= \pm mn; \end{aligned} \quad (54)$$

and, since m and n are always positive, it follows that the upper sign must be used when $v'' - v$ is less than 180° , and the lower sign when $v'' - v$ is greater than 180° . Combining the last equation with (53), the result is

$$3k(t'' - t) = \frac{\sin \frac{1}{2}(v'' - v) \sqrt{rr''}}{\sqrt{2q}} (m^2 + n^2 \pm mn). \quad (55)$$

Now we have

$$\sin \frac{1}{2}(v'' - v) = \sin \frac{1}{2}v'' \cos \frac{1}{2}v - \cos \frac{1}{2}v'' \sin \frac{1}{2}v.$$

Squaring this, and reducing, we get

$$\sin^2 \frac{1}{2}(v'' - v) = \cos^2 \frac{1}{2}v + \cos^2 \frac{1}{2}v'' - 2 \cos \frac{1}{2}v'' \cos \frac{1}{2}v \cos \frac{1}{2}(v'' - v),$$

or, introducing r and q ,

$$\sin^2 \frac{1}{2}(v'' - v) = \frac{q}{r} + \frac{q}{r''} \mp q \frac{mn}{rr''}.$$

Therefore,

$$\sin \frac{1}{2}(v'' - v) = \frac{\sqrt{2q}}{2\sqrt{rr''}} (m \mp n).$$

Introducing this value into equation (55), we find

$$6k(t'' - t) = m^3 \mp n^3.$$

Replacing m and n by their values expressed in terms of r , r'' , and x , this becomes

$$6k(t'' - t) = (r + r'' + x)^{\frac{3}{2}} \mp (r + r'' - x)^{\frac{3}{2}}, \quad (56)$$

the upper sign being used when $v'' - v$ is less than 180° . This equation expresses the relation between the time of describing any parabolic arc and the rectilinear distances of its extremities from each other and from the sun, and enables us at once, when three of these quantities are given, to find the fourth, independent of either the

perihelion distance or the position of the perihelion with respect to the are described.

68. The transcendental form of the equation (56) indicates that, when either of the quantities in the second member is to be found, it must be solved by successive trials; and, to facilitate these approximations, it may be transformed as follows:—

Since the chord x can never exceed $r + r''$, we may put

$$\frac{x}{r + r''} = \sin \gamma', \quad (57)$$

and, since x , r , and r'' are positive, $\sin \gamma'$ must always be positive. The value of γ' must, therefore, be within the limits 0° and 180° . From the last equation we obtain

$$\cos^2 \gamma' = \frac{(r + r'')^2 - x^2}{(r + r'')^2};$$

and substituting for x^2 its value given by

$$x^2 = (r + r'')^2 - 4rr'' \cos^2 \frac{1}{2}(v'' - v),$$

this becomes

$$\cos^2 \gamma' = \frac{4rr'' \cos^2 \frac{1}{2}(v'' - v)}{(r + r'')^2}.$$

Therefore, we have

$$\cos \gamma' = \cos \frac{1}{2}(v'' - v) \frac{2\sqrt{rr''}}{r + r''}, \quad (58)$$

and also

$$\tan \gamma' = \frac{x}{2\sqrt{rr''} \cos \frac{1}{2}(v'' - v)}. \quad (59)$$

Hence it appears that when $v'' - v$ is less than 180° , γ' belongs to the first quadrant, and that when $v'' - v$ is greater than 180° , $\cos \gamma'$ is negative, and γ' belongs to the second quadrant.

If we introduce γ' into the expressions for m^2 and n^2 , they become

$$\begin{aligned} m^2 &= (r + r'')(1 + \sin \gamma'), \\ n^2 &= (r + r'')(1 - \sin \gamma'), \end{aligned}$$

which give

$$\begin{aligned} m^2 &= (r + r'')(\cos \frac{1}{2}\gamma' + \sin \frac{1}{2}\gamma')^2, \\ n^2 &= (r + r'')(\pm \cos \frac{1}{2}\gamma' \mp \sin \frac{1}{2}\gamma')^2; \end{aligned}$$

and, since γ' is greater than 90° when $v'' - v$ exceeds 180° , the equation (56) becomes

$$\frac{6r'}{(r + r'')^{\frac{3}{2}}} = (\cos \frac{1}{2}\gamma' + \sin \frac{1}{2}\gamma')^3 - (\cos \frac{1}{2}\gamma' - \sin \frac{1}{2}\gamma')^3$$

From this equation we get

$$\frac{6\tau'}{(r+r'')^{\frac{3}{2}}} = 6 \cos^2 \frac{1}{2}\gamma' \sin \frac{1}{2}\gamma' + 2 \sin^3 \frac{1}{2}\gamma',$$

or

$$\frac{6\tau'}{(r+r'')^{\frac{3}{2}}} = 6 \sin \frac{1}{2}\gamma' - 4 \sin^3 \frac{1}{2}\gamma';$$

and this, again, may be transformed into

$$\frac{6\tau'}{2^{\frac{3}{2}}(r+r'')^{\frac{3}{2}}} = 3 \left(\frac{\sin \frac{1}{2}\gamma'}{\sqrt{2}} \right) - 4 \left(\frac{\sin \frac{1}{2}\gamma'}{\sqrt{2}} \right)^3. \quad (60)$$

Let us now put

$$\sin x = \frac{\sin \frac{1}{2}\gamma'}{\sqrt{2}}, \quad (61)$$

or

$$\sin \frac{1}{2}\gamma' = \sqrt{2} \sin x,$$

and we have

$$\frac{3\tau'}{\sqrt{2}(r+r'')^{\frac{3}{2}}} = 3 \sin x - 4 \sin^3 x = \sin 3x. \quad (62)$$

When $v'' - v$ is less than 180° , γ' must be less than 90° , and hence, in this case, $\sin x$ cannot exceed the value $\frac{1}{2}$, or x must be within the limits 0° and 30° . When $v'' - v$ is greater than 180° , the angle γ' is within the limits 90° and 180° , and corresponding to these limits, the values of $\sin x$ are, respectively, $\frac{1}{2}$ and $\frac{1}{2}\sqrt{2}$. Hence, in the case that $v'' - v$ exceeds 180° , it follows that x must be within the limits 30° and 45° .

The equation

$$\frac{3\tau'}{\sqrt{2}(r+r'')^{\frac{3}{2}}} = \sin 3x$$

is satisfied by the values $3x$ and $180^\circ - 3x$; but when the first gives x less than 15° , there can be but one solution, the value $180^\circ - 3x$ being in this case excluded by the condition that $3x$ cannot exceed 135° . When x is greater than 15° , the required condition will be satisfied by $3x$ or by $180^\circ - 3x$, and there will be two solutions, corresponding respectively to the cases in which $v'' - v$ is less than 180° , and in which $v'' - v$ is greater than 180° . Consequently, when it is not known whether the heliocentric motion during the interval $t'' - t$ is greater or less than 180° , and we find $3x$ greater than 45° , the same data will be satisfied by these two different solutions. In practice, however, it is readily known which of the

two solutions must be adopted, since, when the interval $t'' - t$ is not very large, the heliocentric motion cannot exceed 180° , unless the perihelion distance is very small; and the known circumstances will generally show whether such an assumption is admissible.

We shall now put

$$\eta = \frac{2\tau'}{(r + r'')^{\frac{3}{2}}}, \quad (63)$$

and we obtain

$$\sin 3x = \frac{3\eta}{\sqrt{8}}. \quad (64)$$

We have, also,

$$\sin \frac{1}{2}\gamma' = \sqrt{2} \sin x,$$

and hence

$$\cos \frac{1}{2}\gamma' = \sqrt{1 - 2 \sin^2 x} = \sqrt{\cos 2x}.$$

Therefore

$$\sin \gamma' = 2^{\frac{3}{2}} \sin x \sqrt{\cos 2x},$$

and, since $\kappa = (r + r'') \sin \gamma'$, we have

$$\kappa = 2^{\frac{3}{2}} (r + r'') \sin x \sqrt{\cos 2x}.$$

If we put

$$\mu = \frac{3 \sin x}{\sin 3x} \sqrt{\cos 2x}, \quad (65)$$

the preceding equation reduces to

$$\kappa = \frac{2\tau'}{\sqrt{(r + r'')}} \mu. \quad (66)$$

From equation (64) it appears that η must be within the limits 0 and $\frac{1}{3}\sqrt{8}$. We may, therefore, construct a table which, with η as the argument, will give the corresponding value of μ , since, with a given value of η , $3x$ may be derived from equation (64), and then the value of μ from (65). Table XI. gives the values of μ corresponding to values of η from 0.0 to 0.9 .

69. In determining an orbit wholly unknown, it will be necessary to make some assumption in regard to the approximate distance of the comet from the sun. In this case the interval $t'' - t$ will generally be small, and, consequently, κ will be small compared with r and r'' . As a first assumption we may take $r = 1$, or $r + r'' = 2$, and $\mu = 1$, and then find κ from the formula

$$\kappa = \tau' \sqrt{2}.$$

With this value of κ we compute d , r , and r'' by means of the equations (52). Having thus found approximate values of r and r'' , we compute η by means of (63), and with this value we enter Table XI. and take out the corresponding value of μ . A second value for κ is then found from (66), with which we recompute r and r'' , and proceed as before, until the values of these quantities remain unchanged. The final values will exactly satisfy the equation (56), and will enable us to complete the determination of the orbit.

After three trials the value of $r + r''$ may be found very nearly correct from the numbers already derived. Thus, let y be the true value of $\log(r + r'')$, and let Δy be the difference between any assumed or approximate value of y and the true value, or

$$y_0 = y + \Delta y.$$

Then if we denote by y'_0 the value which results by direct calculation from the assumed value y_0 , we shall have

$$y'_0 - y_0 = f(y_0) = f(y + \Delta y).$$

Expanding this function, we have

$$y'_0 - y_0 = f(y) + A \Delta y + B \Delta y^2 + \&c.$$

But, since the equations (52) and (66) will be exactly satisfied when the true value of y is used, it follows that

$$f(y) = 0,$$

and hence, when Δy is very small, so that we may neglect terms of the second order, we shall have

$$y'_0 - y_0 = A \Delta y = A (y_0 - y).$$

Let us now denote three successive approximate values of $\log(r + r'')$ by y_0 , y'_0 , y''_0 , and let

$$y'_0 - y_0 = a, \quad y''_0 - y'_0 = a';$$

then we shall have

$$\begin{aligned} a &= A (y_0 - y), \\ a' &= A (y'_0 - y). \end{aligned}$$

Eliminating A from these equations, we get

$$y (a' - a) = a' y_0 - a y'_0,$$

from which

$$y = y'_0 - \frac{aa'}{a' - a} = y''_0 - \frac{a'^2}{a' - a}. \quad (67)$$

Unless the assumed values are considerably in error, the value of y or of $\log(r + r'')$ thus found will be sufficiently exact; but should it be still in error, we may, from the three values which approximate nearest to the truth, derive y with still greater accuracy. In the numerical application of this equation, a and a' may be expressed in units of the last decimal place of the logarithms employed.

The solution of equation (56), to find $t'' - t$ when x is known, is readily effected by means of Table VIII. Thus we have

$$\frac{3\tau'}{\sqrt{2}(r + r'')^{\frac{3}{2}}} = \sin 3x.$$

and, when γ' is less than 90° , if we put

$$N = \frac{\sin 3x}{\sin \gamma'},$$

we get

$$\tau' = \frac{1}{3} \sqrt{2} N \sin \gamma' (r + r'')^{\frac{3}{2}}, \quad (68)$$

or

$$\tau' = \frac{1}{3} \sqrt{2} N x \sqrt{r + r''}.$$

When γ' exceeds 90° , we put

$$N' = \sin 3x,$$

and we have

$$\tau' = \frac{1}{3} \sqrt{2} N' (r + r'')^{\frac{3}{2}}, \quad (69)$$

in which $\log \frac{1}{3} \sqrt{2} = 9.6733937$. With the argument γ' we take from Table VIII. the corresponding value of N or N' , and by means of these equations $\tau' = k(t'' - t)$ is at once derived.

The inverse problem, in which τ' is known and x is required, may also be solved by means of the same table. Thus, we may for a first approximation put

$$x = \tau' \sqrt{2},$$

and with this value of x compute d , r , and r'' . The value of γ' is then found from

$$\sin \gamma' = \frac{x}{r + r''},$$

and the table gives the corresponding value of N or N' . A second approximation to x will be given by the equation

$$x = \frac{3}{\sqrt{2}} \cdot \frac{\tau'}{N \sqrt{r + r''}},$$

or by

$$\kappa = \frac{3}{\sqrt{2}} \cdot \frac{\tau' \sin \gamma'}{N' \sqrt{r + r''}},$$

in which $\log \frac{3}{\sqrt{2}} = 0.3266063$. Then we recompute d , r , and r'' , and proceed as before until κ remains unchanged. The approximations are facilitated by means of equation (67).

It will be observed that d is computed from

$$d = \pm \sqrt{\kappa^2 - A^2},$$

and it should be known whether the positive or negative sign must be used. It is evident from the equation

$$d = \rho h - g \cos \varphi,$$

since ρ , h , and g are positive quantities, that so long as φ (which must be within the limits 0° and 180°) exceeds 90° , the value of d must be positive; and therefore φ must be less than 90° , and $g \cos \varphi$ greater than ρh , in order that d may be negative. The equation (47) shows that when κ is greater than g , we have

$$g \cos \varphi < \frac{1}{2} \rho h,$$

and hence d must in this case be positive. But when κ is less than g , either the positive or the negative value of d will answer to the given value of φ , and the sign to be adopted must be determined from the physical conditions of the problem.

If we suppose the chords g and κ to be proportional to the linear velocities of the earth and comet at the middle observation, we have, the eccentricity of the earth's orbit being neglected,

$$\kappa = g \sqrt{\frac{2}{r''}},$$

which shows that κ is greater than g , and that d is positive, so long as r' is less than 2. The comets are rarely visible at a distance from the earth which much exceeds the distance of the earth from the sun, and a comet whose radius-vector is 2 must be nearly in opposition in order to satisfy this condition of visibility. Hence cases will rarely occur in which d can be negative, and for those which do occur it will generally be easy to determine which sign is to be used. However, if d is very small, it may be impossible to decide which of the two solutions is correct without comparing the resulting elements with other and more distant observations.

70. When the values of r and r'' have been finally determined, as just explained, the exact value of d may be computed, and then we have

$$\begin{aligned} d &= \frac{d + g \cos \varphi}{h}, \\ \rho'' &= M\rho, \end{aligned} \quad (70)$$

from which to find ρ and ρ'' .

According to the equations (90)₁, we have

$$\begin{aligned} r \cos b \cos(l - \odot) &= \rho \cos(\lambda - \odot) - R, \\ r \cos b \sin(l - \odot) &= \rho \sin(\lambda - \odot), \\ r \sin b &= \rho \tan \beta, \end{aligned} \quad (71)$$

and also

$$\begin{aligned} r'' \cos b'' \cos(l'' - \odot'') &= \rho'' \cos(\lambda'' - \odot'') - R'', \\ r'' \cos b'' \sin(l'' - \odot'') &= \rho'' \sin(\lambda'' - \odot''), \\ r'' \sin b'' &= \rho'' \tan \beta'', \end{aligned} \quad (72)$$

in which l and l'' are the heliocentric longitudes and b, b'' the corresponding heliocentric latitudes of the comet. From these equations we find r, r'', l, l'', b , and b'' ; and the values of r and r'' thus found, should agree with the final values already obtained. When l'' is less than l , the motion of the comet is retrograde, or, rather, when the motion is such that the heliocentric longitude is diminishing instead of increasing.

From the equations (82)₁, we have

$$\begin{aligned} \pm \tan i \sin(l - \odot) &= \tan b, \\ \pm \tan i \sin(l'' - \odot) &= \tan b'', \end{aligned} \quad (73)$$

which may be written

$$\begin{aligned} \pm \tan i (\sin(l - x) \cos(x - \odot) + \sin(x - \odot) \cos(l - x)) &= \tan b, \\ \pm \tan i (\sin(l'' - x) \cos(x - \odot) + \sin(x - \odot) \cos(l'' - x)) &= \tan b''. \end{aligned}$$

Multiplying the first of these equations by $\sin(l'' - x)$, and the second by $-\sin(l - x)$, and adding the products, we get

$$\pm \tan i \sin(x - \odot) \sin(l'' - l) = \tan b \sin(l'' - x) - \tan b'' \sin(l - x);$$

and in a similar manner we find

$$\pm \tan i \cos(x - \odot) \sin(l'' - l) = \tan b'' \cos(l - x) - \tan b \cos(l'' - x).$$

Now, since x is entirely arbitrary, we may put it equal to l , and we have

$$\begin{aligned}\tan i \sin (l - \Omega) &= \pm \tan b, \\ \tan i \cos (l - \Omega) &= \pm \frac{\tan b'' - \tan b \cos (l'' - l)}{\sin (l'' - l)},\end{aligned}\quad (74)$$

the lower sign being used when it is desired to introduce the distinction of retrograde motion.

The formulæ will be better adapted to logarithmic calculation if we put $x = \frac{1}{2}(l'' + l)$, whence $l'' - x = \frac{1}{2}(l'' - l)$ and $l - x = \frac{1}{2}(l - l'')$; and we obtain

$$\begin{aligned}\tan i \sin (\tfrac{1}{2}(l'' + l) - \Omega) &= \pm \frac{\sin (b'' + b)}{2 \cos b \cos b'' \cos \tfrac{1}{2}(l'' - l)}, \\ \tan i \cos (\tfrac{1}{2}(l'' + l) - \Omega) &= \pm \frac{\sin (b'' - b)}{2 \cos b \cos b'' \sin \tfrac{1}{2}(l'' - l)}.\end{aligned}\quad (75)$$

These equations may also be derived directly from (73) by addition and subtraction. Thus we have

$$\begin{aligned}\pm \tan i (\sin (l'' - \Omega) + \sin (l - \Omega)) &= \tan b'' + \tan b, \\ \pm \tan i (\sin (l'' - \Omega) - \sin (l - \Omega)) &= \tan b'' - \tan b;\end{aligned}$$

and, since

$$\begin{aligned}\sin (l'' - \Omega) + \sin (l - \Omega) &= 2 \sin \tfrac{1}{2}(l'' + l - 2\Omega) \cos \tfrac{1}{2}(l'' - l), \\ \sin (l'' - \Omega) - \sin (l - \Omega) &= 2 \cos \tfrac{1}{2}(l'' + l - 2\Omega) \sin \tfrac{1}{2}(l'' - l),\end{aligned}$$

these become

$$\begin{aligned}\tan i \sin (\tfrac{1}{2}(l'' + l) - \Omega) &= \pm \frac{\tfrac{1}{2}(\tan b'' + \tan b)}{\cos \tfrac{1}{2}(l'' - l)}, \\ \tan i \cos (\tfrac{1}{2}(l'' + l) - \Omega) &= \pm \frac{\tfrac{1}{2}(\tan b'' - \tan b)}{\sin \tfrac{1}{2}(l'' - l)},\end{aligned}\quad (76)$$

which may be readily transformed into (75). However, since b and b'' will be found by means of their tangents in the numerical application of equations (71) and (72), if addition and subtraction logarithms are used, the equations last derived will be more convenient than in the form (75).

As soon as Ω and i have been computed from the preceding equations, we have, for the determination of the arguments of the latitude u and u'' ,

$$\tan u = \pm \frac{\tan (l - \Omega)}{\cos i}, \quad \tan u'' = \pm \frac{\tan (l'' - \Omega)}{\cos i}. \quad (77)$$

Now we have

$$u = v + \omega,$$

in which $\omega = \pi - \Omega$ in the case of direct motion, and $\omega = \Omega - \pi$

when the distinction of retrograde motion is adopted; and we shall have

$$u'' - u = v'' - v,$$

and, consequently,

$$\kappa^2 = r^2 + r''^2 - 2rr'' \cos(u'' - u), \quad (78)$$

or

$$\kappa^2 = (r'' - r \cos(u'' - u))^2 + r^2 \sin^2(u'' - u). \quad (79)$$

The value of κ derived from this equation should agree with that already found from (66).

We have, further,

$$r = q \sec^2 \frac{1}{2}(u - \omega), \quad r'' = q \sec^2 \frac{1}{2}(u'' - \omega),$$

or

$$\frac{1}{\sqrt{q}} \cos \frac{1}{2}(u - \omega) = \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{q}} \cos \frac{1}{2}(u'' - \omega) = \frac{1}{\sqrt{r''}}$$

By addition and subtraction, we get, from these equations,

$$\begin{aligned} \frac{1}{\sqrt{q}} (\cos \frac{1}{2}(u'' - \omega) + \cos \frac{1}{2}(u - \omega)) &= \frac{1}{\sqrt{r''}} + \frac{1}{\sqrt{r}}, \\ \frac{1}{\sqrt{q}} (\cos \frac{1}{2}(u'' - \omega) - \cos \frac{1}{2}(u - \omega)) &= \frac{1}{\sqrt{r''}} - \frac{1}{\sqrt{r}}, \end{aligned}$$

from which we easily derive

$$\begin{aligned} \frac{2}{\sqrt{q}} \cos \frac{1}{2}(\frac{1}{2}(u'' + u) - \omega) \cos \frac{1}{4}(u'' - u) &= \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{r''}}, \\ \frac{2}{\sqrt{q}} \sin \frac{1}{2}(\frac{1}{2}(u'' + u) - \omega) \sin \frac{1}{4}(u'' - u) &= \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r''}}. \end{aligned} \quad (80)$$

But

$$\frac{1}{\sqrt{r}} \mp \frac{1}{\sqrt{r''}} = \frac{1}{\sqrt{rr''}} \left(\sqrt[4]{\frac{r''}{r}} \mp \sqrt[4]{\frac{r}{r''}} \right),$$

and if we put

$$\tan(45^\circ + \theta') = \sqrt[4]{\frac{r''}{r}},$$

since $\sqrt[4]{\frac{r''}{r}}$ will not differ much from 1, θ' will be a small angle; and we shall have, since $\tan(45^\circ + \theta') - \cot(45^\circ + \theta') = 2 \tan 2\theta'$,

$$\begin{aligned} \sqrt[4]{\frac{r''}{r}} - \sqrt[4]{\frac{r}{r''}} &= 2 \tan 2\theta', \\ \sqrt[4]{\frac{r''}{r}} + \sqrt[4]{\frac{r}{r''}} &= 2 \sec 2\theta', \end{aligned}$$

Therefore, the equations (80) become

$$\begin{aligned} \frac{1}{\sqrt{q}} \sin \frac{1}{2} (\frac{1}{2}(u'' + u) - \omega) &= \frac{\tan 2\theta'}{\sin \frac{1}{4} (u'' - u) \sqrt[4]{rr''}}, \\ \frac{1}{\sqrt{q}} \cos \frac{1}{2} (\frac{1}{2}(u'' + u) - \omega) &= \frac{\sec 2\theta'}{\cos \frac{1}{4} (u'' - u) \sqrt[4]{rr''}}, \end{aligned} \quad (81)$$

from which the values of q and ω may be found. Then we shall have, for the longitude of the perihelion

$$\pi = \omega + \Omega,$$

when the motion is direct, and

$$\pi = \Omega - \omega,$$

when i unrestricted exceeds 90° and the distinction of retrograde motion is adopted.

It remains now to find T , the time of perihelion passage. We have

$$v = u - \omega, \quad v'' = u'' - \omega.$$

With the resulting values of v and v'' we may find, by means of Table VI., the corresponding values of M (which must be distinguished from the symbol M already used to denote the ratio of the curtate distances), and if these values are designated by M and M'' , we shall have

$$t - T = \frac{M}{m}, \quad t'' - T = \frac{M''}{m},$$

or

$$T = t - \frac{M}{m} = t'' - \frac{M''}{m},$$

in which $m = \frac{C_0}{q^{\frac{3}{2}}}$, and $\log C_0 = 9.9601277$. When v is negative, the corresponding value of M is negative. The agreement between the two values of T will be a final proof of the accuracy of the numerical calculation.

The value of T when the true anomaly is small, is most readily and accurately found by means of Table VIII., from which we derive the two values of N and compute the corresponding values of T from the equation

$$T = t - \frac{2}{3k} N r^{\frac{3}{2}} \sin v,$$

in which $\log \frac{2}{3k} = 1.5883273$. When v is greater than 90° , we de-

rive the values of N' from the table, and compute the corresponding values of T from

$$T = t - \frac{2}{3k} N' r^{\frac{3}{2}}.$$

71. The elements q and T may be derived directly from the values of r , r'' , and κ , as derived from the equations (52), without first finding the position of the plane of the orbit and the position of the orbit in its own plane. Thus, the equations (80), replacing u and u'' by their values $v + \omega$ and $v + \omega''$, become

$$\begin{aligned} \frac{2}{\sqrt{q}} \sin \frac{1}{4} (v'' + v) \sin \frac{1}{4} (v'' - v) &= \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r''}}, \\ \frac{2}{\sqrt{q}} \cos \frac{1}{4} (v'' + v) \cos \frac{1}{4} (v'' - v) &= \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{r''}}. \end{aligned} \quad (82)$$

Adding together the squares of these, and reducing, we get

$$\frac{1}{q} = \frac{\frac{1}{r} + \frac{1}{r''} - \frac{2}{\sqrt{rr''}} \cos \frac{1}{2} (v'' - v)}{\sin^2 \frac{1}{2} (v'' - v)},$$

or

$$q = \frac{rr'' \sin^2 \frac{1}{2} (v'' - v)}{r'' + r - 2\sqrt{rr''} \cos \frac{1}{2} (v'' - v)}.$$

Combining this equation with (59), the result is

$$q = \frac{rr'' \sin^2 \frac{1}{2} (v'' - v)}{r + r'' - \kappa \cot \gamma'},$$

and hence, since $\kappa = (r + r'') \sin \gamma'$,

$$q = \frac{rr''}{\kappa} \sin^2 \frac{1}{2} (v'' - v) \cot \frac{1}{2} \gamma'. \quad (83)$$

We have, further, from (78),

$$\kappa^2 = (r'' - r)^2 + 4rr'' \sin^2 \frac{1}{2} (v'' - v),$$

from which, putting

$$\sin \nu = \frac{r'' - r}{\kappa}, \quad (84)$$

we derive

$$\cos \nu = \frac{2\sqrt{rr''}}{\kappa} \sin \frac{1}{2} (v'' - v). \quad (85)$$

Therefore, the equation (83) becomes

$$q = \frac{1}{2} (r + r'') \cos^2 \frac{1}{2} \gamma' \cos^2 \nu, \quad (86)$$

by means of which q is derived directly from r , r'' , and x , the value of ν being found by means of the formula (84), so that $\cos \nu$ is positive.

When γ' cannot be found with sufficient accuracy from the equation

$$\sin \gamma' = \frac{x}{r + r''}$$

we may use another form. Thus, we have

$$1 + \sin \gamma' = \frac{r + r'' + x}{r + r''}, \quad 1 - \sin \gamma' = \frac{r + r'' - x}{r + r''},$$

which give, by division,

$$\tan (45^\circ + \frac{1}{2} \gamma') = \sqrt{\frac{r + r'' + x}{r + r'' - x}}. \quad (87)$$

In a similar manner, we derive

$$\tan (45^\circ + \frac{1}{2} \nu) = \sqrt{\frac{x + (r'' - r)}{x - (r'' - r)}}. \quad (88)$$

In order to find the time of perihelion passage, it is necessary first to derive the values of v and v'' . The equations (59) and (85) give, by multiplication,

$$\tan \frac{1}{2} (v'' - v) = \tan \gamma' \cos \nu, \quad (89)$$

from which $v'' - v$ may be computed. From (82) we get

$$\tan \frac{1}{4} (v'' + v) \tan \frac{1}{4} (v'' - v) = \frac{\sqrt{\frac{r''}{r}} - 1}{\sqrt{\frac{r''}{r}} + 1}.$$

If we put

$$\tan \chi' = \sqrt{\frac{r''}{r}}, \quad (90)$$

this equation reduces to

$$\tan \frac{1}{4} (v'' + v) = \tan (\chi' - 45^\circ) \cot \frac{1}{4} (v'' - v), \quad (91)$$

and the equations (81) give, also,

$$\tan \frac{1}{4} (v'' + v) = \cot \frac{1}{4} (v'' - v) \sin 2\theta',$$

either of which may be used to find $v'' + v$.

From the equations

$$\frac{\cos \frac{1}{2}v}{\sqrt{q}} = \frac{1}{\sqrt{r}}, \quad \frac{\cos \frac{1}{2}v''}{\sqrt{q}} = \frac{1}{\sqrt{r''}}$$

by multiplying the first by $\sin \frac{1}{2}v''$ and the second by $-\sin \frac{1}{2}v$, adding the products and reducing, we easily find

$$\frac{\sin \frac{1}{2}(v'' - v) \sin \frac{1}{2}v}{\sqrt{q}} = \frac{\cos \frac{1}{2}(v'' - v)}{\sqrt{r}} - \frac{1}{\sqrt{r''}}.$$

Hence we have

$$\begin{aligned} \frac{1}{\sqrt{q}} \sin \frac{1}{2}v &= \frac{\cot \frac{1}{2}(v'' - v)}{\sqrt{r}} - \frac{1}{\sqrt{r''} \sin \frac{1}{2}(v'' - v)}, \\ \frac{1}{\sqrt{q}} \cos \frac{1}{2}v &= \frac{1}{\sqrt{r}}, \end{aligned} \quad (92)$$

which may be used to compute q , v , and v'' when $v'' - v$ is known.

When $\frac{1}{2}(v'' - v)$ and $\frac{1}{2}(v'' + v)$, and hence v'' and v , have been determined, the time of perihelion passage must be found, as already explained, by means of Table VI. or Table VIII.

It is evident, therefore, that in the determination of an orbit, as soon as the numerical values of r , r'' , and κ have been derived from the equations (52), instead of completing the calculation of the elements of the orbit, we may find q and T , and then, by means of these, the values of r' and v' may be computed directly. When this has been effected, the values of n and n'' may be found from (3), or that of $\frac{n}{n''}$ from (34). Then we compute ρ by means of the first of equations (70), and the corrected value of M from (33), or, in the special cases already examined, from the equations (37) and (39). In this way, by successive approximations, the determination of parabolic elements from given data may be carried to the limit of accuracy which is consistent with the assumption of parabolic motion. In the case, however, of the equations (37) and (39), the neglected terms may be of the second order, and, consequently, for the final results it will be necessary, in order to attain the greatest possible accuracy, to derive

$$M = \frac{\rho''}{\rho}$$

from (15) and (16). When the final value of M has been found, the determination of the elements is completed by means of the formulæ already given.

72. EXAMPLE.—To illustrate the application of the formulæ for the calculation of the parabolic elements of the orbit of a comet by a numerical example, let us take the following observations of the Fifth Comet of 1863, made at Ann Arbor:—

| Ann Arbor M. T. | α | δ |
|--|--|-----------------|
| 1864 Jan. 10 6 ^h 57 ^m 20 ^s .5 | 19 ^h 14 ^m 4 ^s .92 | + 34° 6' 27".4, |
| 13 6 11 54.7 | 19 25 2.84 | 36 36 52 .8, |
| 16 6 35 11.6 | 19 41 4.54 | + 39 41 26 .9. |

These places are referred to the apparent equinox of the date and are already corrected for parallax and aberration by means of approximate values of the geocentric distances of the comet. But if approximate values of these distances are not already known, the corrections for parallax and aberration may be neglected in the first determination of the approximate elements of the unknown orbit of a comet. If we convert the observed right ascensions and declinations into the corresponding longitudes and latitudes by means of equations (1), and reduce the times of observation to the meridian of Washington, we get

| Washington M. T. | λ | β |
|--|---------------|------------------|
| 1864 Jan. 10 7 ^h 24 ^m 3 ^s | 297° 53' 7".6 | + 55° 46' 58".4, |
| 13 6 38 37 | 302 57 51 .3 | 57 39 35 .9, |
| 16 7 1 54 | 310 31 52 .3 | + 59 38 18 .7. |

Next, we reduce these places by applying the corrections for precession and nutation to the mean equinox of 1864.0, and reduce the times of observation to decimals of a day, and we have

| | | |
|-------------------|-------------------------------------|------------------------------------|
| $t = 10.30837,$ | $\lambda = 297^{\circ} 52' 51''.1,$ | $\beta = + 55^{\circ} 46' 58''.4,$ |
| $t' = 13.27682,$ | $\lambda' = 302 57 34 .4,$ | $\beta' = 57 39 35 .9,$ |
| $t'' = 16.29299,$ | $\lambda'' = 310 31 35 .0,$ | $\beta'' = + 59 38 18 .7.$ |

For the same times we find, from the *American Nautical Almanac*,

| | |
|----------------------------------|------------------------|
| $\odot = 290^{\circ} 6' 27''.4,$ | $\log R = 9.992763,$ |
| $\odot' = 293 7 57 .1,$ | $\log R' = 9.992830,$ |
| $\odot'' = 296 12 15 .7,$ | $\log R'' = 9.992916,$ |

which are referred to the mean equinox of 1864.0. It will generally be sufficient, in a first approximation, to use logarithms of five decimals; but, in order to exhibit the calculation in a more complete form, we shall retain six places of decimals.

Since the intervals are very nearly equal, we may assume

$$\frac{n}{n''} = \frac{\tau}{\tau''} = \frac{N}{N''}.$$

Then we have

$$M = \frac{t'' - t'}{t' - t} \cdot \frac{\tan \beta' \sin (\lambda - \odot') - \tan \beta \sin (\lambda' - \odot')}{\tan \beta'' \sin (\lambda' - \odot') - \tan \beta' \sin (\lambda'' - \odot')},$$

and

$$\begin{aligned} g \sin (G - \odot) &= R'' \sin (\odot'' - \odot), \\ g \cos (G - \odot) &= R'' \cos (\odot'' - \odot) - R; \\ h \cos \zeta \cos (H - \lambda'') &= M - \cos (\lambda'' - \lambda), \\ h \cos \zeta \sin (H - \lambda'') &= \sin (\lambda'' - \lambda), \\ h \sin \zeta &= M \tan \beta'' - \tan \beta; \end{aligned}$$

from which to find M , G , g , H , ζ , and h . Thus we obtain

$$\begin{aligned} \log M &= 9.829827, & H &= 94^\circ 24' 1''.8, \\ G &= 22^\circ 58' 1''.7, & \zeta &= -40 \quad 28 \quad 21 \quad .9, \\ \log h &= 9.019613, & \log h &= 9.688532. \end{aligned}$$

Since $\frac{A''}{A} = M \frac{\cos \beta}{\cos \beta''} = 0.752$, it appears that the comet, at the time of these observations, was rapidly approaching the earth. The quadrants in which $G - \odot$ and $H - \lambda''$ must be taken, are determined by the condition that g and $h \cos \zeta$ must always be positive. The value of M should be checked by duplicate calculation, since an error in this will not be exhibited until the values of λ' and β' are computed from the resulting elements.

Next, from

$$\begin{aligned} \cos \psi &= \cos \beta \cos (\lambda - \odot), & \cos \psi'' &= \cos \beta'' \cos (\lambda'' - \odot''), \\ \cos \varphi &= \cos \zeta \cos (G - H), \end{aligned}$$

we compute $\cos \psi$, $\cos \psi''$, and $\cos \varphi$; and then from

$$\begin{aligned} g \sin \varphi &= A, & h \cos \beta &= b, \\ R \sin \psi &= B, & \frac{h \cos \beta''}{M} &= b'', \\ R'' \sin \psi'' &= B'', \\ g \cos \varphi - bR \cos \psi &= c, & g \cos \varphi - b''R'' \cos \psi'' &= c'', \end{aligned}$$

we obtain A , B , B'' , &c. It will generally be sufficiently exact to find $\sin \psi$ and $\sin \psi''$ from $\cos \psi$ and $\cos \psi''$; but if more accurate values of ψ and ψ'' are required, they may be obtained by means of the equations (42) and (43). Thus we derive

$$\begin{aligned} \log A &= 9.006485, & \log R &= 9.912052, & \log B'' &= 9.933366, \\ \log b &= 9.438524, & \log b'' &= 9.562387, \\ c &= -0.125067, & c'' &= -0.150562. \end{aligned}$$

Then we have

$$\begin{aligned}\tau' &= k(t'' - t), & \eta &= \frac{2\tau'}{(r + r'')^{\frac{3}{2}}}, \\ \kappa &= \frac{2\tau'}{\sqrt{r + r''}}\mu, & d &= \sqrt{\kappa^2 - A^2}, \\ r &= \sqrt{\left(\frac{d + c}{b}\right)^2 + B^2}, & r'' &= \sqrt{\left(\frac{d + c''}{b''}\right)^2 + B'^2},\end{aligned}$$

from which to find, by successive trials, the values of r , r'' , and κ , that of μ being found from Table XI. with the argument η . First, we assume

$$\log \kappa = \log (\tau' \sqrt{2}) = 9.163132,$$

and with this we obtain

$$\log r = 9.913895, \quad \log r'' = 9.938040, \quad \log (r + r'') = 0.227165.$$

This value of $\log (r + r'')$ gives $\eta = 0.094$, and from Table XI. we find $\log \mu = 0.000160$. Hence we derive

$$\begin{aligned}\log \kappa &= 9.200220, & \log r &= 9.912097, & \log r'' &= 9.935187, \\ & & \log (r + r'') &= 0.224825.\end{aligned}$$

Repeating the operation, using the last value of $\log (r + r'')$, we get

$$\begin{aligned}\log \kappa &= 9.201396, & \log r &= 9.912083, & \log r'' &= 9.935117, \\ & & \log (r + r'') &= 0.224783.\end{aligned}$$

The correct value of $\log (r + r'')$ may now be found by means of the equation (67). Thus, we have, in units of the sixth decimal place of the logarithms,

$$a = 224825 - 227165 = -2340, \quad a' = 224783 - 224825 = -42,$$

and the correction to the last value of $\log (r + r'')$ becomes

$$-\frac{a'^2}{a' - a} = -0.8.$$

Therefore,

$$\log (r + r'') = 0.224782,$$

and, recomputing η , μ , κ , r , and r'' , we get, finally,

$$\begin{aligned}\log \kappa &= 9.201419, & \log r &= 9.912083, & \log r'' &= 9.935116, \\ & & \log (r + r'') &= 0.224782.\end{aligned}$$

The agreement of the last value of $\log (r + r'')$ with the preceding one shows that the results are correct. Further, it appears from the

values of r and r'' that the comet had passed its perihelion and was receding from the sun.

By means of the values of r and r'' we might compute approximate values of r' and $\frac{dr'}{dt}$ from the equations (30) and (31), and then a more approximate value of $\frac{n}{n''}$ from (28), that of $\frac{N}{N''}$ being found from (32). But, since r' differs but little from R' , the difference between $\frac{n}{n''}$ and $\frac{N}{N''}$ is very small, so that it is not necessary to consider the second term of the second member of the equation (33); and, since the intervals are very nearly equal, the error of the assumption

$$\frac{n}{n''} = \frac{\tau}{\tau''}$$

is of the third order. It should be observed, however, that an error in the value of M affects H , ζ , h , and hence also A , b , b'' , c , and c'' , and the resulting value of ρ may be affected by an error which considerably exceeds that of M . It is advantageous, therefore, to select observations which furnish intervals as nearly equal as possible in order that the error of M may be small, otherwise it may become necessary to correct M and to repeat the calculation of r , r'' , and κ . We may also compute the perihelion distance and the time of perihelion passage from r , r'' , and κ by means of the equations (86), (89), and (91) in connection with Tables VI. and VIII. Then r' and v' may be computed directly, and the complete expression for M may be employed.

In the first determination of the elements, and especially when the corrections for parallax and aberration have been neglected, it is unnecessary to attempt to arrive at the limit of accuracy attainable, since, when approximate elements have been found, the observations may be more conveniently reduced, and those which include a longer interval may be used in a more complete calculation. Hence, as soon as r , r'' , and κ have been found, the curtate distances are next determined, and then the elements of the orbit. To find ρ and ρ'' , we have

$$d = + 0.122395,$$

the positive sign being used since κ is greater than g , and the formulæ

$$\rho = \frac{d + g \cos \varphi}{h}, \quad \rho'' = M\rho,$$

give

$$\log \rho = 9.480952, \quad \log \rho'' = 9.310779.$$

From these values of ρ and ρ'' , it appears that the comet was very near the earth at the time of the observations.

The heliocentric places are then found by means of the equations (71) and (72). Thus we obtain

$$\begin{aligned} l &= 106^\circ 40' 50''.5, & b &= +33^\circ 1' 10''.6, & \log r &= 9.912082, \\ l'' &= 112^\circ 31' 9.9, & b'' &= +23^\circ 55' 5.8, & \log r'' &= 9.935116. \end{aligned}$$

The agreement of these values of r and r'' with those previously found, checks the accuracy of the calculation. Further, since the heliocentric longitudes are increasing, the motion is *direct*.

The longitude of the ascending node and the inclination of the orbit may now be found by means of the equations (74), (75), or (76); and we get

$$\Omega = 304^\circ 43' 11''.5, \quad i = 64^\circ 31' 21''.7.$$

The values of u and u'' are given by the formulæ

$$\tan u = \frac{\tan(l - \Omega)}{\cos i}, \quad \tan u'' = \frac{\tan(l'' - \Omega)}{\cos i},$$

u and $l - \Omega$ being in the same quadrant in the case of direct motion. Thus we obtain

$$u = 142^\circ 52' 12''.4, \quad u'' = 153^\circ 18' 49''.4.$$

Then the equation

$$x^2 = (r'' - r \cos(u'' - u))^2 + r^2 \sin^2(u'' - u)$$

gives

$$\log x = 9.201423,$$

and the agreement of this value of x with that previously found, proves the calculation of Ω , i , u , and u'' .

From the equations

$$\tan(45^\circ + \theta') = \sqrt[4]{\frac{r''}{r}},$$

$$\frac{1}{\sqrt{q}} \sin \frac{1}{2} \left(\frac{1}{2} (u'' + u) - \omega \right) = \frac{\tan 2\theta'}{\sin \frac{1}{4} (u'' - u) \sqrt[4]{rr''}},$$

$$\frac{1}{\sqrt{q}} \cos \frac{1}{2} \left(\frac{1}{2} (u'' + u) - \omega \right) = \frac{\sec 2\theta'}{\cos \frac{1}{4} (u'' - u) \sqrt[4]{rr''}},$$

we get

$$\theta' = 0^\circ 22' 47''.4, \quad \omega = 115^\circ 40' 6''.3, \quad \log q = 9.887378.$$

Hence we have

$$\pi = \omega + \Omega = 60^\circ 23' 17''.8,$$

and

$$u = u - \omega = 27^\circ 12' 6''.1, \quad v'' = u'' - \omega = 37^\circ 38' 43''.1.$$

Then we obtain

$$\log m = 9.9601277 - \frac{3}{2} \log q = 0.129061,$$

and, corresponding to the values of v and v'' , Table VI. gives

$$\log M = 1.267163, \quad \log M'' = 1.424152.$$

Therefore, for the time of perihelion passage, we have

$$T = t - \frac{M}{m} = t - 13.74364,$$

and

$$T = t'' - \frac{M''}{m} = t'' - 19.72836.$$

The first value gives $T = 1863$ Dec. 27.56473, and the second gives $T = 1863$ Dec. 27.56463. The agreement between these results is the final proof of the calculation of the elements from the adopted value of $M = \frac{\rho''}{\rho}$.

If we find T by means of Table VIII., we have

$$\log N = 0.021616, \quad \log N'' = 0.018210,$$

and the equation

$$T = t - \frac{2}{3k} N r^{\frac{3}{2}} \sin v = t'' - \frac{2}{3k} N'' r''^{\frac{3}{2}} \sin v'',$$

in which $\log \frac{2}{3k} = 1.5883273$, gives for T the values Dec. 27.56473 and Dec. 27.56469.

Collecting together the several results obtained, we have the following elements:

$$\begin{array}{l} T = 1863 \text{ Dec. } 27.56471 \text{ Washington mean time.} \\ \left. \begin{array}{l} \pi = 60^\circ 23' 17''.8 \\ \Omega = 304 \quad 43 \quad 11 \quad .5 \\ i = 64 \quad 31 \quad 21 \quad .7 \end{array} \right\} \begin{array}{l} \text{Ecliptic and Mean} \\ \text{Equinox } 1864.0, \end{array} \\ \log q = 9.887378. \\ \text{Motion Direct.} \end{array}$$

73. The elements thus derived will, in all cases, exactly represent the extreme places of the comet, since these only have been used in finding the elements after ρ and ρ'' have been found. If, by means

of these elements, we compute n and n'' , and correct the value of M , the elements which will then be obtained will approximate nearer the true values; and each successive correction will furnish more accurate results. When the adopted value of M is exact, the resulting elements must by calculation reproduce this value, and since the computed values of λ , λ'' , β , and β'' will be the same as the observed values, the computed values of λ' and β' must be such that when substituted in the equation for M , the same result will be obtained as when the observed values of λ' and β' are used. But, according to the equations (13) and (14), the value of M depends only on the inclination to the ecliptic of a great circle passing through the places of the sun and comet for the time t' , and is independent of the angle at the earth between the sun and comet. Hence, the spherical co-ordinates of any point of the great circle joining these places of the sun and comet will, in connection with those of the extreme places, give the same value of M , and when the exact value of M has been used in deriving the elements, the computed values of λ' and β' must give the same value for w' as that which is obtained from observation. But if we represent by ψ' the angle at the earth between the sun and comet at the time t' , the values of ψ' derived by observation and by computation from the elements will differ, unless the middle place is exactly represented. In general, this difference will be small, and since w' is constant, the equations

$$\begin{aligned}\cos \psi' &= \cos \beta' \cos (\lambda' - \odot'), \\ \sin \psi' \cos w' &= \cos \beta' \sin (\lambda' - \odot'), \\ \sin \psi' \sin w' &= \sin \beta',\end{aligned}\tag{93}$$

give, by differentiation,

$$\begin{aligned}\cos \beta' d\lambda' &= \cos w' \sec \beta' d\psi', \\ d\beta' &= \sin w' \cos (\lambda' - \odot') d\psi' .\end{aligned}\tag{94}$$

From these we get

$$\frac{\cos \beta' d\lambda'}{d\beta'} = \frac{\tan (\lambda' - \odot')}{\sin \beta'},$$

which expresses the ratio of the residual errors in longitude and latitude, for the middle place, when the correct value of M has been used.

Whenever these conditions are satisfied, the elements will be correct on the hypothesis of parabolic motion, and the magnitude of the final residuals in the middle place will depend on the deviation of the actual orbit of the comet from the parabolic form. Further,

when elements have been derived from a value of M which has not been finally corrected, if we compute λ' and β' by means of these elements, and then

$$\tan w' = \frac{\tan \beta'}{\sin (\lambda' - \odot')}, \quad (95)$$

the comparison of this value of $\tan w'$ with that given by observation will show whether any further correction of M is necessary, and if the difference is not greater than what may be due to unavoidable errors of calculation, we may regard M as exact.

To compare the elements obtained in the case of the example given with the middle place, we find

$$v' = 32^\circ 31' 13''.5, \quad u' = 148^\circ 11' 19''.8, \quad \log r' = 9.922836.$$

Then from the equations

$$\begin{aligned} \tan (\ell' - \Omega) &= \cos i \tan u', \\ \tan b' &= \tan i \sin (\ell' - \Omega), \end{aligned}$$

we derive

$$\ell' = 109^\circ 46' 48''.3, \quad b' = 28^\circ 24' 56''.0.$$

By means of these and the values of \odot' and R' , we obtain

$$\lambda' = 302^\circ 57' 41''.1, \quad \beta' = 57^\circ 39' 37''.0;$$

and, comparing these results with the observed values of λ' and β' , the residuals for the middle place are found to be

$$\begin{array}{c} \text{Comp. — Obs.} \\ \cos \beta' \Delta \lambda' = + 3''.6, \quad \Delta \beta = + 1''.1. \end{array}$$

The ratio of these remaining errors, after making due allowance for unavoidable errors of calculation, shows that the adopted value of M is not exact, since the error of the longitude should be less than that of the latitude.

The value of w' given by observation is

$$\log \tan w' = 0.966314,$$

and that given by the computed values of λ' and β' is

$$\log \tan w' = 0.966247.$$

The difference being greater than what can be attributed to errors of calculation, it appears that the value of M requires further cor-

rection. Since the difference is small, we may derive the correct value of M by using the same assumed value of $\frac{n}{n''}$, and, instead of the value of $\tan w'$ derived from observation, a value differing as much from this in a contrary direction as the computed value differs. Thus, in the present example, the computed value of $\log \tan w'$ is 0.000067 less than the observed value, and, in finding the new value of M , we must use

$$\log \tan w' = 0.966381$$

in computing β_0 and β_0'' involved in the first of equations (14). If the first of equations (10) is employed, we must use, instead of $\tan \beta'$ as derived from observation,

$$\tan \beta' = \tan w' \sin (\lambda' - \odot'),$$

or

$$\log \tan \beta' = 0.966381 + \log \sin (\lambda' - \odot') = 0.198559,$$

the observed value of λ' being retained. Thus we derive

$$\log M = 9.829586,$$

and if the elements of the orbit are computed by means of this value, they will represent the middle place in accordance with the condition that the difference between the computed and the observed value of $\tan w'$ shall be zero.

A system of elements computed with the same data from $\log M = 9.822906$ gives for the error of the middle place,

C. — O.

$$\cos \beta' \Delta \lambda' = -1' 26''.2, \quad \Delta \beta' = -40''.1.$$

If we interpolate by means of the residuals thus found for two values of M , it appears that a system of elements computed from

$$\log M = 9.829586$$

will almost exactly represent the middle place, so that the data are completely satisfied by the hypothesis of parabolic motion.

The equations (34) and (32) give

$$\log \frac{n}{n''} = 0.006955, \quad \log \frac{N}{N''} = 0.006831,$$

and from (10) we get

$$\log M' = 9.822906, \quad \log M'' = 9.663729.$$

Then by means of the equation (33) we derive, for the corrected value of M ,

$$\log M = 9.829582,$$

which differs only in the sixth decimal place from the result obtained by varying $\tan w'$ and retaining the approximate values $\frac{n}{n''} = \frac{\tau}{\tau''} = \frac{N}{N''}$.

74. When the approximate elements of the orbit of a comet are known, they may be corrected by using observations which include a longer interval of time. The most convenient method of effecting this correction is by the variation of the geocentric distance for the time of one of the extreme observations, and the formulæ which may be derived for this purpose are applicable, without modification, to any case in which it is possible to determine the elements of the orbit of a comet on the supposition of motion in a parabola. Since there are only five elements to be determined in the case of parabolic motion, if the distance of the comet from the earth corresponding to the time of one complete observation is known, one additional complete observation will enable us to find the elements of the orbit. Therefore, if the elements are computed which result from two or more assumed values of A differing but little from the correct value, by comparison of intermediate observations with these different systems of elements, we may derive that value of the geocentric distance of the comet for which the resulting elements will best represent the observations.

In order that the formulæ may be applicable to the case of any fundamental plane, let us consider the equator as this plane, and, supposing the data to be three complete observations, let A, A', A'' be the right ascensions, and D, D', D'' the declinations of the sun for the times t, t', t'' . The co-ordinates of the first place of the earth referred to the third are

$$\begin{aligned} x &= R'' \cos D'' \cos A'' - R \cos D \cos A, \\ y &= R'' \cos D'' \sin A'' - R \cos D \sin A, \\ z &= R'' \sin D'' \quad \quad \quad - R \sin D. \end{aligned}$$

If we represent by g the chord of the earth's orbit between the places for the first and third observations, and by G and K , respectively, the right ascension and declination of the first place of the earth as seen from the third, we shall have

$$\begin{aligned} x &= g \cos K \cos G, \\ y &= g \cos K \sin G, \\ z &= g \sin K. \end{aligned}$$

and, consequently,

$$\begin{aligned} g \cos K \cos (G - A) &= R'' \cos D'' \cos (A'' - A) - R \cos D, \\ g \cos K \sin (G - A) &= R'' \cos D'' \sin (A'' - A), \\ g \sin K &= R'' \sin D'' - R \sin D, \end{aligned} \quad (96)$$

from which g , K , and G may be found.

If we designate by x , y , z , the co-ordinates of the first place of the comet referred to the third place of the earth, we shall have

$$\begin{aligned} x &= \Delta \cos \delta \cos \alpha + g \cos K \cos G, \\ y &= \Delta \cos \delta \sin \alpha + g \cos K \sin G, \\ z &= \Delta \sin \delta + g \sin K. \end{aligned}$$

Let us now put

$$\begin{aligned} x &= h' \cos \zeta' \cos H', \\ y &= h' \cos \zeta' \sin H', \\ z &= h' \sin \zeta', \end{aligned}$$

and we get

$$\begin{aligned} h' \cos \zeta' \cos (H' - G) &= \Delta \cos \delta \cos (\alpha - G) + g \cos K, \\ h' \cos \zeta' \sin (H' - G) &= \Delta \cos \delta \sin (\alpha - G), \\ h' \sin \zeta' &= \Delta \sin \delta + g \sin K, \end{aligned} \quad (97)$$

from which to determine H' , ζ' , and h' .

If we represent by ϕ' the angle at the third place of the earth between the actual first and third places of the comet in space, we obtain

$$\cos \phi' = \cos \zeta' \cos H' \cos \delta'' \cos \alpha'' + \cos \zeta' \sin H' \cos \delta'' \sin \alpha'' + \sin \zeta' \sin \delta'',$$

or

$$\cos \phi' = \cos \zeta' \cos \delta'' \cos (\alpha'' - H') + \sin \zeta' \sin \delta''; \quad (98)$$

and if we put

$$\begin{aligned} e \sin f &= \sin \delta'', \\ e \cos f &= \cos \delta'' \cos (\alpha'' - H') \end{aligned}$$

this becomes

$$\cos \phi' = e \cos (\zeta' - f). \quad (99)$$

Then we shall have

$$\kappa^2 = h'^2 + \Delta'^2 - 2h'\Delta' \cos \phi'$$

or

$$\kappa^2 = (\Delta' - h' \cos \phi')^2 + h'^2 \sin^2 \phi', \quad (100)$$

in which Δ' is the distance of the comet from the earth corresponding to the last observation. We have, also, from equations (44) and (45),

$$\begin{aligned} r^2 &= (\Delta - R \cos \psi)^2 + R^2 \sin^2 \psi, \\ r'^2 &= (\Delta' - R' \cos \psi')^2 + R'^2 \sin^2 \psi', \end{aligned} \quad (101)$$

in which ψ is the angle at the earth between the sun and comet at the time t , and ψ'' the same angle at the time t'' . To find their values, we have

$$\begin{aligned}\cos \psi &= \cos D \cos \delta \cos (\alpha - A) + \sin D \sin \delta, \\ \cos \psi'' &= \cos D'' \cos \delta'' \cos (\alpha'' - A'') + \sin D'' \sin \delta'',\end{aligned}\quad (102)$$

which may be still further reduced by the introduction of auxiliary angles as in the case of equation (98).

Let us now put

$$\begin{aligned}h' \sin \varphi' &= C, & h' \cos \varphi' &= c, \\ R \sin \psi &= B, & R \cos \psi &= b, \\ R'' \sin \psi'' &= B'', & R'' \cos \psi'' &= b'',\end{aligned}\quad (103)$$

and we shall have

$$\begin{aligned}\kappa &= \sqrt{(\Delta'' - c)^2 + C^2}, \\ r &= \sqrt{(\Delta - b)^2 + B^2}, \\ r'' &= \sqrt{(\Delta'' - b'')^2 + B''^2}.\end{aligned}\quad (104)$$

These equations, together with (56), will enable us to determine Δ'' by successive trials when Δ is given.

We may, therefore, assume an approximate value of Δ'' by means of the approximate elements known, and find r'' from the last of these equations, the value of r having been already found from the assumed value of Δ . Then κ is obtained from the equation

$$\kappa = \frac{2c'}{\sqrt{r + r''}} \mu,$$

μ being found by means of Table XI., and a second approximation to the value of Δ'' from

$$\Delta'' = c \pm \sqrt{\kappa^2 - C^2}.\quad (105)$$

The approximate elements will give Δ'' near enough to show whether the upper or lower sign must be used. With the value of Δ'' thus found we recompute r'' and κ as before, and in a similar manner find a still closer approximation to the correct value of Δ'' . A few trials will generally give the correct result.

When Δ'' has thus been determined, the heliocentric places are found by means of the formulæ

$$\begin{aligned}r \cos b \cos (l - A) &= \Delta \cos \delta \cos (\alpha - A) - R \cos D, \\ r \cos b \sin (l - A) &= \Delta \cos \delta \sin (\alpha - A), \\ r \sin b &= \Delta \sin \delta - R \sin D;\end{aligned}\quad (106)$$

$$\begin{aligned} r'' \cos b'' \cos (l'' - A'') &= \Delta'' \cos \delta'' \cos (\alpha'' - A'') - R'' \cos D'', \\ r'' \cos b'' \sin (l'' - A'') &= \Delta'' \cos \delta'' \sin (\alpha'' - A''), \\ r'' \sin b'' &= \Delta'' \sin \delta'' - R'' \sin D'', \end{aligned} \quad (107)$$

in which b , b'' , l , l'' are the heliocentric spherical co-ordinates referred to the equator as the fundamental plane. The values of r and r'' found from these equations must agree with those obtained from (104).

The elements of the orbit may now be determined by means of the equations (75), (77), and (81), in connection with Tables VI. and VIII., as already explained. The elements thus derived will be referred to the equator, or to a plane passing through the centre of the sun and parallel to the earth's equator, and they may be transformed into those for the ecliptic as the fundamental plane by means of the equations (109)₁.

75. With the resulting elements we compute the place of the comet for the time t' and compare it with the corresponding observed place, and if we denote the computed right ascension and declination by α'_0 and δ'_0 , respectively, we shall have

$$\alpha' + \alpha' = \alpha'_0, \quad \delta' + d' = \delta'_0,$$

in which α' and d' denote the differences between computation and observation. Next we assume a second value of Δ , which we represent by $\Delta + \delta\Delta$, and compute the corresponding system of elements. Then we have

$$\alpha' + \alpha'' = \alpha'_0, \quad \delta' + d'' = \delta'_0,$$

α'' and d'' denoting the differences between computation and observation for the second system of elements. We also compute a third system of elements with the distance $\Delta - \delta\Delta$, and denote the differences between computation and observation by a and d ; then we shall have

$$a = f(\Delta - \delta\Delta), \quad \alpha' = f(\Delta), \quad \alpha'' = f(\Delta + \delta\Delta),$$

and similarly for d , d' , and d'' . If these three numbers are exactly represented by the expression

$$m + n \frac{x}{\delta\Delta} + o\left(\frac{x}{\delta\Delta}\right)^2,$$

in which $\Delta + x$ is the general value of the argument, since the values of a , α' , and α'' will be such that the third differences may be neglected, this formula may be assumed to express exactly any value of the function corresponding to a value of the argument not differing

much from Δ , or within the limits $x = -\delta\Delta$ and $x = +\delta\Delta$, the assumed values $\Delta - \delta\Delta$, Δ , and $\Delta + \delta\Delta$ being so taken that the correct value of Δ shall be either within these limits or very nearly so.

To find the coefficients m , n , and o , we have

$$m - n + o = a, \quad m = a', \quad m + n + o = a'',$$

whence

$$m = a', \quad n = \frac{1}{2}(a'' - a), \quad o = \frac{1}{2}(a'' + a) - a'.$$

Now, in order that the middle place may be exactly represented in right ascension, we must have

$$o\left(\frac{x}{\delta\Delta}\right)^2 + n\left(\frac{x}{\delta\Delta}\right) + m = 0,$$

from which we find

$$\frac{x}{\delta\Delta} = -\frac{1}{2o}(n - \sqrt{n^2 - 4mo}) = p,$$

or

$$x - p\delta\Delta = 0.$$

In the same manner, the condition that the middle place shall be exactly represented in declination, gives

$$x - p'\delta\Delta = 0.$$

In order that the orbit shall exactly represent the middle place, both conditions must be satisfied simultaneously; but it will rarely happen that this can be effected, and the correct value of x must be found from those obtained by the separate conditions. The arithmetical mean of the two values of x will not make the sum of the squares of the residuals a minimum, and, therefore, give the most probable value, unless the variation of $\cos \delta' \Delta \alpha'$, for a given increment assigned to Δ , is the same as that of $\Delta \delta'$. But if we denote the value of x for which the error in α' is reduced to zero by x' , and that for which $\Delta \delta' = 0$, by x'' , the most probable value of x will be

$$x = \frac{n^1 x' + n'^2 x''}{n^2 + n'^2}, \quad (108)$$

in which $n = \frac{1}{2}(a'' - a)$ and $n' = \frac{1}{2}(d'' - d)$. It should be observed that, in order that the differences in right ascension and declination shall have equal influence in determining the value of x , the values of a , a' , and a'' must be multiplied by $\cos \delta'$. The value of $\delta\Delta$ is most conveniently expressed in units of the last decimal place of the logarithms employed.

If the elements are already known so approximately that the first assumed value of Δ differs so little from the true value that the second differences of the residuals may be neglected, two assumptions in regard to the value of Δ will suffice. Then we shall have $o = 0$, and hence

$$m = a', \quad n = a'' - a'.$$

The condition that the middle place shall be exactly represented, gives the two equations

$$\begin{aligned} (a'' - a')x + a'\delta\Delta &= 0, \\ (d'' - d')x + d'\delta\Delta &= 0. \end{aligned} \quad (109)$$

The combination of these equations according to the method of least squares will give the most probable value of x , namely, that for which the sum of the squares of the residuals will be a minimum.

Having thus determined the most probable value of x , a final system of elements computed with the geocentric distance $\Delta + x$, corresponding to the time t , will represent the extreme places exactly, and will give the least residuals in the middle place consistent with the supposition of parabolic motion. It is further evident that we may use any number of intermediate places to correct the assumed value of Δ , each of which will furnish two equations of condition for the determination of x , and thus the elements may be found which will represent a series of observations.

76. EXAMPLE.—The formulæ thus derived for the correction of approximate parabolic elements by varying the geocentric distance, are applicable to the case of any fundamental plane, provided that α , δ , A , D , &c. have the same signification with respect to this plane that they have in reference to the equator. To illustrate their numerical application, let us take the following normal places of the Great Comet of 1858, which were derived by comparing an ephemeris with several observations made during a few days before and after the date of each normal, and finding the mean difference between computation and observation:

| Washington M. T. | α | δ |
|------------------|----------------|------------------|
| 1858 June 11.0 | 141° 18' 30".9 | + 24° 46' 25".4, |
| July 13.0 | 144 32 49 .7 | 27 48 0 .8, |
| Aug. 14.0 | 152 14 12 .0 | + 31 21 47 .9, |

which are referred to the apparent equinox of the date. These places are free from aberration.

We shall take the ecliptic for the fundamental plane, and converting these right ascensions and declinations into longitudes and latitudes, and reducing to the ecliptic and mean equinox of 1858.0, the times of observation being expressed in days from the beginning of the year, we get

$$\begin{array}{lll} t = 162.0, & \lambda = 135^\circ 51' 44''.2, & \beta = + 9^\circ 6' 57''.8, \\ t' = 194.0, & \lambda' = 137 39 41 .2, & \beta' = 12 55 9 .0, \\ t'' = 226.0, & \lambda'' = 142 51 31 .8, & \beta'' = + 18 36 28 .7. \end{array}$$

From the *American Nautical Almanac* we obtain, for the true places of the sun,

$$\begin{array}{ll} \odot = 80^\circ 24' 32''.4, & \log R = 0.006774, \\ \odot' = 110 55 51 .2, & \log R' = 0.007101, \\ \odot'' = 141 33 2 .0, & \log R'' = 0.005405, \end{array}$$

the longitudes being referred to the mean equinox 1858.0.

When the ecliptic is the fundamental plane, we have, neglecting the sun's latitude, $D = 0$, and we must write λ and β in place of α and δ , and \odot in place of A , in the equations which have been derived for the equator as the fundamental plane. Therefore, we have

$$\begin{aligned} g \cos (G - \odot) &= R'' \cos (\odot'' - \odot) - R, \\ g \sin (G - \odot) &= R'' \sin (\odot'' - \odot); \\ \cos \psi &= \cos \beta \cos (\lambda - \odot), & \cos \psi'' &= \cos \beta'' \cos (\lambda'' - \odot'') \\ R \cos \psi &= b, & R'' \cos \psi'' &= b'', \\ R \sin \psi &= B, & R'' \sin \psi'' &= B'', \end{aligned}$$

from which to find G , g , b , B , b'' , and B'' , all of which remain unchanged in the successive trials with assumed values of Δ . Thus we obtain

$$\begin{array}{lll} G = 201^\circ 7' 57''.4, & \log B = 9.925092, & b = + 0.568719, \\ \log g = 0.013500, & \log B'' = 9.510309, & b'' = + 0.959342. \end{array}$$

Then we assume, by means of approximate elements already known,

$$\log \Delta = 0.397800,$$

and from

$$\begin{aligned} h' \cos \zeta' \cos (H' - G) &= \Delta \cos \beta \cos (\lambda - G) + g, \\ h' \cos \zeta' \sin (H' - G) &= \Delta \cos \beta \sin (\lambda - G), \\ h' \sin \zeta' &= \Delta \sin \beta, \end{aligned}$$

we find H' , ζ' , and h' . These give

$$H' = 153^\circ 46' 20''.5, \quad \zeta' = + 7^\circ 24' 16''.4, \quad \log h' = 0.487484.$$

Next, from

$$\begin{aligned}\cos \varphi' &= \cos \zeta' \cos \beta'' \cos (\lambda'' - H') + \sin \zeta' \sin \beta'', \\ h' \cos \varphi' &= c, & h' \sin \varphi' &= C,\end{aligned}$$

we get

$$\log C = 9.912519, \quad c = + 2.961673;$$

and from

$$r = \sqrt{(\Delta - b)^2 + B^2},$$

we find

$$\log r = 0.323446.$$

Then we have

$$\begin{aligned}\Delta'' &= c \pm \sqrt{x^2 - C^2}, & r'' &= \sqrt{(\Delta'' - b'')^2 + B''^2}, \\ \tau' &= k(t' - t), & \eta &= \frac{2\tau'}{(r + r'')^{\frac{3}{2}}}, & \kappa &= \frac{2\tau'}{\sqrt{r + r''}} \mu,\end{aligned}$$

from which to find Δ'' , r'' , and κ . First, by means of the approximate elements, we assume

$$\log \Delta'' = 0.310000,$$

which gives $\log r'' = 0.053000$, and hence we have

$$\eta = 0.3783, \quad \log \mu = 0.002706, \quad \log \kappa = 0.090511.$$

With this value of κ we obtain from the expression for Δ'' , the lower sign being used, since Δ'' is less than c ,

$$\log \Delta'' = 0.309717.$$

Repeating the calculation of r'' , μ , and κ , and then finding Δ'' again, the result is

$$\log \Delta'' = 0.309647.$$

Then, by means of the formula (67), we may find the correct value. Thus we have, in units of the sixth decimal place,

$$a = 309717 - 310000 = -283, \quad a' = 309647 - 309717 = -70,$$

and for the correction to the last result for $\log \Delta''$ we have

$$-\frac{a'^2}{a' - a} = -23.$$

Therefore,

$$\log \Delta'' = 0.309624.$$

By means of this value we get

$$\log r'' = 0.052350, \quad \log \kappa = 0.090628,$$

and this value of κ gives, finally,

$$\log \Delta'' = 0.309623, \quad \log r'' = 0.052348.$$

The heliocentric places of the comet are now found from the equations (71) and (72), writing $\Delta \cos \beta$ and $\Delta'' \cos \beta''$ for ρ and ρ'' , respectively. Thus we obtain

$$\begin{aligned} l &= 159^\circ 43' 14''.2, & b &= +10^\circ 50' 14''.0, & \log r &= 0.323447, \\ l'' &= 144^\circ 17' 47''.8, & b'' &= +35^\circ 14' 28''.7, & \log r'' &= 0.052347. \end{aligned}$$

The agreement of these results for r and r'' with those already obtained, proves the accuracy of the calculation. Since the heliocentric longitudes are diminishing, the motion is *retrograde*.

Then from (74) we get

$$\Omega = 165^\circ 17' 30''.3, \quad i = 63^\circ 6' 32''.5;$$

and from

$$\tan u = -\frac{\tan(l - \Omega)}{\cos i}, \quad \tan u'' = -\frac{\tan(l'' - \Omega)}{\cos i},$$

we obtain

$$u = 12^\circ 10' 12''.6, \quad u'' = 40^\circ 18' 51''.2,$$

the values of $-u$ and $l - \Omega$ being in the same quadrant when the motion is retrograde. The equation (79) gives $\log \kappa = 0.090630$, which agrees with the value already found.

The formulæ (81) give

$$\omega = 129^\circ 6' 46''.3, \quad \log q = 9.760326,$$

and hence we have

$$v = u - \omega = -116^\circ 56' 33''.7, \quad v'' = u'' - \omega = -88^\circ 47' 55''.1,$$

from which we get

$$T = 1858 \text{ Sept. } 29.4274.$$

From these elements we find

$$\log r' = 0.212844, \quad v' = -107^\circ 7' 34''.0, \quad u' = 21^\circ 59' 12''.3,$$

and from

$$\begin{aligned} \tan(l' - \Omega) &= -\cos i \tan u', \\ \tan b' &= -\tan i \sin(l' - \Omega), \end{aligned}$$

we get

$$l' = 154^\circ 56' 33''.4, \quad b' = +19^\circ 30' 22''.1.$$

By means of these and the values of \odot' and R' , we obtain

$$\lambda' = 137^\circ 39' 13''.3, \quad \beta' = +12^\circ 54' 45''.3,$$

and comparing these results with observation, we have, for the error of the middle place,

$$\begin{array}{c} \text{C. — O.} \\ \cos \beta' \Delta \lambda' = -27''.2, \quad \Delta \beta' = -23''.7. \end{array}$$

From the relative positions of the sun, earth, and comet at the time t'' it is easily seen that, in order to diminish these residuals, the geocentric distance must be increased, and therefore we assume, for a second value of Δ ,

$$\log \Delta = 0.398500,$$

from which we derive

$$\begin{array}{lll} H' = 153^\circ 44' 57''.6, & \zeta' = +7^\circ 24' 26''.1, & \log h' = 0.488026, \\ \log C = 9.912587, & \log c = 0.472115, & \log r = 0.324207, \\ \log \Delta'' = 0.311054, & \log r'' = 0.054824, & \log \kappa = 0.089922 \end{array}$$

Then we find the heliocentric places

$$\begin{array}{lll} l = 159^\circ 40' 33''.8, & b = +10^\circ 50' 8''.6, & \log r = 0.324207, \\ l'' = 144 \ 17 \ 12 \ .1, & b'' = +35 \ 8 \ 37 \ .8, & \log r'' = 0.054825, \end{array}$$

and from these,

$$\begin{array}{ll} \Omega = 165^\circ 15' 41''.1, & i = 63^\circ 2' 49''.2, \\ u = 12 \ 10 \ 30 \ .8, & u'' = 40 \ 13 \ 26 \ .0, \\ \omega = 128 \ 54 \ 44 \ .4, & \log q = 9.763620, \\ T = 1858 \text{ Sept. } 29.8245, & \log r' = 0.214116, \\ v' = -106^\circ 55' 43''.8, & u' = 21^\circ 59' 0''.6, \\ l' = 154 \ 53 \ 32 \ .3, & b' = +19 \ 29 \ 31 \ .9, \\ \lambda' = 137 \ 39 \ 39 \ .7, & \beta' = +12 \ 55 \ 2 \ .9. \end{array}$$

Therefore, for the second assumed value of Δ , we have

$$\begin{array}{c} \text{C. — O.} \\ \cos \beta' \Delta \lambda' = -1''.5, \quad \Delta \beta' = -6''.1. \end{array}$$

Since these residuals are very small, it will not be necessary to make a third assumption in regard to Δ , but we may at once derive the correction to be applied to the last assumed value by means of the equations (109). Thus we have

$$\begin{array}{llll} a' = -1.5, & a'' = -27.2, & d' = -6.1, & d'' = -23.7, \\ & \delta \log \Delta = -0.000700, & & \end{array}$$

and, expressing $\delta \log A$ in units of the sixth decimal place, these equations give

$$25.7x - 1050 = 0.$$

$$17.6x - 4270 = 0.$$

Combining these according to the method of least squares, we get

$$x = \frac{105 \times 2.57 + 427 \times 1.76}{(2.57)^2 + (1.76)^2} = +106.$$

Hence the corrected value of $\log A$ is

$$\log A = 0.398500 + 0.000106 = 0.398606.$$

With this value of $\log A$ the final elements are computed as already illustrated, and the following system is obtained:—

$$T = 1858 \text{ Sept. } 29.88617 \text{ Washington mean time.}$$

$$\left. \begin{array}{l} \pi = 36^\circ 22' 36''.9 \\ \Omega = 165 \quad 15 \quad 24 \quad .8 \\ i = 63 \quad 2 \quad 14 \quad .2 \end{array} \right\} \text{Mean Equinox } 1858.0.$$

$$\log q = 9.764142$$

Motion Retrograde.

If the distinction of retrograde motion is not adopted, and we regard i as susceptible of any value from 0° to 180° , we shall have

$$\pi = 294^\circ 8' 12''.7,$$

$$i = 116 \quad 57 \quad 45 \quad .8,$$

the other elements remaining the same.

The comparison of the middle place with these final elements gives the following residuals:—

$$C. - O.$$

$$\cos \beta \Delta \lambda = + 0''.2,$$

$$\Delta \beta = - 4''.3.$$

These errors are so small that the orbit indicated by the observed places on which the elements are based differs very little from a parabola.

When, instead of a single place, a series of intermediate places is employed to correct the assumed value of A , it is best to adopt the equator as the fundamental plane, since an error in α or δ will affect both λ and β ; and, besides, incomplete observations may also be used

when the fundamental plane is that to which the observations are directly referred. Further, the entire group of equations of condition for the determination of x , according to the formulæ (109), must be combined by multiplying each equation by the coefficient of x in that equation and taking the sum of all the equations thus formed as the final equation from which to find x , the observations being supposed equally good.

CHAPTER IV.

DETERMINATION, FROM THREE COMPLETE OBSERVATIONS, OF THE ELEMENTS OF THE ORBIT OF A HEAVENLY BODY, INCLUDING THE ECCENTRICITY OR FORM OF THE CONIC SECTION.

77. THE formulæ which have thus far been derived for the determination of the elements of the orbit of a heavenly body by means of observed places, do not suffice, in the form in which they have been given, to determine an orbit entirely unknown, except in the particular case of parabolic motion, for which one of the elements becomes known. In the general case, it is necessary to derive at least one of the curtate distances without making any assumption as to the form of the orbit, after which the others may be found. But, preliminary to a complete investigation of the elements of an unknown orbit by means of three complete observations of the body, it is necessary to provide for the corrections due to parallax and aberration, so that they may be applied in as advantageous a manner as possible.

When the elements are entirely unknown, we cannot correct the observed places directly for parallax and aberration, since both of these corrections require a knowledge of the distance of the body from the earth. But in the case of the aberration we may either correct the time of observation for the time in which the light from the body reaches the earth, or we may consider the observed place corrected for the actual aberration due to the combined motion of the earth and of light as the true place at the instant when the light left the planet or comet, but as seen from the place which the earth occupies at the time of the observation. When the distance is unknown, the latter method must evidently be adopted, according to which we apply to the observed apparent longitude and latitude the actual aberration of the fixed stars, and regard this place as corresponding to the time of observation corrected for the time of aberration, to be effected when the distances shall have been found, but using for the place of the earth that corresponding to the time of observation. It will appear, therefore, that only that part of the calculation of the

elements which involves the times of observation will have to be repeated after the corresponding distances of the body from the earth have been found. First, then, by means of the apparent obliquity of the ecliptic, the observed apparent right ascension and declination must be converted into apparent longitude and latitude. Let λ_0 and β_0 , respectively, denote the observed apparent longitude and latitude; and let \odot_0 be the true longitude of the sun, Σ_0 its latitude, and R_0 its distance from the earth, corresponding to the time of observation. Then, if λ and β denote the longitude and latitude of the planet or comet corrected for the actual aberration of the fixed stars, we shall have

$$\begin{aligned}\lambda - \lambda_0 &= + 20''.445 \cos(\lambda - \odot_0) \sec \beta + 0''.343 \cos(\lambda - 281^\circ) \sec \beta, \\ \beta - \beta_0 &= - 20''.445 \sin(\lambda - \odot_0) \sin \beta - 0''.343 \sin(\lambda - 281^\circ) \sin \beta.\end{aligned}\quad (1)$$

In computing the numerical values of these corrections, it will be sufficiently accurate to use λ_0 and β_0 instead of λ and β in the second members of these equations, and the last terms may, in most cases, be neglected. The values of λ and β thus derived give the true place of the body at the time $t - 497^s.784$, but as seen from the place of the earth at the time t .

When the distance of the planet or comet is unknown, it is impossible to reduce the observed place to the centre of the earth; but if we conceive a line to be drawn from the body through the true place of observation, it is evident that were an observer at the point of intersection of this line with the plane of the ecliptic, or at any point in the line, the body would be seen in the same direction as from the actual place of observation. Hence, instead of applying any correction for parallax directly to the observed apparent place, we may conceive the place of the observer to be changed from the actual place to this point of intersection with the ecliptic, and, therefore, it becomes necessary to determine the position of this point by means of the data furnished by observation.

Let θ_0 be the sidereal time corresponding to the time t_0 of observation, φ' the geocentric latitude of the place of observation, and ρ_0 the radius of the earth at the place of observation, expressed in parts of the equatorial radius as unity. Then θ_0 is the right ascension and φ' the declination of the zenith at the time t_0 . Let l_0 and b_0 denote these quantities converted into longitude and latitude, or the longitude and latitude of the geocentric zenith at the time t_0 . The rectangular co-ordinates of the place of observation referred to the centre of the

earth and expressed in parts of the mean distance of the earth from the sun as the unit, will be

$$\begin{aligned}x_0 &= \rho_0 \sin \pi_0 \cos b_0 \cos l_0, \\y_0 &= \rho_0 \sin \pi_0 \cos b_0 \sin l_0, \\z_0 &= \rho_0 \sin \pi_0 \sin b_0,\end{aligned}$$

in which $\pi_0 = 8''.57116$.

Let Δ_0 be the distance of the planet or comet from the true place of the observer, and Δ , its distance from the point in the ecliptic to which the observation is to be reduced. Then will the co-ordinates of the place of observation, referred to this point in the ecliptic, be

$$\begin{aligned}x, &= (\Delta, - \Delta_0) \cos \beta \cos \lambda, \\y, &= (\Delta, - \Delta_0) \cos \beta \sin \lambda, \\z, &= (\Delta, - \Delta_0) \sin \beta,\end{aligned}$$

the axis of x being directed to the vernal equinox. Let us now designate by \odot the longitude of the sun as seen from the point of reference in the ecliptic, and by R its distance from this point. Then will the heliocentric co-ordinates of this point be

$$\begin{aligned}X &= -R \cos \odot, \\Y &= -R \sin \odot, \\Z &= 0.\end{aligned}$$

The heliocentric co-ordinates of the centre of the earth are

$$\begin{aligned}X_0 &= -R_0 \cos \Sigma_0 \cos \odot_0, \\Y_0 &= -R_0 \cos \Sigma_0 \sin \odot_0, \\Z_0 &= -R_0 \sin \Sigma_0.\end{aligned}$$

But the heliocentric co-ordinates of the true place of observation will be

$$\begin{array}{lll}X + x, & Y + y, & Z + z, \\ \text{or} & & \\X_0 + x_0, & Y_0 + y_0, & Z_0 + z_0,\end{array}$$

and, consequently, we shall have

$$\begin{aligned}R \cos \odot - (\Delta, - \Delta_0) \cos \beta \cos \lambda &= R_0 \cos \Sigma_0 \cos \odot_0 - \rho_0 \sin \pi_0 \cos b_0 \cos l_0, \\R \sin \odot - (\Delta, - \Delta_0) \cos \beta \sin \lambda &= R_0 \cos \Sigma_0 \sin \odot_0 - \rho_0 \sin \pi_0 \cos b_0 \sin l_0, \\-(\Delta, - \Delta_0) \sin \beta &= R_0 \sin \Sigma_0 - \rho_0 \sin \pi_0 \sin b_0.\end{aligned}$$

If we suppose the axis of x to be directed to the point whose longitude is \odot_0 , these become

$$\begin{aligned}
 R \cos (\odot - \odot_0) - (\Delta, - \Delta_0) \cos \beta \cos (\lambda - \odot_0) = \\
 R_0 \cos \Sigma_0 - \rho_0 \sin \pi_0 \cos b_0 \cos (l_0 - \odot_0), \\
 R \sin (\odot - \odot_0) - (\Delta, - \Delta_0) \cos \beta \sin (\lambda - \odot_0) = \\
 - \rho_0 \sin \pi_0 \cos b_0 \sin (l_0 - \odot_0), \\
 - (\Delta, - \Delta_0) \sin \beta = R_0 \sin \Sigma_0 - \rho_0 \sin \pi_0 \sin b_0,
 \end{aligned} \tag{2}$$

from which R and \odot may be determined. Let us now put

$$(\Delta, - \Delta_0) \cos \beta = D; \tag{3}$$

then, since π_0 , Σ_0 , and $\odot - \odot_0$ are small, these equations may be reduced to

$$\begin{aligned}
 R &= D \cos (\lambda - \odot_0) - \pi_0 \rho_0 \cos b_0 \cos (l_0 - \odot_0) + R_0, \\
 R (\odot - \odot_0) &= D \sin (\lambda - \odot_0) - \pi_0 \rho_0 \cos b_0 \sin (l_0 - \odot_0), \\
 0 &= D \tan \beta - \pi_0 \rho_0 \sin b_0 + R_0 \Sigma_0.
 \end{aligned}$$

Hence we shall have, if π_0 and Σ_0 are expressed in seconds of arc,

$$\begin{aligned}
 D &= \frac{\pi_0 \rho_0 \sin b_0 - R_0 \Sigma_0}{206264.8} \cot \beta, \\
 R &= R_0 + D \cos (\lambda - \odot_0) - \frac{\pi_0 \rho_0 \cos b_0 \cos (l_0 - \odot_0)}{206264.8}, \\
 \odot &= \odot_0 + \frac{206264.8 D \sin (\lambda - \odot_0) - \pi_0 \rho_0 \cos b_0 \sin (l_0 - \odot_0)}{R},
 \end{aligned} \tag{4}$$

from which we may derive the values of \odot and R which are to be used throughout the calculation of the elements as the longitude and distance of the sun, instead of the corresponding places referred to the centre of the earth. The point of reference being in the plane of the ecliptic, the latitude of the sun as seen from this point is zero, which simplifies some of the equations of the problem, since, if the observations had been reduced to the centre of the earth, the sun's latitude would be retained.

We may remark that the body would not be seen, at the instant of observation, from the point of reference in the direction actually observed, but at a time different from t_0 , to be determined by the interval which is required for the light to pass over the distance $\Delta, - \Delta_0$. Consequently we ought to add to the time of observation the quantity

$$(\Delta, - \Delta_0) 497.78 = 497.78 D \sec \beta, \tag{5}$$

which is called the *reduction of the time*; but unless the latitude of the body should be very small, this correction will be insensible.

The value of λ derived from equations (1) and the longitude \odot

derived from (4) should be reduced by applying the correction for nutation to the mean equinox of the date, and then both these and the latitude β should be reduced by applying the correction for precession to the ecliptic and mean equinox of a fixed epoch, for which the beginning of the year is usually chosen.

In this way each observed apparent longitude and latitude is to be corrected for the aberration of the fixed stars, and the corresponding places of the sun, referred to the point in which the line drawn from the body through the place of observation on the earth's surface intersects the plane of the ecliptic, are derived from the equations (4). Then the places of the sun and of the planet or comet are reduced to the ecliptic and mean equinox of a fixed date, and the results thus obtained, together with the times of observation, furnish the data for the determination of the elements of the orbit.

When the distance of the body corresponding to each of the observations shall have been determined, the times of observation may be corrected for the time of aberration. This correction is necessary, since the adopted places of the body are the true places for the instant when the light was emitted, corresponding respectively to the times of observation diminished by the time of aberration, but as seen from the places of the earth at the actual times of observation, respectively.

When $\beta = 0$, the equations (4) cannot be applied, and when the latitude is so small that the reduction of the time and the correction to be applied to the place of the sun are of considerable magnitude, it will be advisable, if more suitable observations are not available, to neglect the correction for parallax and derive the elements, using the uncorrected places. The distances of the body from the earth which may then be derived, will enable us to apply the correction for parallax directly to the observed places of the body.

When the approximate distances of the body from the earth are already known, and it is required to derive new elements of the orbit from given observed places or from normal places derived from many observations, the observations may be corrected directly for parallax, and the times corrected for the time of aberration. We shall then have the true places of the body as seen from the centre of the earth, and if these places are adopted, it will be necessary, for the most accurate solution possible, to retain the latitude of the sun in the formulæ which may be required. But since some of these formulæ acquire greater simplicity when the sun's latitude is not introduced, if, in this case, we reduce the geocentric places to the

point in which a perpendicular let fall from the centre of the earth to the plane of the ecliptic cuts that plane, the longitude of the sun will remain unchanged, the latitude will be zero, and the distance R will also be unchanged, since the greatest geocentric latitude of the sun does not exceed $1''$. Then the longitude of the planet or comet as seen from this point in the ecliptic will be the same as seen from the centre of the earth, and if Δ , is the distance of the body from this point of reference, and β , its latitude as seen from this point, we shall have

$$\begin{aligned}\Delta \cos \beta &= \Delta \cos \beta, \\ \Delta \sin \beta &= \Delta \sin \beta - R_0 \sin \Sigma_0,\end{aligned}$$

from which we easily derive the correction $\beta, -\beta$, or $\Delta\beta$, to be applied to the geocentric latitude. Thus, we find

$$\Delta\beta = -\frac{R_0 \Sigma_0}{\Delta} \cos \beta, \quad (6)$$

Σ_0 being expressed in seconds. This correction having been applied to the geocentric latitude, the latitude of the sun becomes

$$\Sigma = 0.$$

The correction to be applied to the time of observation (already diminished by the time of aberration) due to the distance $\Delta, -\Delta_0$ will be absolutely insensible, its maximum value not exceeding 0.002 . It should be remarked also that before applying the equation (6), the latitude Σ_0 should be reduced to the fixed ecliptic which it is desired to adopt for the definition of the elements which determine the position of the plane of the orbit.

78. When these preliminary corrections have been applied to the data, we are prepared to proceed with the calculation of the elements of the orbit, the necessary formulæ for which we shall now investigate. For this purpose, let us resume the equations (6)₃; and, if we multiply the first of these equations by $\tan \beta \sin \lambda'' - \tan \beta'' \sin \lambda$, the second by $\tan \beta'' \cos \lambda - \tan \beta \cos \lambda''$, and the third by $\sin(\lambda - \lambda'')$, and add the products, we shall have

$$\begin{aligned}0 &= nR (\tan \beta'' \sin(\lambda - \odot) - \tan \beta \sin(\lambda'' - \odot)) \\ &\quad - \rho' (\tan \beta \sin(\lambda'' - \lambda') - \tan \beta' \sin(\lambda'' - \lambda) + \tan \beta'' \sin(\lambda' - \lambda)) \\ &\quad - R' (\tan \beta'' \sin(\lambda - \odot') - \tan \beta \sin(\lambda'' - \odot')) \\ &\quad + n'' R'' (\tan \beta'' \sin(\lambda - \odot'') - \tan \beta \sin(\lambda'' - \odot'')).\end{aligned} \quad (7)$$

It should be observed that when the correction for parallax is applied

to the place of the sun, ρ' is the projection, on the plane of the ecliptic, of the distance of the body from the point of reference to which the observation has been reduced.

Let us now designate by K the longitude of the ascending node, and by I the inclination to the ecliptic, of a great circle passing through the first and third observed places of the body, and we have

$$\begin{aligned}\tan \beta &= \sin (\lambda - K) \tan I, \\ \tan \beta'' &= \sin (\lambda'' - K) \tan I.\end{aligned}\quad (8)$$

Introducing these values of $\tan \beta$ and $\tan \beta''$ into the equation (7), since

$$\begin{aligned}\sin (\lambda - \odot) \sin (\lambda'' - K) - \sin (\lambda'' - \odot) \sin (\lambda - K) &= \\ &= -\sin (\lambda'' - \lambda) \sin (\odot - K), \\ \sin (\lambda' - \lambda) \sin (\lambda'' - K) + \sin (\lambda'' - \lambda') \sin (\lambda - K) &= \\ &= +\sin (\lambda'' - \lambda) \sin (\lambda' - K), \\ \sin (\lambda - \odot') \sin (\lambda'' - K) - \sin (\lambda'' - \odot') \sin (\lambda - K) &= \\ &= -\sin (\lambda'' - \lambda) \sin (\odot' - K), \\ \sin (\lambda - \odot'') \sin (\lambda'' - K) - \sin (\lambda'' - \odot'') \sin (\lambda - K) &= \\ &= -\sin (\lambda'' - \lambda) \sin (\odot'' - K),\end{aligned}$$

we obtain, by dividing through by $\sin (\lambda'' - \lambda) \tan I$,

$$\begin{aligned}0 &= nR \sin (\odot - K) + \rho' (\sin (\lambda' - K) - \tan \beta' \cot I) \\ &\quad - R' \sin (\odot' - K) + n'' R'' \sin (\odot'' - K).\end{aligned}\quad (9)$$

Let β_0 denote the latitude of that point of the great circle passing through the first and third places which corresponds to the longitude λ' , then

$$\tan \beta_0 = \sin (\lambda' - K) \tan I,$$

and the coefficient of ρ' in equation (9) becomes

$$\frac{\sin (\beta_0 - \beta')}{\cos \beta_0 \cos \beta' \tan I}$$

Therefore, if we put

$$a_0 = \frac{\sin (\beta' - \beta_0)}{\cos \beta_0 \tan I}, \quad (10)$$

we shall have

$$\begin{aligned}\rho' \sec \beta' &= -\frac{R' \sin (\odot' - K)}{a_0} + n \frac{R \sin (\odot - K)}{a_0} \\ &\quad + n'' \frac{R'' \sin (\odot'' - K)}{a_0}\end{aligned}\quad (11)$$

This formula will give the value of ρ' , or of \mathcal{A}' , when the values of n and n'' have been determined, since a_0 and K are derived from the data furnished by observation.

To find K and I , we obtain from equations (8) by a transformation precisely similar to that by which the equations (75)₃ were derived,

$$\begin{aligned}\tan I \sin \left(\frac{1}{2} (\lambda'' + \lambda) - K \right) &= \frac{\sin (\beta'' + \beta)}{2 \cos \beta \cos \beta''} \sec \frac{1}{2} (\lambda'' - \lambda), \\ \tan I \cos \left(\frac{1}{2} (\lambda'' + \lambda) - K \right) &= \frac{\sin (\beta'' - \beta)}{2 \cos \beta \cos \beta''} \operatorname{cosec} \frac{1}{2} (\lambda'' - \lambda).\end{aligned}\quad (12)$$

We may also compute K and I from the equations which may be derived from (74)₃ and (76)₃ by making the necessary changes in the notation, and using only the upper sign, since I is to be taken always less than 90° .

Before proceeding further with the discussion of equation (11), let us derive expressions for ρ and ρ'' in terms of ρ' , the signification of ρ and ρ'' , when the corrections for parallax are applied to the places of the sun, being as already noticed in the case of ρ' .

79. If we multiply the first of equations (6)₃ by $\sin \odot'' \tan \beta''$, the second by $-\cos \odot'' \tan \beta''$, and the third by $\sin (\lambda'' - \odot'')$, and add the products, we get

$$\begin{aligned}0 &= n\rho (\tan \beta'' \sin (\odot'' - \lambda) - \tan \beta \sin (\odot'' - \lambda'')) - nR \tan \beta'' \sin (\odot'' - \odot) \\ &\quad - \rho' (\tan \beta'' \sin (\odot'' - \lambda') - \tan \beta' \sin (\odot'' - \lambda'')) + R' \tan \beta'' \sin (\odot'' - \odot'),\end{aligned}\quad (13)$$

which may be written

$$\begin{aligned}0 &= n\rho (\tan \beta \sin (\lambda'' - \odot'') - \tan \beta'' \sin (\lambda - \odot'')) - nR \tan \beta'' \sin (\odot'' - \odot) \\ &\quad + \rho' (\tan \beta'' \sin (\lambda' - \odot'') - \tan \beta_0 \sin (\lambda'' - \odot'')) \\ &\quad - \rho' (\tan \beta' - \tan \beta_0) \sin (\lambda'' - \odot'') + R' \tan \beta'' \sin (\odot'' - \odot').\end{aligned}$$

Introducing into this the values of $\tan \beta$, $\tan \beta''$, and $\tan \beta_0$ in terms of I and K , and reducing, the result is

$$\begin{aligned}0 &= n\rho \sin (\lambda'' - \lambda) \sin (\odot'' - K) - nR \sin (\odot'' - \odot) \sin (\lambda'' - K) \\ &\quad - \rho' \sin (\lambda'' - \lambda') \sin (\odot'' - K) - \rho' a_0 \sec \beta' \sin (\lambda'' - \odot'') \\ &\quad + R' \sin (\odot'' - \odot') \sin (\lambda'' - K).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\rho &= \frac{\rho'}{n} \left(\frac{\sin (\lambda'' - \lambda')}{\sin (\lambda'' - \lambda)} + \frac{a_0 \sec \beta'}{\sin (\lambda'' - \lambda)} \cdot \frac{\sin (\lambda'' - \odot'')}{\sin (\odot'' - K)} \right) \\ &\quad - \frac{\sin (\lambda'' - K)}{n} \cdot \frac{R' \sin (\odot'' - \odot') - nR \sin (\odot'' - \odot)}{\sin (\lambda'' - \lambda) \sin (\odot'' - K)}.\end{aligned}$$

But, by means of the equations (9)₃, we derive

$$R' \sin (\odot'' - \odot') - nR \sin (\odot'' - \odot) = (N - n) R \sin (\odot'' - \odot),$$

and the preceding equation reduces to

$$\rho = \frac{\rho'}{n} \left(\frac{\sin(\lambda'' - \lambda')}{\sin(\lambda'' - \lambda)} + \frac{a_0 \sec \beta'}{\sin(\lambda'' - \lambda)} \cdot \frac{\sin(\lambda'' - \odot'')}{\sin(\odot'' - K)} \right) + \left(1 - \frac{N}{n} \right) \frac{R \sin(\odot'' - \odot) \sin(\lambda'' - K)}{\sin(\lambda'' - \lambda) \sin(\odot'' - K)}. \quad (14)$$

To obtain an expression for ρ'' in terms of ρ' , if we multiply the first of equations (6)₃ by $\sin \odot \tan \beta$, the second by $-\cos \odot \tan \beta$, and the third by $\sin(\lambda - \odot)$, and add the products, we shall have

$$0 = n'' \rho'' (\tan \beta \sin(\lambda'' - \odot) - \tan \beta' \sin(\lambda - \odot)) - n'' R'' \tan \beta \sin(\odot'' - \odot) - \rho' (\tan \beta \sin(\lambda' - \odot) - \tan \beta' \sin(\lambda - \odot)) + R' \tan \beta \sin(\odot' - \odot). \quad (15)$$

Introducing the values of $\tan \beta$, $\tan \beta'$, and $\tan \beta''$ in terms of K and I , and reducing precisely as in the case of the formula already found for ρ , we obtain

$$\rho'' = \frac{\rho'}{n''} \left(\frac{\sin(\lambda' - \lambda)}{\sin(\lambda'' - \lambda)} - \frac{a_0 \sec \beta'}{\sin(\lambda'' - \lambda)} \cdot \frac{\sin(\lambda - \odot)}{\sin(\odot - K)} \right) + \left(1 - \frac{N''}{n''} \right) \frac{R'' \sin(\odot'' - \odot) \sin(\lambda - K)}{\sin(\lambda'' - \lambda) \sin(\odot - K)}. \quad (16)$$

Let us now put, for brevity,

$$\begin{aligned} b &= \frac{R \sin(\odot - K)}{a_0}, & c &= \frac{R' \sin(\odot' - K)}{a_0}, \\ d &= \frac{R'' \sin(\odot'' - K)}{a_0}, & f &= \frac{\sec \beta'}{\sin(\lambda'' - \lambda)}, & h &= \frac{R R'' \sin(\odot'' - \odot)}{a_0 \sin(\lambda'' - \lambda)}, \\ M_1 &= \frac{\sin(\lambda'' - \lambda')}{\sin(\lambda'' - \lambda)} + f \frac{R' \sin(\lambda'' - \odot'')}{d}, & (17) \\ M_1'' &= \frac{\sin(\lambda' - \lambda)}{\sin(\lambda'' - \lambda)} - f \frac{R \sin(\lambda - \odot)}{b}, \\ M_2 &= \frac{h \sin(\lambda'' - K)}{d}, & M_2'' &= \frac{h \sin(\lambda - K)}{b}, \end{aligned}$$

and the equations (11), (14), and (16) become

$$\begin{aligned} \rho' \sec \beta' &= -c + nb + n''d, \\ \rho &= M_1 \frac{\rho'}{n} + M_2 \left(1 - \frac{N}{n} \right), \\ \rho'' &= M_1'' \frac{\rho'}{n''} + M_2'' \left(1 - \frac{N''}{n''} \right). \end{aligned} \quad (18)$$

If n and n'' are known, these equations will, in most cases, be sufficient to determine ρ , ρ' , and ρ'' .

80. It will be apparent, from a consideration of the equations which have been derived for ρ , ρ' , and ρ'' , that under certain circumstances they are inapplicable in the form in which they have been given, and that in some cases they become indeterminate. When the great circle passing through the first and third observed places of the body passes also through the second place, we have $a_0 = 0$, and equation (11) reduces to

$$n''R'' \sin(\odot'' - K) + nR \sin(\odot - K) = R' \sin(\odot' - K).$$

If the ratio of n'' to n is known, this equation will determine the quantities themselves, and from these the radius-vector r' for the middle place may be found. But if the great circle which thus passes through the three observed places passes also through the second place of the sun, we shall have $K = \odot'$, or $K = 180^\circ + \odot'$, and hence

$$n''R'' \sin(\odot'' - \odot') - nR \sin(\odot' - \odot) = 0,$$

or

$$\frac{n''}{n} = \frac{R \sin(\odot' - \odot)}{R'' \sin(\odot'' - \odot')},$$

from which it appears that the solution of the problem is in this case impossible.

If the first and third observed places coincide, we have $\lambda = \lambda''$ and $\beta = \beta''$, and each term of equation (7) reduces to zero, so that the problem becomes absolutely indeterminate. Consequently, if the data are nearly such as to render the solution impossible, according to the conditions of these two cases of indetermination, the elements which may be derived will be greatly affected by errors of observation. If, however, λ is equal to λ'' and β'' differs from β , it will be possible to derive ρ' , and hence ρ and ρ'' ; but the formulæ which have been given require some modification in this particular case. Thus, when $\lambda = \lambda''$, we have $K = \lambda'' = \lambda$, $I = 90^\circ$, and $\beta_0 = 90^\circ$, and hence a_0 , as determined by equation (10), becomes $\frac{0}{0}$. Still, in this case it is not indeterminate, since, by recurring to the original equation (9), the coefficient of ρ' , which is $-a_0 \sec \beta'$, gives

$$a_0 = \sin \beta' \cot I - \cos \beta' \sin(\lambda' - K), \quad (19)$$

and when $\lambda = \lambda''$, it becomes simply

$$a_0 = -\cos \beta' \sin(\lambda' - K).$$

Whenever, therefore, the difference $\lambda'' - \lambda$ is very small compared with the motion in latitude, a_0 should be computed by means of the equation (19) or by means of the expression which is obtained directly from the coefficient of ρ' in equation (7).

When $\lambda = \lambda'' = K$, the values of M_1 , M_1'' , M_2 , and M_2'' cannot be found by means of the equations (17); but if we use the original form of the expressions for ρ and ρ'' in terms of ρ' , as given by equations (13) and (15), without introducing the auxiliary angles, we shall have

$$\begin{aligned}\rho &= \frac{\rho'}{n} \cdot \frac{\tan \beta' \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda' - \odot'')}{\tan \beta \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda - \odot'')} \\ &\quad + \left(1 - \frac{N}{n}\right) \frac{R \tan \beta'' \sin(\odot'' - \odot)}{\tan \beta \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda - \odot'')}, \\ \rho'' &= \frac{\rho'}{n''} \cdot \frac{\tan \beta \sin(\lambda' - \odot) - \tan \beta' \sin(\lambda - \odot)}{\tan \beta \sin(\lambda'' - \odot) - \tan \beta'' \sin(\lambda - \odot)} \\ &\quad + \left(1 - \frac{N''}{n''}\right) \frac{R'' \tan \beta \sin(\odot'' - \odot)}{\tan \beta \sin(\lambda'' - \odot) - \tan \beta'' \sin(\lambda - \odot)}.\end{aligned}$$

Hence

$$\begin{aligned}M_1 &= \frac{\tan \beta' \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda' - \odot'')}{\tan \beta \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda - \odot'')}, \\ M_1'' &= \frac{\tan \beta \sin(\lambda' - \odot) - \tan \beta' \sin(\lambda - \odot)}{\tan \beta \sin(\lambda'' - \odot) - \tan \beta'' \sin(\lambda - \odot)}, \\ M_2 &= \frac{R \tan \beta'' \sin(\odot'' - \odot)}{\tan \beta \sin(\lambda'' - \odot'') - \tan \beta'' \sin(\lambda - \odot'')}, \\ M_2'' &= \frac{R'' \tan \beta \sin(\odot'' - \odot)}{\tan \beta \sin(\lambda'' - \odot) - \tan \beta'' \sin(\lambda - \odot)},\end{aligned}\tag{20}$$

are the expressions for M_1 , M_1'' , M_2 , and M_2'' which must be used when $\lambda = \lambda''$ or when λ is very nearly equal to λ'' ; and then ρ and ρ'' will be obtained from equations (18). These expressions will also be used when $\lambda'' - \lambda = 180^\circ$, this being an analogous case.

When the great circle passing through the first and third observed places of the body also passes through the first or the third place of the sun, the last two of the equations (18) become indeterminate, and other formulæ must be derived. If we multiply the second of equations (7)₃ by $\tan \beta''$ and the fourth by $-\sin(\lambda'' - \odot')$, and add the products, then multiply the second of these equations by $\tan \beta$ and the fourth by $-\sin(\lambda - \odot')$, and add, and finally reduce by means of the relation

$$NR \sin(\odot' - \odot) = N''R'' \sin(\odot'' - \odot'),$$

we get

$$\begin{aligned} \rho &= \frac{\rho'}{n} \cdot \frac{\tan \beta'' \sin (\lambda' - \odot') - \tan \beta' \sin (\lambda'' - \odot')}{\tan \beta'' \sin (\lambda - \odot') - \tan \beta \sin (\lambda'' - \odot')} \\ &\quad + \left(\frac{n''}{n} - \frac{N''}{N} \right) \frac{R'' \tan \beta'' \sin (\odot'' - \odot')}{\tan \beta'' \sin (\lambda - \odot') - \tan \beta \sin (\lambda'' - \odot')} \\ \rho'' &= \frac{\rho'}{n''} \cdot \frac{\tan \beta' \sin (\lambda - \odot') - \tan \beta \sin (\lambda' - \odot')}{\tan \beta'' \sin (\lambda - \odot') - \tan \beta \sin (\lambda'' - \odot')} \\ &\quad + \left(\frac{n}{n''} - \frac{N}{N''} \right) \frac{R \tan \beta \sin (\odot' - \odot)}{\tan \beta'' \sin (\lambda - \odot') - \tan \beta \sin (\lambda'' - \odot')} \end{aligned} \quad (21)$$

These equations are convenient for determining ρ and ρ'' from ρ' ; but they become indeterminate when the great circle passing through the extreme places of the body also passes through the second place of the sun. Therefore they will generally be inapplicable for the cases in which the equations (18) fail.

If we eliminate ρ'' from the first and second of the equations (6)₃, we get

$$0 = n\rho \sin (\lambda'' - \lambda) - nR \sin (\lambda'' - \odot) - \rho' \sin (\lambda'' - \lambda') \\ + R' \sin (\lambda'' - \odot') - n''R'' \sin (\lambda'' - \odot''),$$

from which we derive

$$\begin{aligned} \rho &= \frac{\rho'}{n} \cdot \frac{\sin (\lambda'' - \lambda')}{\sin (\lambda'' - \lambda)} \\ &\quad + \frac{nR \sin (\lambda'' - \odot) - R' \sin (\lambda'' - \odot') + n''R'' \sin (\lambda'' - \odot'')}{n \sin (\lambda'' - \lambda)}. \end{aligned} \quad (22)$$

Eliminating ρ between the same equations, the result is

$$\begin{aligned} \rho'' &= \frac{\rho'}{n''} \cdot \frac{\sin (\lambda' - \lambda)}{\sin (\lambda'' - \lambda)} \\ &\quad - \frac{nR \sin (\lambda - \odot) - R' \sin (\lambda - \odot') + n''R'' \sin (\lambda - \odot'')}{n'' \sin (\lambda'' - \lambda)}. \end{aligned} \quad (23)$$

These formulæ will enable us to determine ρ and ρ'' from ρ' in the special cases in which the equations (18) and (21) are inapplicable; but, since they do not involve the third of equations (6)₃, they are not so well adapted to a complete solution of the problem as the formulæ previously given whenever these may be applied.

If we eliminate successively ρ'' and ρ between the first and fourth of the equations (7)₃, we get

$$\begin{aligned} \rho &= \frac{\rho'}{n} \cdot \frac{\tan \beta'' \cos (\lambda' - \odot') - \tan \beta' \cos (\lambda'' - \odot')}{\tan \beta'' \cos (\lambda - \odot') - \tan \beta \cos (\lambda'' - \odot')} \\ &\quad + \frac{\tan \beta''}{n} \cdot \frac{nR \cos (\odot' - \odot) - R' + n''R'' \cos (\odot'' - \odot')}{\tan \beta'' \cos (\lambda - \odot') - \tan \beta \cos (\lambda'' - \odot')}, \\ \rho'' &= \frac{\rho'}{n''} \cdot \frac{\tan \beta' \cos (\lambda - \odot') - \tan \beta \cos (\lambda' - \odot')}{\tan \beta'' \cos (\lambda - \odot') - \tan \beta \cos (\lambda'' - \odot')} \\ &\quad - \frac{\tan \beta}{n''} \cdot \frac{nR \cos (\odot' - \odot) - R' + n''R'' \cos (\odot'' - \odot')}{\tan \beta'' \cos (\lambda - \odot') - \tan \beta \cos (\lambda'' - \odot')}, \end{aligned} \quad (24)$$

which may also be used to determine ρ and ρ'' when the equations (18) and (21) cannot be applied. When the motion in latitude is greater than in longitude, these equations are to be preferred instead of (22) and (23.)

81. It would appear at first, without examining the quantities involved in the formula for ρ' , that the equations (26)₃ will enable us to find n and n'' by successive approximations, assuming first that

$$n = \frac{\tau}{\tau'}, \quad n'' = \frac{\tau''}{\tau'},$$

and from the resulting value of ρ' determining n' , and then carrying the approximation to the values of n and n'' one step farther, so as to include terms of the second order with reference to the intervals of time between the observations. But if we consider the equation (10), we observe that a_0 is a very small quantity depending on the difference $\beta' - \beta_0$, and therefore on the deviation of the observed path of the body from the arc of a great circle, and, as this appears in the denominator of terms containing n and n'' in the equation (11), it becomes necessary to determine to what degree of approximation these quantities must be known in order that the resulting value of ρ' may not be greatly in error.

To determine the relation of a_0 to the intervals of time between the observations, we have, from the coefficient of ρ' in equation (7),

$$a_0 \sec \beta' = \tan \beta \sin (\lambda'' - \lambda') - \tan \beta' \sin (\lambda'' - \lambda) + \tan \beta'' \sin (\lambda' - \lambda).$$

We may put

$$\begin{aligned} \tan \beta &= \tan \beta' - A\tau'' + B\tau''^2 - \dots, \\ \tan \beta'' &= \tan \beta' + A\tau + B\tau^2 + \dots, \end{aligned}$$

and hence we have

$$a_0 \sec \beta' = (\sin (\lambda'' - \lambda') - \sin (\lambda'' - \lambda) + \sin (\lambda' - \lambda)) \tan \beta' + (\tau \sin (\lambda' - \lambda) - \tau'' \sin (\lambda'' - \lambda')) A + (\tau^2 \sin (\lambda' - \lambda) + \tau''^2 \sin (\lambda'' - \lambda')) B + \dots,$$

which is easily transformed into

$$a_0 \sec \beta' = 4 \sin \frac{1}{2} (\lambda' - \lambda) \sin \frac{1}{2} (\lambda'' - \lambda') \sin \frac{1}{2} (\lambda'' - \lambda) \tan \beta' \quad (25) + (\tau \sin (\lambda' - \lambda) - \tau'' \sin (\lambda'' - \lambda')) A + (\tau^2 \sin (\lambda' - \lambda) + \tau''^2 \sin (\lambda'' - \lambda')) B + \dots$$

If we suppose the intervals to be small, we may also put

$$\sin \frac{1}{2} (\lambda'' - \lambda) = \frac{1}{2} (\lambda'' - \lambda),$$

and

$$\sin (\lambda'' - \lambda) = \lambda'' - \lambda, \quad \sin (\lambda' - \lambda) = \lambda' - \lambda.$$

Further, we may put

$$\begin{aligned}\lambda &= \lambda' - A'\tau'' + B'\tau''^2 - \dots, \\ \lambda'' &= \lambda' + A'\tau + B'\tau^2 + \dots\end{aligned}$$

Substituting these values in the equation (25), neglecting terms of the fourth order with respect to τ , and reducing, we get

$$a_0 = \tau\tau'\tau'' \left(\frac{1}{2}A'^3 \tan \beta' + A'B - AB' \right) \cos \beta'.$$

It appears, therefore, that a_0 is at least of the third order with reference to the intervals of time between the observations, and that an error of the second order in the assumed values of n and n'' may produce an error of the order zero in the value of ρ' as derived from equation (11) even under the most favorable circumstances. Hence, in general, we cannot adopt the values

$$n = \frac{\tau}{\tau'}, \quad n'' = \frac{\tau''}{\tau'},$$

omitting terms of the second order, without affecting the resulting value of ρ' to such an extent that it cannot be regarded even as an approximation to the true value; and terms of at least the second order must be included in the first assumed values of n and n'' .

The equation (28)₃ gives

$$\frac{n''}{n} = \frac{\tau''}{\tau} \left(1 + \frac{1}{6} \frac{\tau^2 - \tau''^2}{r'^3} \right), \quad (26)$$

omitting the term multiplied by $\frac{dr'}{dt}$, which term is of the third order with respect to the times; and hence in this value of $\frac{n}{n''}$ only terms of at least the fourth order are neglected. Again, from the equations (26)₃ we derive, since $\tau' = \tau + \tau''$,

$$n + n'' = 1 + \frac{\tau\tau''}{2r'^3}, \quad (27)$$

in which only terms of the fourth order have been neglected. Now the first of equations (18) may be written:

$$\rho' \sec \beta' = (n + n'') \frac{b + \frac{n''}{n}d}{1 + \frac{n''}{n}} - c, \quad (28)$$

in which, if we introduce the values of $\frac{n''}{n}$ and $n + n''$ as given by (26) and (27), only terms of the fourth order with respect to the

times will be neglected, and consequently the resulting value of ρ' will be affected with only an error of the second order when a_0 is of the third order. Further, if the intervals between the observations are not very unequal, $\tau^2 - \tau'^2$ will be a quantity of an order superior to τ^2 , and when these intervals are equal, we have, to terms of the fourth order,

$$\frac{n''}{n} = \frac{\tau''}{\tau}.$$

The equation (27) gives

$$2r'^3(n + n'' - 1) = \tau\tau''.$$

Hence, if we put

$$\begin{aligned} P &= \frac{n''}{n}, \\ Q &= 2r'^3(n + n'' - 1), \end{aligned} \quad (29)$$

we may adopt, for a first approximation to the value of ρ' ,

$$P = \frac{\tau''}{\tau}, \quad Q = \tau\tau'', \quad (30)$$

and ρ' will be affected with an error of the first order when the intervals are unequal; but of the second order only when the intervals are equal. It is evident, therefore, that, in the selection of the observations for the determination of an unknown orbit, the intervals should be as nearly equal as possible, since the nearer they approach to equality the nearer the truth will be the first assumed values of P and Q , thus facilitating the successive approximations; and when a_0 is a very small quantity, the equality of the intervals is of the greatest importance.

From the equations (29) we get

$$\begin{aligned} n &= \frac{1}{1+P} \left(1 + \frac{Q}{2r'^3} \right), \\ n'' &= nP; \end{aligned} \quad (31)$$

and introducing P and Q into (28), there results

$$\rho' \sec \beta' = \left(1 + \frac{Q}{2r'^3} \right) \frac{b + Pd}{1 + P} - c. \quad (32)$$

This equation involves both ρ' and r' as unknown quantities, but by means of another equation between these quantities ρ' may be eliminated, thus giving a single equation from which r' may be found, after which ρ' may also be determined.

82. Let ψ' represent the angle at the earth between the sun and planet or comet at the second observation, and we shall have, from the equations (93)₃,

$$\begin{aligned}\tan w' &= \frac{\tan \beta'}{\sin (\lambda' - \odot')}, \\ \tan \psi' &= \frac{\tan (\lambda' - \odot')}{\cos w'}, \\ \cos \psi' &= \cos \beta' \cos (\lambda' - \odot'),\end{aligned}\tag{33}$$

by means of which we may determine ψ' , which cannot exceed 180° . Since $\cos \beta'$ is always positive, $\cos \psi'$ and $\cos (\lambda' - \odot')$ must have the same sign.

We also have

$$r'^2 = \Delta'^2 + R'^2 - 2\Delta'R'\cos\psi',$$

which may be put in the form

$$r'^2 = (\rho' \sec \beta' - R' \cos \psi')^2 + R'^2 \sin^2 \psi',$$

from which we get

$$\rho' \sec \beta' = R' \cos \psi' \pm \sqrt{r'^2 - R'^2 \sin^2 \psi'}.\tag{34}$$

Substituting for $\rho' \sec \beta'$ its value given by equation (32), we have

$$\left(1 + \frac{Q}{2r'^3}\right) \frac{b + Pd}{1 + P} - c = R' \cos \psi' \pm \sqrt{r'^2 - R'^2 \sin^2 \psi'}.$$

For brevity, let us put

$$c_0 = \frac{b + Pd}{1 + P},$$

$$c_0 - c = k_0,\tag{35}$$

$$2c_0Q = l_0,$$

and we shall have

$$k_0 - \frac{l_0}{r'^3} = R' \cos \psi' \pm \sqrt{r'^2 - R'^2 \sin^2 \psi'}.\tag{36}$$

When the values of P and Q have been found, this equation will give the value of r' in terms of quantities derived directly from the data furnished by observation. We shall now represent by z' the angle at the planet between the sun and earth at the time of the second observation, and we shall have

$$r' = \frac{R' \sin \psi'}{\sin z'}.\tag{37}$$

Substituting this value of r' , in the preceding equation, there results

$$(k_0 - R' \cos \psi') \sin z' \mp R' \sin \psi' \cos z' = \frac{l_0 \sin^4 z'}{R'^3 \sin^3 \psi'} \quad (38)$$

and if we put

$$\begin{aligned} \eta_0 \sin \zeta &= R' \sin \psi', \\ \eta_0 \cos \zeta &= k_0 - R' \cos \psi', \\ m_0 &= \frac{l_0}{\eta_0 R'^3 \sin^3 \psi'}, \end{aligned} \quad (39)$$

the condition being imposed that m_0 shall always be positive, we have, finally,

$$\sin(z' \mp \zeta) = m_0 \sin^4 z'. \quad (40)$$

In order that m_0 may be positive, the quadrant in which ζ is taken must be such that η_0 shall have the same sign as l_0 , since $\sin \psi'$ is always positive.

From equation (37) it appears that $\sin z'$ must always be positive, or $z' < 180^\circ$; and further, in the plane triangle formed by joining the actual places of the earth, sun, and planet or comet corresponding to the middle observation, we have

$$\Delta' = \frac{r' \sin(z' + \psi')}{\sin \psi'} = \frac{R' \sin(z' + \psi')}{\sin z'}.$$

Therefore,

$$\rho' = \frac{R' \sin(z' + \psi')}{\sin z'} \cos \beta', \quad (41)$$

and, since ρ' is always positive, it follows that $\sin(z' + \psi')$ must be positive, or that z' cannot exceed $180^\circ - \psi'$.

When the planet or comet at the time of the middle observation is both in the node and in opposition or conjunction with the sun, we shall have $\beta' = 0$, $\psi' = 180^\circ$ when the body is in opposition, and $\psi' = 0^\circ$ when it is in conjunction. Consequently, it becomes impossible to determine r' by means of the angle z' ; but in this case the equation (36) gives

$$k_0 - \frac{l_0}{r'^3} = -R' + r',$$

when the body is in opposition, the lower sign being excluded by the condition that the value of the first member of the equation must be positive, and for $\psi' = 0$,

$$k_0 - \frac{l_0}{r'^3} = R' \pm r',$$

the upper sign being used when the sun is between the earth and the

planet, and the lower sign when the planet is between the earth and the sun. It is hardly necessary to remark that the case of an observation at the superior conjunction when $\beta' = 0$, is physically impossible. The value of r' may be found from these equations by trial; and then we shall have

$$\rho' = r' - R'$$

when the body is in opposition, and

$$\rho' = R' - r'$$

when it is in inferior conjunction with the sun.

For the case in which the great circle passing through the extreme observed places of the body passes also through the middle place, which gives $\alpha_0 = 0$, let us divide equation (32) through by c , and we have

$$\left(1 + \frac{Q}{2r'^3}\right) \frac{\frac{b}{c} + P\frac{d}{c}}{1 + P} - 1 = \frac{\rho' \sec \beta'}{c},$$

The equations (17) give

$$\frac{b}{c} = \frac{R \sin(\odot - K)}{R' \sin(\odot' - K)}, \quad \frac{d}{c} = \frac{R'' \sin(\odot'' - K)}{R' \sin(\odot' - K)},$$

and if we put

$$\frac{\frac{b}{c} + P\frac{d}{c}}{1 + P} = C_0,$$

we shall have

$$\left(1 + \frac{Q}{2r'^3}\right) C_0 - 1 = 0,$$

since $c = \infty$ when $\alpha_0 = 0$. Hence we derive

$$r' = \sqrt[3]{\frac{\frac{1}{2} C_0 Q}{1 - C_0}}. \quad (42)$$

But when the great circle passing through the three observed places passes also through the second place of the sun, both c and C_0 become indeterminate, and thus the solution of the problem, with the given data, becomes impossible.

83. The equation (40) must give four roots corresponding to each sign, respectively; but it may be shown that of these eight roots at least four will, in every case, be imaginary. Thus, the equation may be written

$$m_0 \sin^4 z' - \sin z' \cos \zeta = \mp \cos z' \sin \zeta,$$

and, by squaring and reducing, this becomes

$$m_0^2 \sin^8 z' - 2m_0 \cos \zeta \sin^5 z' + \sin^2 z' - \sin^2 \zeta = 0.$$

When ζ is within the limits -90° and $+90^\circ$, $\cos \zeta$ will be positive, and, m_0 being always positive, it appears from the algebraic signs of the terms of the equation, according to the theory of equations, that in this case there cannot be more than four real roots, of which three will be positive and one negative. When ζ exceeds the limits -90° and $+90^\circ$, $\cos \zeta$ will be negative, and hence, in this case also, there cannot be more than four real roots, of which one will be positive and three negative. Further, since $\sin^2 \zeta$ is real and positive, there must be at least two real roots—one positive and the other negative—whether $\cos \zeta$ be negative or positive.

We may also remark that, in finding the roots of the equation (40), it will only be necessary to solve the equation

$$\sin(z' - \zeta) = m_0 \sin^4 z', \quad (43)$$

since the lower sign in (40) follows directly from this by substituting $180^\circ - z'$ in place of z' ; and hence the roots derived from this will comprise all the real roots belonging to the general form of the equation.

The observed places of the heavenly body only give the direction in space of right lines passing through the places of the earth and the corresponding places of the body, and any three points, one in each of these lines, which are situated in a plane passing through the centre of the sun, and which are at such distances as to fulfil the condition that the areal velocity shall be constant, according to the relation expressed by the equation (30), must satisfy the analytical conditions of the problem. It is evident that the three places of the earth may satisfy these conditions; and hence there may be one root of equation (43) which will correspond to the orbit of the earth, or give

$$\rho' = 0.$$

Further, it follows from the equation (37) that this root must be

$$z' = 180^\circ - \psi';$$

and such would be strictly the case if, instead of the assumed values of P and Q , their exact values for the orbit of the earth were adopted, and if the observations were referred directly to the centre of the earth, in the correction for parallax, neglecting also the perturbations in the motion of the earth.

In the case of the earth,

$$n'' = N'' = \frac{RR' \sin(\odot' - \odot)}{RR'' \sin(\odot'' - \odot)},$$

$$n = N = \frac{R'R'' \sin(\odot'' - \odot')}{RR'' \sin(\odot'' - \odot)},$$

and the complete values of P and Q become

$$P = \frac{RR' \sin(\odot' - \odot)}{R'R'' \sin(\odot'' - \odot')},$$

$$Q = 2R'^3 \left(\frac{RR' \sin(\odot' - \odot) + R'R'' \sin(\odot'' - \odot')}{RR'' \sin(\odot'' - \odot)} - 1 \right);$$

and since the approximate values

$$P = \frac{\tau''}{\tau}, \quad Q = \tau\tau''$$

differ but little from these, as will appear from the equations (27), there will be one root of equation (43) which gives z' nearly equal to $180^\circ - \psi'$. This root, however, cannot satisfy the physical conditions of the problem, which will require that the rays of light in coming from the planet or comet to the earth shall proceed from points which are at a considerable distance from the eye of the observer. Further, the negative values of $\sin z'$ are excluded by the nature of the problem, since r' must be positive, or $z' < 180^\circ$; and of the three positive roots which may result from equation (43), that being excluded which gives z' very nearly equal to $180^\circ - \psi'$, there will remain two, of which one will be excluded if it gives z' greater than $180^\circ - \psi'$, and the remaining one will be that which belongs to the orbit of the planet or comet. It may happen, however, that neither of these two roots is greater than $180^\circ - \psi'$, in which case both will satisfy the physical conditions of the problem, and hence the observations will be satisfied by two wholly different systems of elements. It will then be necessary to compare the elements computed from each of the two values of z' with other observations in order to decide which actually belongs to the body observed.

In the other case, in which $\cos \zeta$ is negative, the negative roots being excluded by the condition that r' is positive, the positive root must in most cases belong to the orbit of the earth, and the three observations do not then belong to the same body. However, in the case of the orbit of a comet, when the eccentricity is large, and the intervals between the observations are of considerable magnitude, if

the approximate values of P and Q are computed directly, by means of approximate elements already known, from the equations

$$\begin{aligned} P &= \frac{rr' \sin(u' - u)}{r'r'' \sin(u'' - u')}, \\ Q &= 2r'^3 \left(\frac{rr' \sin(u' - u) + r'r'' \sin(u'' - u')}{rr'' \sin(u'' - u)} - 1 \right), \end{aligned} \quad (44)$$

it may occur that $\cos \zeta$ is negative, and the positive root will actually belong to the orbit of the comet. The condition that one value of z' shall be very nearly equal to $180^\circ - \psi'$, requires that the adopted values of P and Q shall differ but little from those derived directly from the places of the earth; and in the case of orbits of small eccentricity this condition will always be fulfilled, unless the intervals between the observations and the distance of the planet from the sun are both very great. But if the eccentricity is large, the difference may be such that no root will correspond to the orbit of the earth.

84. We may find an expression for the limiting values of m_0 and ζ , within which equation (43) has four real roots, and beyond which there are only two, one positive and one negative. This change in the number of real roots will take place when there are two equal roots, and, consequently, if we proceed under the supposition that equation (43) has two equal roots, and find the values of m_0 and ζ which will accord with this supposition, we may determine the limits required.

Differentiating equation (43) with respect to z' , we get

$$\cos(z' - \zeta) = 4m_0 \sin^3 z' \cos z';$$

and, in the case of equal roots, the value of z' as derived from this must also satisfy the original equation

$$\sin(z' - \zeta) = m_0 \sin^4 z'.$$

To find the values of m_0 and ζ which will fulfil this condition, if we eliminate m_0 between these equations, we have

$$\sin z' \cos(z' - \zeta) = 4 \cos z' \sin(z' - \zeta),$$

from which we easily find

$$\sin(2z' - \zeta) = \frac{5}{3} \sin \zeta. \quad (45)$$

This gives the value of ζ in terms of z' for which equation (43) has

equal roots, and at which it ceases to have four real roots. To find the corresponding expression for m_0 , we have

$$m_0 = \frac{\sin(z' - \zeta)}{\sin^4 z'} = \frac{\cos(z' - \zeta)}{4 \sin^3 z' \cos z'}$$

in which we must use the value of ζ given by the preceding equation. Now, since $\sin(2z' - \zeta)$ must be within the limits -1 and $+1$, the limiting values of $\sin \zeta$ will be $+\frac{5}{8}$ and $-\frac{5}{8}$, or ζ must be within the limits $+36^\circ 52'.2$ and $-36^\circ 52'.2$, or $143^\circ 7'.8$ and $216^\circ 52'.2$. If ζ is not contained within these limits, the equation cannot have equal roots, whatever may be the value of m_0 , and hence there can only be two real roots, of which one will be positive and one negative. If for a given value of ζ we compute z' from equation (45), and call this z'_0 , or

$$\sin(2z'_0 - \zeta) = \frac{5}{8} \sin \zeta,$$

we may find the limits of the values of m_0 , within which equation (43) has four real roots. The equation for z'_0 will be satisfied by the values

$$2z'_0 - \zeta, \quad 180^\circ - (2z'_0 - \zeta);$$

and hence there will be two values of m_0 , which we will denote by m_1 and m_2 , for which, with a given value of ζ , equation (43) will have equal roots. Thus we shall have

$$m_1 = \frac{\sin(z'_0 - \zeta)}{\sin^4 z'_0},$$

and, putting in this equation $180^\circ - (2z'_0 - \zeta)$ instead of $2z'_0 - \zeta$, or $90^\circ - (z'_0 - \zeta)$ in place of z'_0 ,

$$m_2 = \frac{\cos z'_0}{\cos^4(z'_0 - \zeta)}.$$

It follows, therefore, that for any given value of ζ , if m_0 is not within the limits assigned by the values of m_1 and m_2 , equation (43) will only have two real roots, one positive and one negative, of which the latter is excluded by the nature of the problem, and the former may belong to the orbit of the earth. But if P and Q differ so much from their values in the case of the orbit of the earth that z' is not very nearly equal to $180^\circ - \psi'$, the positive root, when ζ exceeds the limits $+36^\circ 52'.2$ and $-36^\circ 52'.2$, may actually satisfy the conditions of the problem, and belong to the orbit of the body observed.

When ζ is within the limits $143^\circ 7'.8$ and $216^\circ 52'.2$, there will be four real roots, one positive and three negative, if m_0 is within the limits m_1 and m_2 ; but, if m_0 surpasses these limits, there will be only two real roots.

Table XII. contains for values of ζ from $-36^\circ 52'.2$ to $+36^\circ 52'.2$ the values of m_1 and m_2 , and also the values of the four real roots corresponding respectively to m_1 and m_2 .

In every case in which equation (43) has three positive roots and one negative root, the value of m_0 must be within the limits indicated by m_1 and m_2 , and the values of z' will be within the limits indicated by the quantities corresponding to m_1 and m_2 for each root, which we designate respectively by z'_1, z'_2, z'_3 , and z'_4 . The table will show, from the given values of m_0 and $180^\circ - \psi'$, whether the problem admits of two distinct solutions, since, excluding the value of z' , which is nearly equal to $180^\circ - \psi'$, and corresponds to the orbit of the earth, and also that which exceeds 180° , it will appear at once whether one or both of the remaining two values of z' will satisfy the condition that z' shall be less than $180^\circ - \psi'$. The table will also indicate an approximate value of z' , by means of which the equation (43) may be solved by a few trials.

For the root of the equation (43) which corresponds to the orbit of the earth, we have $\rho' = 0$, and hence from (36) we derive

$$k_0 = \frac{l_0}{R'^3}.$$

Substituting this value for k_0 in the general equation (32), we have

$$\rho' \sec \beta' = l_0 \left(\frac{1}{R'^3} - \frac{1}{r'^3} \right);$$

and, since ρ' must be positive, the algebraic sign of the numerical value of l_0 will indicate whether r' is greater or less than R' . It is easily seen, from the formulæ for l_0, b, d , &c., that in the actual application of these formulæ, the intervals between the observations not being very large, l_0 will be positive when $\beta' - \beta_0$ and $\sin(\odot' - K)$ have contrary signs, and negative when $\beta' - \beta_0$ has the same sign as $\sin(\odot' - K)$. Hence, when $\odot' - K$ is less than 180° , r' must be less than R' if $\beta' - \beta_0$ is positive, but greater than R' if $\beta' - \beta_0$ is negative. When $\odot' - K$ exceeds 180° , r' will be greater than R' if $\beta' - \beta_0$ is positive, and less than R' if $\beta' - \beta_0$ is negative. We may, therefore, by means of a celestial globe, determine by inspection whether the distance of a comet from the sun is greater or less than

that of the earth from the sun. Thus, if we pass a great circle through the two extreme observed places of the comet, r' must be greater than R' when the place of the comet for the middle observation is on the same side of this great circle as the point of the ecliptic which corresponds to the place of the sun. But when the middle place and the point of the ecliptic corresponding to the place of the sun are on opposite sides of the great circle passing through the first and third places of the comet, r' must be less than R' .

85. From the values of ρ' and r' derived from the assumed values $P = \frac{\tau''}{\tau}$ and $Q = \tau\tau''$, we may evidently derive more approximate values of these quantities, and thus, by a repetition of the calculation, make a still closer approximation to the true value of ρ' . To derive other expressions for P and Q which are exact, provided that r' and ρ' are accurately known, let us denote by s'' the ratio of the sector of the orbit included by r and r' to the triangle included by the same radii-vectores and the chord joining the first and second places; by s' the same ratio with respect to r and r'' , and by s this ratio with respect to r' and r'' . These ratios s, s', s'' must necessarily be greater than 1, since every part of the orbit is concave toward the sun. According to the equation (30), we have for the areas of the sectors, neglecting the mass of the body,

$$\frac{1}{2}\tau''\sqrt{p}, \quad \frac{1}{2}\tau'\sqrt{p}, \quad \frac{1}{2}\tau\sqrt{p},$$

and therefore we obtain

$$s''[rr'] = \tau''\sqrt{p}, \quad s'[rr''] = \tau'\sqrt{p}, \quad s[r'r''] = \tau\sqrt{p}. \quad (46)$$

Then, since

$$n = \frac{[r'r'']}{[rr'']}, \quad n'' = \frac{[rr']}{[r'r']},$$

we shall have

$$n = \frac{\tau}{\tau'} \cdot \frac{s'}{s}, \quad n'' = \frac{\tau''}{\tau} \cdot \frac{s'}{s''} \quad (47)$$

and, consequently,

$$P = \frac{\tau''}{\tau} \cdot \frac{s}{s''}, \quad (48)$$

$$Q = \frac{\tau\tau''}{ss''} \left(\frac{s's''}{\tau'\tau''} + \frac{ss'}{\tau\tau'} - \frac{ss''}{\tau\tau''} \right) 2r'^3.$$

Substituting for $s, s',$ and s'' their values from (46), we have

$$Q = 2pr'^3 \frac{[r'r''] + [rr'] - [r'r'']}{[rr'] \cdot [rr''] \cdot [r'r']} \cdot \frac{\tau\tau''}{ss''}. \quad (49)$$

The angular distance between the perihelion and node being denoted by ω , the polar equation of the conic section gives

$$\begin{aligned}\frac{p}{r} &= 1 + e \cos(u - \omega), \\ \frac{p}{r'} &= 1 + e \cos(u' - \omega), \\ \frac{p}{r''} &= 1 + e \cos(u'' - \omega).\end{aligned}\tag{50}$$

If we multiply the first of these equations by $\sin(u'' - u')$, the second by $-\sin(u'' - u)$, and the third by $\sin(u' - u)$, add the products and reduce, we get

$$\begin{aligned}\frac{p}{r} \sin(u'' - u') - \frac{p}{r'} \sin(u'' - u) + \frac{p}{r''} \sin(u' - u) &= \sin(u'' - u') \\ &\quad - \sin(u'' - u) + \sin(u' - u);\end{aligned}$$

and, since

$$\begin{aligned}\sin(u'' - u') &= 2 \sin \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u'), \\ \sin(u'' - u) - \sin(u' - u) &= 2 \sin \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' + u' - 2u),\end{aligned}$$

the second member reduces to

$$4 \sin \frac{1}{2}(u'' - u') \sin \frac{1}{2}(u'' - u) \sin \frac{1}{2}(u' - u).$$

Therefore, we shall have

$$p = \frac{4r'r'' \sin \frac{1}{2}(u'' - u') \sin \frac{1}{2}(u'' - u) \sin \frac{1}{2}(u' - u)}{r'r'' \sin(u'' - u') - rr'' \sin(u'' - u) + rr' \sin(u' - u)}.$$

If we multiply both numerator and denominator of this expression by

$$2rr'r'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u),$$

it becomes, introducing $[rr']$, $[rr'']$, and $[r'r'']$,

$$p = \frac{[r'r''] \cdot [rr''] \cdot [rr']}{[r'r''] + [rr'] - [rr'']} \cdot \frac{1}{2rr'r'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u)}.$$

Substituting this value of p in equation (49), it reduces to

$$Q = \frac{\tau\tau''}{ss'} \cdot \frac{r'^2}{rr'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u)}.\tag{51}$$

86. If we compare the equations (47) with the formula (28)₃, we derive

$$\frac{g''}{s} = 1 - \frac{1}{6} \frac{\tau^2 - \tau'^2}{r'^3} + \frac{1}{4} \frac{(\tau^3 + \tau'^3)}{kr'^4} \cdot \frac{dr'}{dt} \dots\tag{52}$$

Consequently, in the first approximation, we may take

$$\frac{s''}{s} = 1.$$

If the intervals of the times are not very unequal, this assumption will differ from the truth only in terms of the third order with respect to the time, and in terms of the fourth order if the intervals are equal, as has already been shown. Hence, we adopt for the first approximation,

$$P = \frac{\tau''}{\tau}, \quad Q = \tau\tau'',$$

the values of τ and τ'' being computed from the uncorrected times of observation, which may be denoted by t_0 , t'_0 , and t''_0 . With the values of P and Q thus found, we compute r' , and from this ρ' , ρ , and ρ'' , by means of the formulæ already derived.

The heliocentric places for the first and third observations may now be found from the formulæ (71)₃ and (72)₃, and then the angle $u'' - u$ between the radii-vectores r and r'' may be obtained in various ways, precisely as the distance between two points on the celestial sphere is obtained from the spherical co-ordinates of these points. When $u'' - u$ has been found, we have

$$\begin{aligned} \sin(u'' - u') &= \frac{nr}{r'} \sin(u'' - u), \\ \sin(u' - u) &= \frac{n'r''}{r'} \sin(u'' - u), \end{aligned} \tag{53}$$

from which $u'' - u'$ and $u' - u$ may be computed. From these results the ratios s and s'' may be computed, and then new and more approximate values of P and Q . The value of $u'' - u$, found by taking the sum of $u'' - u'$ and $u' - u$ as derived from (53), should agree with that used in the second members of these equations, within the limits of the errors which may be attributed to the logarithmic tables.

The most advantageous method of obtaining the angles between the radii-vectores is to find the position of the plane of the orbit directly from l , l'' , b , and b'' , and then compute u , u' , and u'' directly from Ω and i , according to the first of equations (82)₁. It will be expedient also to compute r' , l' and b' from ρ' , λ' , and β' , and the agreement of the value of r' , thus found, with that already obtained from equation (37), will check the accuracy of part of the numerical

calculation. Further, since the three places of the body must be in a plane passing through the centre of the sun, whether P and Q are exact or only approximate, we must also have

$$\tan b' = \tan i \sin (\ell' - \Omega),$$

and the value of b' derived from this equation must agree with that computed directly from ρ' , or at least the difference should not exceed what may be due to the unavoidable errors of logarithmic calculation.

We may now compute n and n'' directly from the equations

$$n = \frac{r'r'' \sin (u'' - u')}{rr'' \sin (u'' - u)}, \quad n'' = \frac{rr' \sin (u' - u)}{rr'' \sin (u'' - u)}; \quad (54)$$

but when the values of u , u' , and u'' are those which result from the assumed values of P and Q , the resulting values of n and n'' will only satisfy the condition that the plane of the orbit passes through the centre of the sun. If substituted in the equations (29), they will only reproduce the assumed values of P and Q , from which they have been derived, and hence they cannot be used to correct them. If, therefore, the numerical calculation be correct, the values of n and n'' obtained from (54) must agree with those derived from equations (31), within the limits of accuracy admitted by the logarithmic tables.

The differences $u'' - u'$ and $u' - u$ will usually be small, and hence a small error in either of these quantities may considerably affect the resulting values of n and n'' . In order to determine whether the error of calculation is within the limits to be expected from the logarithmic tables used, if we take the logarithms of both members of the equations (54) and differentiate, supposing only n , n'' , and u' to vary, we get

$$\begin{aligned} d \log n &= - \cot (u'' - u') du', \\ d \log n'' &= + \cot (u' - u) du'. \end{aligned}$$

Multiplying these by 0.434294, the modulus of the common system of logarithms, and expressing du' in seconds of arc, we find, in units of the seventh decimal place of common logarithms,

$$\begin{aligned} d \log n &= - 21.055 \cot (u'' - u') du', \\ d \log n'' &= + 21.055 \cot (u' - u) du'. \end{aligned}$$

If we substitute in these the differences between $\log n$ and $\log n''$ as found from the equations (54), and the values already obtained by

means of (31), the two resulting values of du' should agree, and the magnitude of du' itself will show whether the error of calculation exceeds the unavoidable errors due to the limited extent of the logarithmic tables. When the agreement of the two results for n and n'' is in accordance with these conditions, and no error has been made in computing n and n'' from P and Q by means of the equations (31), the accuracy of the entire calculation, both of the quantities which depend on the assumed values of P and Q , and of those which are obtained independently from the data furnished by observation, is completely proved.

87. Since the values of n and n'' derived from equations (54) cannot be used to correct the assumed values of P and Q , from which r , r' , u , u' , &c. have been computed, it is evidently necessary to compute the values for a second approximation by means of the series given by the equations (26)₃, or by means of the ratios s and s'' . The expressions for n and n'' arranged in a series with respect to the time involve the differential coefficients of r' with respect to t , and, since these are necessarily unknown, and cannot be conveniently determined, it is plain that if the ratios s and s'' can be readily found from r , r' , r'' , u , u' , u'' , and τ , τ' , τ'' , so as to involve the relation between the times of observation and the places in the orbit, they may be used to obtain new values of P and Q by means of equations (48) and (51), to be used in a second approximation.

Let us now resume the equation

$$M = E - e \sin E,$$

or

$$\frac{k(t - T)}{a^{\frac{3}{2}}} = E - e \sin E,$$

and also for the third place

$$\frac{k(t'' - T)}{a^{\frac{3}{2}}} = E'' - e \sin E''.$$

Subtracting, we get

$$\frac{\tau'}{a^{\frac{3}{2}}} = E'' - E - 2e \sin \frac{1}{2}(E'' - E) \cos \frac{1}{2}(E'' + E). \quad (55)$$

This equation contains three unknown quantities, a , e , and the difference $E'' - E$. We can, however, by means of expressions involving r , r'' , u , and u'' , eliminate a and e . Thus, since $p = a(1 - e^2)$, we have

$$\tau \sqrt{p} = a^2 \sqrt{1 - e^2} (E'' - E - 2e \sin \frac{1}{2}(E'' - E) \cos \frac{1}{2}(E'' + E)). \quad (56)$$

From the equations

$$\begin{aligned}\sqrt{r} \sin \tfrac{1}{2}v &= \sqrt{a(1+e)} \sin \tfrac{1}{2}E, & \sqrt{r''} \sin \tfrac{1}{2}v'' &= \sqrt{a(1+e)} \sin \tfrac{1}{2}E'', \\ \sqrt{r} \cos \tfrac{1}{2}v &= \sqrt{a(1-e)} \cos \tfrac{1}{2}E, & \sqrt{r''} \cos \tfrac{1}{2}v'' &= \sqrt{a(1-e)} \cos \tfrac{1}{2}E'',\end{aligned}$$

since $v'' - v = u'' - u$, we easily derive

$$\sqrt{rr''} \sin \tfrac{1}{2}(u'' - u) = a\sqrt{1-e^2} \sin \tfrac{1}{2}(E'' - E), \quad (57)$$

and also

$$a \cos \tfrac{1}{2}(E'' - E) - ae \cos \tfrac{1}{2}(E'' + E) = \sqrt{rr''} \cos \tfrac{1}{2}(u'' - u),$$

or

$$e \cos \tfrac{1}{2}(E'' + E) = \cos \tfrac{1}{2}(E'' - E) - \frac{\sqrt{rr''} \cos \tfrac{1}{2}(u'' - u)}{a}. \quad (58)$$

Substituting this value of $e \cos \tfrac{1}{2}(E'' + E)$ in equation (56), we get

$$\begin{aligned}\tau' \sqrt{p} &= a^2 \sqrt{1-e^2} (E'' - E - \sin(E'' - E)) \\ &\quad + 2a\sqrt{1-e^2} \sin \tfrac{1}{2}(E'' - E) \cos \tfrac{1}{2}(u'' - u) \sqrt{rr''},\end{aligned}$$

and substituting, in the last term of this, for $a\sqrt{1-e^2}$, its value from (57), the result is

$$\tau' \sqrt{p} = a^2 \sqrt{1-e^2} (E'' - E - \sin(E'' - E)) + rr'' \sin(u'' - u). \quad (59)$$

From (57) we obtain

$$a^2 \sqrt{1-e^2} = \frac{1}{p} \cdot \frac{(\sqrt{rr''} \sin \tfrac{1}{2}(u'' - u))^3}{\sin^3 \tfrac{1}{2}(E'' - E)},$$

or

$$a^2 \sqrt{1-e^2} = \left(\frac{rr'' \sin(u'' - u)}{2\sqrt{rr''} \cos \tfrac{1}{2}(u'' - u)} \right)^3 \frac{1}{p \sin^3 \tfrac{1}{2}(E'' - E)}.$$

Therefore, the equation (59) becomes

$$\tau' \sqrt{p} = \frac{1}{p} \cdot \frac{E'' - E - \sin(E'' - E)}{\sin^3 \tfrac{1}{2}(E'' - E)} \left(\frac{[rr'']}{2\sqrt{rr''} \cos \tfrac{1}{2}(u'' - u)} \right)^3 + [rr'']. \quad (60)$$

Let κ' be the chord of the orbit between the first and third places, and we shall have

$$\kappa'^2 = (r + r'')^2 - 4rr'' \cos^2 \tfrac{1}{2}(u'' - u).$$

Now, since the chord κ' can never exceed $r + r''$, we may put

$$\kappa' = (r + r'') \sin \gamma', \quad (61)$$

and from this, in combination with the preceding equation, we derive

$$2\sqrt{rr''} \cos \tfrac{1}{2}(u'' - u) = (r + r'') \cos \gamma'. \quad (62)$$

Substituting this value, and $[rr''] = \frac{\tau'}{s} \sqrt{p}$, in equation (60), it reduces to

$$\frac{E'' - E - \sin(E'' - E)}{\sin^{\frac{3}{2}}(E'' - E)} \cdot \frac{\tau'^2}{(r + r'')^3 \cos^3 \gamma'} \cdot \frac{1}{s^3} + \frac{1}{s} = 1. \quad (63)$$

The elements a and e are thus eliminated, but the resulting equation involves still the unknown quantities $E'' - E$ and s' . It is necessary, therefore, to derive an additional equation involving the same unknown quantities in order that $E'' - E$ may be eliminated, and that thus the ratio s' , which is the quantity sought, may be found.

From the equations

$$r = a - ae \cos E, \quad r'' = a - ae \cos E'',$$

we get

$$r'' + r = 2a - 2ae \cos \frac{1}{2}(E'' + E) \cos \frac{1}{2}(E'' - E).$$

Substituting in this the value of $e \cos \frac{1}{2}(E'' + E)$ from (58), we have

$$r'' + r = 2a \sin^2 \frac{1}{2}(E'' - E) + 2\sqrt{rr''} \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(E'' - E),$$

and substituting for $\sin \frac{1}{2}(E'' - E)$ its value from (57), there results

$$r'' + r = \frac{2rr'' \sin^2 \frac{1}{2}(u'' - u)}{p} + 2\sqrt{rr''} \cos \frac{1}{2}(u'' - u) (1 - 2\sin^2 \frac{1}{4}(E'' - E)).$$

But, since

$$\frac{2rr'' \sin^2 \frac{1}{2}(u'' - u)}{p} = \frac{([rr''])^2}{2prr'' \cos^2 \frac{1}{2}(u'' - u)} = \frac{2\tau'^2}{s'^2} \left(\frac{1}{2\sqrt{rr''} \cos \frac{1}{2}(u'' - u)} \right)^2,$$

we have

$$r + r'' = \frac{2\tau'^2}{s'^2} \cdot \frac{1}{(r + r'')^2 \cos^2 \gamma'} + (r + r'') \cos \gamma' (1 - 2\sin^2 \frac{1}{4}(E'' - E)),$$

from which we derive

$$\sin^2 \frac{1}{4}(E'' - E) = \frac{1}{s'^2} \cdot \frac{\tau'^2}{(r + r'')^3 \cos^3 \gamma'} - \frac{\sin^2 \frac{1}{2}\gamma'}{\cos \gamma'}, \quad (64)$$

which is the additional equation required, involving $E'' - E$ and s' as unknown quantities.

Let us now put

$$\begin{aligned} y &= \frac{\sin^3 \frac{1}{2}(E'' - E)}{E'' - E - \sin(E'' - E)}, \\ m &= \frac{\tau'^2}{(r + r'')^3 \cos^3 \gamma'}, \\ j &= \frac{\sin^2 \frac{1}{2}\gamma'}{\cos \gamma'}, \\ s' &= \sin^2 \frac{1}{4}(E'' - E), \end{aligned} \quad (65)$$

and the equations (63) and (64) become

$$\begin{aligned}\frac{m'}{y'} \cdot \frac{1}{s^3} + \frac{1}{s'} &= 1, \\ x' &= \frac{m'}{s'^2} - j'.\end{aligned}\tag{66}$$

When the value of y' is known, the first of these equations will enable us to determine s' , and hence the value of x' , or $\sin^2 \frac{1}{4}(E'' - E)$, from the last equation.

The calculation of γ' may be facilitated by the introduction of an additional auxiliary quantity. Thus, let

$$\tan \chi' = \sqrt{\frac{r''}{r}},\tag{67}$$

and from (62) we find

$$\cos \gamma' = \cos \frac{1}{2}(u'' - u) \frac{2\sqrt{rr''}}{r + r''} = 2 \cos \frac{1}{2}(u'' - u) \cos^2 \chi' \tan \chi',$$

or

$$\cos \gamma' = \sin 2\chi' \cos \frac{1}{2}(u'' - u).\tag{68}$$

We have, also,

$$x'^2 = (r + r'')^2 - 4rr'' \cos^2 \frac{1}{2}(u'' - u),$$

which gives

$$x'^2 = (r - r'')^2 + 4rr'' \sin^2 \frac{1}{2}(u'' - u).$$

Multiplying this equation by $\cos^2 \frac{1}{2}(u'' - u)$ and the preceding one by $\sin^2 \frac{1}{2}(u'' - u)$, and adding, we get

$$x'^2 = (r + r'')^2 \sin^2 \frac{1}{2}(u'' - u) + (r - r'')^2 \cos^2 \frac{1}{2}(u'' - u).\tag{69}$$

From (67) we get

$$\cos^2 \chi' = \frac{r}{r + r''}, \quad \sin^2 \chi' = \frac{r''}{r + r''}$$

and, therefore,

$$\cos 2\chi' = \frac{r - r''}{r + r''}$$

so that equation (69) may be written

$$\frac{x'^2}{(r + r'')^2} = \sin^2 \gamma' = \sin^2 \frac{1}{2}(u'' - u) + \cos^2 2\chi' \cos^2 \frac{1}{2}(u'' - u).$$

We may, therefore, put

$$\begin{aligned}\sin \gamma' \cos G' &= \sin \frac{1}{2}(u'' - u), \\ \sin \gamma' \sin G' &= \cos \frac{1}{2}(u'' - u) \cos 2\chi', \\ \cos \gamma' &= \cos \frac{1}{2}(u'' - u) \sin 2\chi',\end{aligned}\tag{70}$$

from which γ' may be derived by means of its tangent, so that $\sin \gamma'$ shall be positive. The auxiliary angle G' will be of subsequent use in determining the elements of the orbit from the final hypothesis for P and Q .

88. We shall now consider the auxiliary quantity y' introduced into the first of equations (66). For brevity, let us put

$$g = \frac{1}{2}(E'' - E),$$

and we shall have

$$y' = \frac{\sin^3 g}{2g - \sin 2g}.$$

This gives, by differentiation,

$$\frac{dy'}{y'} = 3 \cot g \, dg - \frac{4 \sin^2 g \, dg}{2g - \sin 2g},$$

or

$$\frac{dy'}{dg} = 3y' \cot g - 4y'^2 \operatorname{cosec} g.$$

The last of equations (65) gives $x' = \sin^2 \frac{1}{2}g$, and hence

$$\frac{dg}{dx'} = 2 \operatorname{cosec} g.$$

Therefore we have

$$\frac{dy'}{dx'} = \frac{6y' \cos g - 8y'^2}{\sin^2 g} = \frac{3(1 - 2x')y' - 4y'^3}{2x'(1 - x')}.$$

It is evident that we may expand y' into a series arranged in reference to the ascending powers of x' , so that we shall have

$$y' = \alpha + \beta x' + \gamma x'^2 + \delta x'^3 + \varepsilon x'^4 + \zeta x'^5 + \&c.$$

Differentiating, we get

$$\frac{dy'}{dx'} = \beta + 2\gamma x' + 3\delta x'^2 + 4\varepsilon x'^3 + 5\zeta x'^4 + \&c.,$$

and substituting for $\frac{dy'}{dx'}$ the value already obtained, there results

$$\begin{aligned} 2\beta x' + (4\gamma - 2\beta)x'^2 + (6\delta - 4\gamma)x'^3 + (8\varepsilon - 6\delta)x'^4 + (10\zeta - 8\varepsilon)x'^5 + \&c. \\ = (3\alpha - 4\alpha^2) + (3\beta - 6\alpha - 8\alpha\beta)x' + (3\gamma - 6\beta - 4\beta^2 - 8\alpha\gamma)x'^2 \\ + (3\delta - 6\gamma - 8\beta\gamma - 8\alpha\delta)x'^3 + (3\varepsilon - 6\delta - 4\gamma^2 - 8\beta\delta - 8\alpha\varepsilon)x'^4 \\ + (3\zeta - 6\varepsilon - 8\gamma\delta - 8\beta\varepsilon - 8\alpha\zeta)x'^5 + \&c. \end{aligned}$$

Since the coefficients of like powers of x' must be equal, we have

$$\begin{aligned} 3\alpha - 4\alpha^2 &= 0, & 3\beta - 6\alpha - 8\alpha\beta &= 2\beta, \\ 3\gamma - 6\beta - 4\beta^2 - 8\alpha\gamma &= 2(2\gamma - \beta), \&c.; \end{aligned}$$

and hence we derive

$$\begin{aligned} \alpha &= \frac{3}{4}, & \beta &= -\frac{9}{10}, & \gamma &= \frac{9}{175}, & \delta &= \frac{26}{875}, \\ \epsilon &= \frac{6228}{336875}, & \zeta &= \frac{265896}{21896875}, & \eta &= \frac{19139024}{2299171875}. \end{aligned}$$

Therefore we have

$$y' = \frac{3}{4} - \frac{9}{10}x' + \frac{9}{175}x'^2 + \frac{26}{875}x'^3 + \frac{6228}{336875}x'^4 + \frac{265896}{21896875}x'^5 + \frac{19139024}{2299171875}x'^6 + \&c. \quad (71)$$

If we multiply through by $\frac{1}{9}$, and put

$$\xi' = \frac{2}{35}x'^2 + \frac{52}{1575}x'^3 + \frac{1384}{67375}x'^4 + \frac{59088}{4379375}x'^5 + \frac{38278048}{4138509375}x'^6 + \&c., \quad (72)$$

we obtain

$$\frac{1}{9}y' - \frac{5}{6} + x' = \xi'. \quad (73)$$

Combining this with the second of equations (66), the result is

$$\frac{1}{9}y' + \frac{m'}{s'^2} = \frac{5}{6} + j' + \xi'.$$

If we put

$$\eta' = \frac{m'}{\frac{5}{6} + j' + \xi'}, \quad (74)$$

we shall have

$$\frac{1}{9}\frac{m'}{y'} = \frac{\eta's'^2}{s'^2 - \eta'}.$$

But from the first of equations (66) we get

$$\frac{m'}{y'} = s'^2(s' - 1);$$

and therefore we have

$$\eta' = \frac{s'^2(s' - 1)}{s' + \frac{1}{9}}. \quad (75)$$

As soon as η' is known, this equation will give the corresponding value of s' .

Since ξ' is a quantity of the fourth order in reference to the difference $\frac{1}{2}(E'' - E)$, we may evidently, for a first approximation to the value of η' , take

$$\eta' = \frac{m'}{\frac{5}{6} + j''},$$

and with this find s' from (75), and the corresponding value of x' from the last of equations (66). With this value of x' we find the corresponding value of ξ' , and recompute η' , s' , and x' ; and, if the

value of ξ' derived from the last value of x' differs from that already used, the operation must be repeated.

It will be observed that the series (72) for ξ' converges with great rapidity, and that for $E'' - E = 94^\circ$ the term containing x'^6 amounts to only one unit of the seventh decimal place in the value of ξ' . Table XIV. gives the values of ξ' corresponding to values of x' from 0.0 to 0.3, or from $E'' - E = 0$ to $E'' - E = 132^\circ 50'.6$. Should a case occur in which $E'' - E$ exceeds this limit, the expression

$$y' = \frac{\sin^3 \frac{1}{2} (E'' - E)}{E'' - E - \sin (E'' - E)}$$

may then be computed accurately by means of the logarithmic tables ordinarily in use. An approximate value of x' may be easily found with which y' may be computed from this equation, and then ξ' from (73). With the value of ξ' thus found, η' may be computed from (74), and thus a more approximate value of x' is immediately obtained.

The equation (75) is of the third degree, and has, therefore, three roots. Since s' is always positive, and cannot be less than 1, it follows from this equation that η' is always a positive quantity. The equation may be written thus:

$$s'^3 - s'^2 - \eta' s' - \frac{1}{9} \eta' = 0,$$

and there being only one variation of sign, there can be only one positive root, which is the one to be adopted, the negative roots being excluded by the nature of the problem. Table XIII. gives the values of $\log s'^2$ corresponding to values of η' from $\eta' = 0$ to $\eta' = 0.6$. When η' exceeds the value 0.6, the value of s' must be found directly from the equation (75).

89. We are now enabled to determine whether the orbit is an ellipse, parabola, or hyperbola. In the ellipse $x = \sin^2 \frac{1}{4} (E'' - E)$ is positive. In the parabola the eccentric anomaly is zero, and hence $x = 0$. In the hyperbola the angle which we call the eccentric anomaly, in the case of elliptic motion, becomes imaginary, and hence, since $\sin \frac{1}{4} (E'' - E)$ will be imaginary, x' must be negative. It follows, therefore, that if the value of x' derived from the equation

$$x' = \frac{m'}{s'^2} - j'$$

is positive, the orbit is an ellipse; if equal to zero, the orbit is a parabola; and if negative, it is a hyperbola.

For the case of parabolic motion we have $x' = 0$, and the second of equations (66) gives

$$s'^2 = \frac{m'}{j'}. \quad (76)$$

If we eliminate s' by means of both equations, since, in this case, $y' = \frac{3}{4}$, we get

$$m'^{\frac{1}{2}} = j'^{\frac{1}{2}} + \frac{4}{3}j'^{\frac{3}{2}}.$$

Substituting in this the values of m and l given by (65), we obtain

$$\frac{3\tau'}{(r + r'')^{\frac{3}{2}}} = 3 \sin \frac{1}{2}\gamma' \cos \gamma' + 4 \sin^3 \frac{1}{2}\gamma',$$

which gives

$$\frac{6\tau'}{(r + r'')^{\frac{3}{2}}} = 6 \sin \frac{1}{2}\gamma' \cos^2 \frac{1}{2}\gamma' + 2 \sin^3 \frac{1}{2}\gamma',$$

or

$$\frac{6\tau'}{(r + r'')^{\frac{3}{2}}} = (\sin \frac{1}{2}\gamma' + \cos \frac{1}{2}\gamma')^3 + (\sin \frac{1}{2}\gamma' - \cos \frac{1}{2}\gamma')^3.$$

This may evidently be written

$$\frac{6\tau'}{(r + r'')^{\frac{3}{2}}} = (1 + \sin \gamma')^{\frac{3}{2}} \mp (1 - \sin \gamma')^{\frac{3}{2}},$$

the upper sign being used when γ' is less than 90° , and the lower sign when it exceeds 90° . Multiplying through by $(r + r'')^{\frac{3}{2}}$, and replacing $(r + r'') \sin \gamma$ by κ , we obtain

$$6\tau' = (r + r'' + \kappa)^{\frac{3}{2}} \mp (r + r'' - \kappa)^{\frac{3}{2}},$$

which is identical with the equation (56)₃ for the special case of parabolic motion.

Since x' is negative in the case of hyperbolic motion, the value of ξ' determined by the series (72) will be different from that in the case of elliptic motion. Table XIV. gives the value of ξ' corresponding to both forms; but when x' exceeds the limits of this table, it will be necessary, in the case of the hyperbola also, to compute the value of ξ' directly, using additional terms of the series, or we may modify the expression for y' in terms of E'' and E so as to be applicable.

If we compare equations (44)₁ and (56)₁, we get

$$\tan \frac{1}{2}E = \sqrt{-1} \tan \frac{1}{2}F;$$

and hence, from (58)₁,

$$\tan \frac{1}{2}E = \frac{\sigma - 1}{\sigma + 1} \sqrt{-1}.$$

We have, also, by comparing (65)₁ with (41)₁, since a is negative in the hyperbola,

$$\cos E = \frac{\sigma^2 + 1}{2\sigma},$$

which gives

$$\sin E = \frac{\sigma^2 - 1}{2\sigma} \sqrt{-1}.$$

Now, since

$$\cos E + \sqrt{-1} \sin E = e^{E\sqrt{-1}},$$

in which e is the base of Napierian logarithms, we have

$$E\sqrt{-1} = \log_e (\cos E + \sqrt{-1} \sin E),$$

which reduces to

$$E\sqrt{-1} = \log_e \frac{1}{\sigma},$$

or

$$E = \sqrt{-1} \log_e \sigma.$$

By means of these relations between E and σ , the expression for y' may be transformed so as not to involve imaginary quantities. Thus we have

$$E'' - E = (\log_e \sigma'' - \log_e \sigma) \sqrt{-1} = \sqrt{-1} \log_e \frac{\sigma''}{\sigma},$$

$$\sin (E'' - E) = \sin E'' \cos E - \cos E'' \sin E = \frac{\sigma''^2 - \sigma^2}{2\sigma\sigma''} \sqrt{-1}.$$

From the value of $\cos E$ we easily derive

$$\sin \frac{1}{2}E = \frac{\sigma - 1}{2\sqrt{\sigma}} \sqrt{-1}, \quad \cos \frac{1}{2}E = \frac{\sigma + 1}{2\sqrt{\sigma}},$$

and hence

$$\sin \frac{1}{2}(E'' - E) = \frac{\sigma'' - \sigma}{2\sqrt{\sigma\sigma''}} \sqrt{-1}.$$

Therefore the expression for y' becomes

$$y' = - \frac{(\sigma'' - \sigma)^3}{8(\sqrt{\sigma\sigma''})^3 \log_e \frac{\sigma''}{\sigma} - 4\sqrt{\sigma\sigma''}(\sigma''^2 - \sigma^2)}.$$

Since the auxiliary quantity σ in the hyperbola is always positive, let us now put

$$\frac{\sigma''}{\sigma} = A^2,$$

and we have

$$y' = \frac{\frac{1}{4} (A - \frac{1}{A})^2}{A^2 - \frac{1}{A^2} - 4 \log_e A}, \quad (77)$$

from which y' may be derived when A is known.

We have, further,

$$\sin^2 \frac{1}{4} (E'' - E) = \frac{1}{2} (1 - \cos \frac{1}{2} (E'' - E)) = \frac{1}{2} \left(1 - \frac{\sigma'' + \sigma}{2 \sqrt{\sigma \sigma''}} \right);$$

and therefore

$$x' = - \frac{(\sqrt{\sigma''} - \sqrt{\sigma})^2}{4 \sqrt{\sigma \sigma''}}, \quad (78)$$

or

$$x' = - \frac{1}{4} \left(\sqrt{A} - \frac{1}{\sqrt{A}} \right)^2. \quad (79)$$

These expressions for y' and x' enable us to find ξ' when x' exceeds the limits of the table. Thus, we obtain an approximate value of x' by putting

$$\eta' = \frac{m'}{\frac{5}{6} + j''}$$

from which we first find s' and then x' from the second of equations (66). Then we compute A from the formula (79), which gives

$$A = 1 - 2x' + 2\sqrt{x'^2 - x'}, \quad (80)$$

y' from (77), and ξ' from (73). A repetition of the calculation, using the value of ξ' thus found, will give a still closer approximation to the correct values of x' and s' ; and this process should be continued until ξ' remains unchanged.

90. The formulæ for the calculation of s' will evidently give the value of s if we use τ , r' , r'' , u' , and u'' , the necessary changes in the notation being indicated at once; and in the same manner using τ'' , r , r' , u , and u' , we obtain s'' . From the values of s and s'' thus found, more accurate values of P and Q may be computed by means of the equations (48) and (51). We may remark, however, that if the times of the observations have not been already corrected for the

time of aberration, as in the case of the determination of an unknown orbit, this correction may now be applied as determined by means of the values of ρ , ρ' , and ρ'' already obtained. Thus, if t_0 , t'_0 , and t''_0 are the uncorrected times of observation, the corrected values will be

$$\begin{aligned} t &= t_0 - C\rho \sec \beta, \\ t' &= t'_0 - C\rho' \sec \beta', \\ t'' &= t''_0 - C\rho'' \sec \beta'', \end{aligned} \quad (81)$$

in which $\log C = 7.760523$, expressed in parts of a day; and from these values of t , t' , t'' we recompute τ , τ' , and τ'' , which values will require no further correction, since ρ , ρ' , and ρ'' , derived from the first approximation, are sufficient for this purpose. With the new values of P and Q we recompute r , r' , r'' , and u , u' , u'' as before, and thence again P and Q , and if the last values differ from the preceding, we proceed in the same manner to a third approximation, which will usually be sufficient unless the interval of time between the extreme observations is considerable. If it be found necessary to proceed further with the approximations to P and Q after the calculation of these quantities in the third approximation has been effected, instead of employing these directly for the next trial, we may derive more accurate values from those already obtained. Thus, let x and y be the true values of P and Q respectively, with which, if the calculation be repeated, we should derive the same values again. Let Δx and Δy be the differences between any assumed values of x and y and the true values, or

$$x_0 = x + \Delta x, \quad y_0 = y + \Delta y.$$

Then, if we denote by x'_0 , y'_0 the values which result by direct calculation from the assumed values x_0 and y_0 , we shall have

$$x'_0 - x_0 = f(x_0, y_0) = f(x + \Delta x, y + \Delta y).$$

Expanding this function, we get

$$x'_0 - x_0 = f(x, y) + A\Delta x + B\Delta y + C\Delta x^2 + D\Delta x \Delta y + E\Delta y^2 + \dots,$$

and if Δx and Δy are very small, we may neglect terms of the second order. Further, since the employment of x and y will reproduce the same values, we have

$$f(x, y) = 0,$$

and hence, since $\Delta x = x_0 - x$ and $\Delta y = y_0 - y$,

$$x'_0 - x_0 = A(x_0 - x) + B(y_0 - y).$$

In a similar manner, we obtain

$$y_0' - y_0 = A'(x_0 - x) + B'(y_0 - y).$$

Let us now denote the values resulting from the first assumption for P and Q by P , and Q_1 , those resulting from P_1 , Q_1 by P_2 , Q_2 , and from P_2 , Q_2 by P_3 , Q_3 ; and, further, let

$$\begin{aligned} P_1 - P &= a, & P_2 - P_1 &= a', & P_3 - P_2 &= a'', \\ Q_1 - Q &= b, & Q_2 - Q_1 &= b', & Q_3 - Q_2 &= b''. \end{aligned}$$

Then, by means of the equations for $x_0' - x_0$ and $y_0' - y_0$, we shall have

$$\begin{aligned} a &= A(P - x) + B(Q - y), & b &= A'(P - x) + B'(Q - y), \\ a' &= A(P_1 - x) + B(Q_1 - y), & b' &= A'(P_1 - x) + B'(Q_1 - y), \\ a'' &= A(P_2 - x) + B(Q_2 - y), & b'' &= A'(P_2 - x) + B'(Q_2 - y). \end{aligned}$$

If we eliminate A , B , A' , and B' from these equations, the results are

$$\begin{aligned} x &= \frac{P(a'b'' - a''b') + P_1(a''b - ab'') + P_2(ab' - a'b)}{(a'b'' - a''b') + (a''b - ab'') + (ab' - a'b)}, \\ y &= \frac{Q(a'b'' - a''b') + Q_1(a''b - ab'') + Q_2(ab' - a'b)}{(a'b'' - a''b') + (a''b - ab'') + (ab' - a'b)}, \end{aligned}$$

from which we get

$$\begin{aligned} x &= P_3 - \frac{(a'' + a')(a'b'' - a''b') + a''(a''b - ab'')}{(a'b'' - a''b') + (a''b - ab'') + (ab' - a'b)}, \\ y &= Q_3 - \frac{(b'' + b')(a'b'' - a''b') + b''(a''b - ab'')}{(a'b'' - a''b') + (a''b - ab'') + (ab' - a'b)}. \end{aligned} \quad (82)$$

In the numerical application of these formulæ it will be more convenient to use, instead of the numbers P , P_1 , P_2 , Q , Q_1 , &c., the logarithms of these quantities, so that $a = \log P_1 - \log P$, $b = \log Q_1 - \log Q$, and similarly for a' , b' , a'' , b'' ,—which may also be expressed in units of the last decimal place of the logarithms employed,—and we shall thus obtain the values of $\log x$ and $\log y$. With these values of $\log x$ and $\log y$ for $\log P$ and $\log Q$ respectively, we proceed with the final calculation of r , r' , r'' , and u , u' , u'' .

When the eccentricity is small and the intervals of time between the observations are not very great, it will not be necessary to employ the equations (82); but if the eccentricity is considerable, and if, in addition to this, the intervals are large, they will be required. It may also occur that the values of P and Q derived from the last hypothesis as corrected by means of these formulæ, will differ so

much from the values found for x and y , on account of the neglected terms of the second order, that it will be necessary to recompute these quantities, using these last values of P and Q in connection with the three preceding ones in the numerical solution of the equations (82).

91. It remains now to complete the determination of the elements of the orbit from these final values of P and Q . As soon as Ω , i , and u , u' , u'' have been found, the remaining elements may be derived by means of r , r' , and $u'' - u$, and also from r' , r'' , and $u'' - u'$; or, which is better, we will obtain them from the extreme places, and, if the approximation to P and Q is complete, the results thus found will agree with those resulting from the combination of the middle place with either extreme.

We must, therefore, determine s' and x' from r , r'' , and $u'' - u$, by means of the formulæ already derived, and then, from the second of equations (46), we have

$$p = \left(\frac{s' r r'' \sin(u'' - u)}{\tau'} \right)^2, \quad (83)$$

from which to obtain p . If we compute s and s'' also, we shall have

$$p = \left(\frac{s r' r'' \sin(u'' - u')}{\tau} \right)^2 = \left(\frac{s' r r' \sin(u' - u)}{\tau''} \right)^2,$$

and the mean of the two values of p obtained from this expression should agree with that found from (83), thus checking the calculation and showing the degree of accuracy to which the approximation to P and Q has been carried.

The last of equations (65) gives

$$\sin \frac{1}{4}(E'' - E) = \sqrt{x'}, \quad (84)$$

from which $E'' - E$ may be computed. Then, from equation (57), since $e = \sin \varphi$, we have

$$a \cos \varphi = \frac{\sin \frac{1}{2}(u'' - u)}{\sin \frac{1}{2}(E'' - E)} \sqrt{r r'}, \quad (85)$$

for the calculation of $a \cos \varphi$. But $p = a(1 - e^2) = a \cos^2 \varphi$, whence

$$\cos \varphi = \frac{p}{a \cos \varphi}, \quad (86)$$

which may be used to determine φ when e is very nearly equal to unity; and then e may be found from

$$e = 1 - 2 \sin^2(45^\circ - \frac{1}{2}\varphi).$$

The equations (50) give

$$e \cos(u - \omega) = \frac{p}{r} - 1,$$

$$e \cos(u'' - \omega) = \frac{p}{r''} - 1,$$

and from these, by addition and subtraction, we derive

$$\begin{aligned} 2e \cos \frac{1}{2}(u'' - u) \cos \left(\frac{1}{2}(u'' + u) - \omega \right) &= \frac{p}{r} + \frac{p}{r''} - 2, \\ 2e \sin \frac{1}{2}(u'' - u) \sin \left(\frac{1}{2}(u'' + u) - \omega \right) &= \frac{p}{r} - \frac{p}{r''}, \end{aligned} \quad (87)$$

by means of which e and ω may be found.

Since

$$\cos 2\chi' = \frac{r - r''}{r + r''}, \quad \sin 2\chi' = \frac{2\sqrt{rr''}}{r + r''},$$

we have

$$\begin{aligned} \frac{p}{r} + \frac{p}{r''} - 2 &= \frac{2p}{\sqrt{rr''} \sin 2\chi'} - 2, \\ \frac{p}{r} - \frac{p}{r''} &= -\frac{2p \cot 2\chi'}{\sqrt{rr''}}, \end{aligned}$$

and from equations (70),

$$\cot 2\chi' = \frac{\sin \frac{1}{2}(u'' - u) \tan G'}{\cos \gamma'}, \quad \sin 2\chi' = \frac{\cos \gamma'}{\cos \frac{1}{2}(u'' - u)}.$$

Therefore the formulæ (87) reduce to

$$\begin{aligned} e \sin \left(\omega - \frac{1}{2}(u'' + u) \right) &= \frac{p}{\cos \gamma' \sqrt{rr''}} \tan G', \\ e \cos \left(\omega - \frac{1}{2}(u'' + u) \right) &= \frac{p}{\cos \gamma' \sqrt{rr''}} - \sec \frac{1}{2}(u'' - u), \end{aligned} \quad (88)$$

from which also e and ω may be derived. Then

$$\sin \varphi = e,$$

and the agreement of $\cos \varphi$ as derived from this value of φ with that given by (86) will serve as a further proof of the calculation. The longitude of the perihelion will be given by

$$\pi = \omega + \Omega,$$

or, when i exceeds 90° , and the distinction of retrograde motion is adopted, by $\pi = \Omega - \omega$.

To find a , we have

$$a = \frac{p}{\cos^2 \varphi} = \frac{(a \cos \varphi)^2}{p},$$

or it may be computed directly from the equation

$$a = \frac{\tau'^2}{4s'^2 r r'' \cos^2 \frac{1}{2}(u'' - u) \sin^2 \frac{1}{2}(E'' - E)}, \quad (89)$$

which results from the substitution, in the last term of the preceding equation, of the expressions for $a \cos \varphi$ and p given by (83) and (85). Then for the mean daily motion we have

$$\mu = \frac{k}{a^{\frac{3}{2}}}.$$

We have now only to find the mean anomaly corresponding to any epoch, and the elements are completely determined. For the true anomalies we have

$$v = u - \omega, \quad v' = u' - \omega, \quad v'' = u'' - \omega;$$

and if we compute r, r', r'' from these by means of the polar equation of the conic section, the results should agree with the values of the same quantities previously obtained. According to the equation (45)₁, we have

$$\begin{aligned} \tan \frac{1}{2} E &= \tan(45^\circ - \frac{1}{2} \varphi) \tan \frac{1}{2} v, \\ \tan \frac{1}{2} E' &= \tan(45^\circ - \frac{1}{2} \varphi) \tan \frac{1}{2} v', \\ \tan \frac{1}{2} E'' &= \tan(45^\circ - \frac{1}{2} \varphi) \tan \frac{1}{2} v'', \end{aligned} \quad (90)$$

from which to find $E, E',$ and E'' . The difference $E'' - E$ should agree with that derived from equation (84) within the limits of accuracy afforded by the logarithmic tables. Then, to find the mean anomalies, we have

$$\begin{aligned} M &= E - e \sin E, \\ M' &= E' - e \sin E', \\ M'' &= E'' - e \sin E''; \end{aligned} \quad (91)$$

and, if M_0 denotes the mean anomaly corresponding to any epoch T , we have, also,

$$\begin{aligned} M_0 &= M - \mu(t - T) \\ &= M' - \mu(t' - T) \\ &= M'' - \mu(t'' - T), \end{aligned}$$

in the application of which the values of $t, t',$ and t'' must be those which have been corrected for the time of aberration. The agree-

ment of the three values of M_0 will be a final test of the accuracy of the entire calculation. If the final values of P and Q are exact, this proof will be complete within the limits of accuracy admitted by the logarithmic tables.

When the eccentricity is such that the equations (91) cannot be solved with the requisite degree of accuracy, we must proceed according to the methods already given for finding the time from the perihelion in the case of orbits differing but little from the parabola. For this purpose, Tables IX. and X. will be employed. As soon as v , v' , and v'' have been determined, we may find the auxiliary angle V for each observation by means of Table IX.; and, with V as the argument, the quantities M , M' , M'' (which are not the mean anomalies) must be obtained from Table VI. Then, the perihelion distance having been computed from

$$q = \frac{p}{1+e},$$

we shall have

$$T = t - \frac{Mq^{\frac{3}{2}}}{C_0} \sqrt{\frac{2}{1+e}} = t' - \frac{M'q^{\frac{3}{2}}}{C_0} \sqrt{\frac{2}{1+e}} = t'' - \frac{M''q^{\frac{3}{2}}}{C_0} \sqrt{\frac{2}{1+e}} \quad (92)$$

in which $\log C_0 = 9.96012771$, for the determination of the time of perihelion passage. The times t , t' , t'' must be those which have been corrected for the time of aberration, and the agreement of the three values of T is a final proof of the numerical calculation.

If Table X. is used, as soon as the true anomalies have been found, the corresponding values of $\log B$ and $\log C$ must be derived from the table. Then w is computed from

$$\tan \frac{1}{2}w = \frac{\tan \frac{1}{2}v}{C} \sqrt{\frac{1+9e}{5(1+e)}},$$

and similarly for w' and w'' ; and, with these as arguments, we derive M , M' , M'' from Table VI. Finally, we have

$$T = t - \frac{MBq^{\frac{3}{2}}}{C_0 \sqrt{\frac{1}{10}(1+9e)}} = t' - \frac{M'B'q^{\frac{3}{2}}}{C_0 \sqrt{\frac{1}{10}(1+9e)}} = t'' - \frac{M''B''q^{\frac{3}{2}}}{C_0 \sqrt{\frac{1}{10}(1+9e)}}, \quad (93)$$

for the time of perihelion passage, the value of C_0 being the same as in (92).

When the orbit is a parabola, $e = 1$ and $p = 2q$, and the elements q and ω can be derived from r , r'' , u , and u'' by means of the equa-

tions (76), (83), and (88), or by means of the formulæ already given for the special case of parabolic motion.

92. Since certain quantities which are real in the ellipse and parabola become imaginary in the case of the hyperbola, the formulæ already given for determining the elements from r , r'' , u , and u'' require some modification when applied to a hyperbolic orbit.

When s' and x' have been found, p , e , and ω may be derived from equations (83) and (87) or (88) precisely as in the case of an elliptic orbit. Since $x' = \sin^2 \frac{1}{2} (E'' - E)$, we easily find

$$\sin \frac{1}{2} (E'' - E) = 2 \sqrt{x' - x'^2},$$

and equation (85) becomes

$$a \cos \varphi = \frac{\sin \frac{1}{2} (u'' - u) \sqrt{rr''}}{2 \sqrt{x' - x'^2}}. \quad (94)$$

But in the hyperbola x' is negative, and hence $\sqrt{x' - x'^2}$ will be imaginary; and, further, comparing the values of p in the ellipse and hyperbola, we have $\cos^2 \varphi = -\tan^2 \psi$, or

$$\cos \varphi = \sqrt{-1} \tan \psi.$$

Therefore the equation for $a \cos \varphi$ becomes

$$a \tan \psi = \frac{\sin \frac{1}{2} (u'' - u) \sqrt{rr''}}{2 \sqrt{x'^2 - x'}}, \quad (95)$$

if a is considered as being positive, from which $a \tan \psi$ may be obtained. Then, since $p = a \tan^2 \psi$, we have

$$\tan \psi = \frac{p}{a \tan \psi}, \quad (96)$$

for the determination of ψ , and the value of e computed from

$$e = \sec \psi = \sqrt{1 + \tan^2 \psi}$$

should agree with that derived from equation (88). When e differs but little from unity, it is conveniently and accurately computed from

$$e = 1 + 2 \sin^2 \frac{1}{2} \psi \sec \psi.$$

The value of a may be found from

$$a = p \cot^2 \psi = \frac{(a \tan \psi)^2}{p}, \quad (97)$$

or from

$$a = \frac{\tau'^2}{16s'^2 r r'' \cos^2 \frac{1}{2} (u'' - u) (x'^2 - x'')},$$

which is derived directly from (89), observing that the elliptic semi-transverse axis becomes negative in the case of the hyperbola.

As soon as ω has been found, we derive from u , u' , and u'' the corresponding values of v , v' , and v'' , and then compute the values of F , F' , and F'' by means of the formula (57)₁; after which, by means of the equation (69)₁, the corresponding values of N , N' , and N'' will be obtained. Finally, the time of perihelion passage will be given by

$$T = t - \frac{a^{\frac{3}{2}}}{\lambda_0 k} N = t' - \frac{a'^{\frac{3}{2}}}{\lambda_0 k} N' = t'' - \frac{a''^{\frac{3}{2}}}{\lambda_0 k} N''$$

wherein $\log \lambda_0 k = 7.87336575$.

The cases of hyperbolic orbits are rare, and in most of those which do occur the eccentricity will not differ much from that of the parabola, so that the most accurate determination of T will be effected by means of Tables IX. and X. as already illustrated.

93. EXAMPLE.—To illustrate the application of the principal formulæ which have been derived in this chapter, let us take the following observations of *Eurynome* ②9 :

| Ann Arbor M. T. | ②9 α | ②9 φ |
|--|---|------------------|
| 1863 Sept. 14 15 ^h 53 ^m 37 ^s .2 | 1 ^h 0 ^m 44 ^s .91 | + 9° 53' 30''.8, |
| 21 9 46 18.0 | 0 57 3.57 | 9 13 5.5, |
| 28 8 49 29.2 | 0 52 18.90 | + 8 22 8.7. |

The apparent obliquity of the ecliptic for these dates was, respectively, 23° 27' 20''.75, 23° 27' 20''.71, and 23° 27' 20''.65; and, by means of these, converting the observed right ascensions and declinations into apparent longitudes and latitudes, we get—

| Ann Arbor M. T. | Longitude. | Latitude. |
|--|-----------------|------------------|
| 1863 Sept. 14 15 ^h 53 ^m 37 ^s .2 | 17° 47' 37''.60 | + 3° 8' 43''.19, |
| 21 9 46 18.0 | 16 41 36.20 | 2 52 27.46, |
| 28 8 49 29.2 | 15 16 56.35 | + 2 32 42.98. |

For the same dates we obtain from the *American Nautical Almanac* the following places of the sun :

| True Longitude. | Latitude. | $\log R_0$. |
|-----------------|-----------|--------------|
| 172° 1' 42".1 | — 0.07 | 0.0022140, |
| 178 37 17 .2 | + 0.77 | 0.0013857, |
| 185 26 54 .8 | + 0.67 | 0.0005174. |

Since the elements are supposed to be wholly unknown, the places of the planet must be corrected for the aberration of the fixed stars as given by equations (1). Thus we find for the corrections to be applied to the longitudes, respectively,

$$-18''.48, \quad -19''.49, \quad -20''.8,$$

and for the latitudes,

$$+0''.47, \quad +0''.30, \quad +0''.14.$$

When these corrections are applied, we obtain the true places of the planet for the instants when the light was emitted, but as seen from the places of the earth at the instants of observation.

Next, each place of the sun must be reduced from the centre of the earth to the point in which a line drawn from the planet through the place of the observer cuts the plane of the ecliptic. For this purpose we have, for Ann Arbor,

$$\varphi' = 42^\circ 5'.4, \quad \log \rho_0 = 9.99935;$$

and the mean time of observation being converted into sidereal time gives, for the three observations,

$$\theta_0 = 3^h 29^m 1^s, \quad \theta'_0 = 21^h 48^m 17^s, \quad \theta''_0 = 21^h 18^m 55^s,$$

which are the right ascensions of the geocentric zenith, of which φ' is in each case the declination. From these we derive the longitude and latitude of the zenith for each observation, namely,

$$\begin{array}{lll} l_0 = 60^\circ 33'.9, & l'_0 = 350^\circ 35'.2, & l''_0 = 342^\circ 59'.2, \\ b_0 = +22 \ 25.0, & b'_0 = +50 \ 50.9, & b''_0 = +53 \ 41.6. \end{array}$$

Then, by means of equations (4), we obtain

$$\begin{array}{lll} \Delta \odot_0 = -18''.92, & \Delta \odot' = -36''.94, & \Delta \odot'' = -25''.76, \\ \Delta \log R_0 = -0.0001084, & \Delta \log R'_0 = -0.0002201, & \\ & \Delta \log R''_0 = -0.0002796. & \end{array}$$

For the reduction of time, we have the values $+0^s.15$, $+0^s.28$, and $+0^s.34$, which are so small that they may be neglected.

Finally, the longitudes of both the sun and planet are reduced to the mean equinox of 1863.0 by applying the corrections

$$-50''.95, \quad -51''.52, \quad -52''.14;$$

and the latitudes of the planet are reduced to the ecliptic of the same date by applying the corrections $-0''.15$, $-0''.14$, and $-0''.14$, respectively.

Collecting together and applying the several corrections thus obtained for the places of the sun and of the planet, reducing the uncorrected times of observation to the meridian of Washington, and expressing them in days from the beginning of the year, we have the following data:—

$$\begin{array}{lll} t_0 = 257.68079, & \lambda = 17^\circ 46' 28''.17, & \beta = +3^\circ 8' 43''.51, \\ t'_0 = 264.42570, & \lambda' = 16 \ 40 \ 25 \ .19, & \beta' = \ 2 \ 52 \ 27 \ .62, \\ t''_0 = 271.38625, & \lambda'' = 15 \ 15 \ 44 \ .03, & \beta'' = +2 \ 32 \ 42 \ .98, \\ \odot = 172^\circ \ 0' \ 32''.23, & \log R = 0.0021056, & \\ \odot' = 178 \ 35 \ 48 \ .74, & \log R' = 0.0011656, & \\ \odot'' = 185 \ 25 \ 36 \ .90, & \log R'' = 0.0002378. & \end{array}$$

The numerical values of the several corrections to be applied to the data furnished by observation and by the solar tables should be checked by duplicate calculation, since an error in any of these reductions will not be indicated until after the entire calculation of the elements has been effected.

By means of the equations

$$\begin{array}{ll} N = \frac{R'R'' \sin(\odot'' - \odot')}{RR'' \sin(\odot'' - \odot)}, & N'' = \frac{RR' \sin(\odot' - \odot)}{RR'' \sin(\odot'' - \odot)}, \\ \tan w' = \frac{\tan \beta'}{\sin(\lambda' - \odot')}, & \tan \psi' = \frac{\tan(\lambda' - \odot')}{\cos w'}, \end{array}$$

we obtain

$$\begin{array}{ll} \log N = 9.7087449, & \log N'' = 9.6950091, \\ \psi' = 161^\circ 42' 13''.16, & \\ \log(R' \sin \psi') = 9.4980010, & \log(R' \cos \psi') = 9.9786355. \end{array}$$

The quadrant in which ψ' must be taken is determined by the conditions that ψ' must be less than 180° , and that $\cos \psi'$ and $\cos(\lambda' - \odot')$ must have the same sign. Then from

$$\tan I \sin \left(\frac{1}{2} (\lambda'' + \lambda) - K \right) = \frac{\sin (\beta'' + \beta)}{2 \cos \beta \cos \beta'}, \sec \frac{1}{2} (\lambda'' - \lambda),$$

$$\tan I \cos \left(\frac{1}{2} (\lambda'' + \lambda) - K \right) = \frac{\sin (\beta'' - \beta)}{2 \cos \beta \cos \beta'}, \operatorname{cosec} \frac{1}{2} (\lambda'' - \lambda);$$

$$\tan \beta_0 = \sin (\lambda' - K) \tan I, \quad a_0 = \frac{\sin (\beta' - \beta_0)}{\cos \beta_0 \tan I},$$

$$b = \frac{R \sin (\odot - K)}{a_0}, \quad c = \frac{R' \sin (\odot' - K)}{a_0},$$

$$d = \frac{R'' \sin (\odot'' - K)}{a_0}, \quad f = \frac{\sec \beta'}{\sin (\lambda'' - \lambda)}, \quad h = \frac{RR'' \sin (\odot'' - \odot)}{a_0 \sin (\lambda'' - \lambda)}$$

we compute K , I , β_0 , a_0 , b , c , d , f , and h . The angle I must be less than 90° , and the value of β_0 must be determined with the greatest possible accuracy, since on this the accuracy of the resulting elements principally depends. Thus we obtain

$$\begin{aligned} K &= 4^\circ 47' 29''.48, & \log \tan I &= 9.3884640, \\ \beta_0 &= 2^\circ 52' 59'' \frac{2 \frac{1}{4}}{1 \frac{5}{9}}, & \log a_0 &= 6.8013583_n, \\ \log b &= 2.5456342_n, & \log c &= 2.2328550_n, \\ \log d &= 1.2437914, & \log f &= 1.3587437_n, & \log h &= 3.9247691. \end{aligned}$$

The formulæ

$$\begin{aligned} M_1 &= \frac{\sin (\lambda'' - \lambda')}{\sin (\lambda'' - \lambda)} + f \frac{R'' \sin (\lambda'' - \odot'')}{d}, \\ M_1'' &= \frac{\sin (\lambda' - \lambda)}{\sin (\lambda'' - \lambda)} - f \frac{R \sin (\lambda - \odot)}{b}, \\ M_2 &= \frac{h \sin (\lambda'' - K)}{d}, & M_2'' &= \frac{h \sin (\lambda - K)}{b}, \end{aligned}$$

give

$$\begin{aligned} \log M_1 &= 9.8946712, & \log M_1'' &= 9.6690383, \\ \log M_2 &= 1.9404111, & \log M_2'' &= 0.7306625_n. \end{aligned}$$

The quantities thus far obtained remain unchanged in the successive approximations to the values of P and Q .

For the first hypothesis, from

$$\begin{aligned} \tau &= k(t_0'' - t_0'), & \tau' &= k(t_0' - t_0), \\ P &= \frac{\tau''}{\tau}, & Q &= \tau \tau'', \\ c_0 &= \frac{b + Pd}{1 + P}, & k_0 &= c_0 - c, & l_0 &= -\frac{1}{2} c_0 Q, \\ \eta_0 \sin \zeta &= R' \sin \psi', \\ \eta_0 \cos \zeta &= k_0 - R' \cos \psi', \\ m_0 &= \frac{l_0}{\eta_0 R'^3 \sin^3 \psi'} \end{aligned}$$

we obtain

$$\begin{array}{ll} \log \tau = 9.0782249, & \log \tau'' = 9.0645575, \\ \log P = 9.9863326, & \log Q = 8.1427824, \\ \log c_0 = 2.2298567_n, & \log k_0 = 0.0704470, \\ \log l_0 = 0.0716091, & \log \gamma_0 = 0.3326925, \\ \zeta = 8^\circ 24' 49''.74, & \log m_0 = 1.2449136. \end{array}$$

The quadrant in which ζ must be situated is determined by the condition that γ_0 shall have the same sign as l_0 .

The value of z' must now be found by trial from the equation

$$\sin(z' - \zeta) = m_0 \sin^4 z'.$$

Table XII. shows that of the four roots of this equation one exceeds 180° , and is therefore excluded by the condition that $\sin z'$ must be positive, and that two of these roots give z' greater than $180^\circ - \psi'$, and are excluded by the condition that z' must be less than $180^\circ - \psi'$. The remaining root is that which belongs to the orbit of the planet, and it is shown to be approximately $10^\circ 40'$; but the correct value is found from the last equation by a few trials to be

$$z' = 9^\circ 1' 22''.96.$$

The root which corresponds to the orbit of the earth is $18^\circ 20' 41''$, and differs very little from $180^\circ - \psi'$.

Next, from

$$\begin{aligned} r' &= \frac{R' \sin \psi'}{\sin z'}, & \rho' &= \frac{R' \sin(z' + \psi')}{\sin z'} \cos \beta', \\ n &= \frac{1}{1+P} \left(1 + \frac{Q}{2r'^3} \right), & n'' &= nP, \\ \rho &= M_1 \frac{\rho'}{n} + M_2 \left(1 - \frac{N}{n} \right), \\ \rho'' &= M_1'' \frac{\rho'}{n''} + M_2'' \left(1 - \frac{N''}{n''} \right), \end{aligned}$$

we derive

$$\begin{array}{ll} \log r' = 0.3025672, & \log \rho' = 0.0123991, \\ \log n = 9.7061229, & \log n'' = 9.6924555, \\ \log \rho = 0.0254823, & \log \rho'' = 0.0028859. \end{array}$$

The values of the curtate distances having thus been found, the heliocentric places for the three observations are now computed from

$$\begin{aligned}
r \cos b \cos (l - \odot) &= \rho \cos (\lambda - \odot) - R, \\
r \cos b \sin (l - \odot) &= \rho \sin (\lambda - \odot), \\
r \sin b &= \rho \tan \beta; \\
r' \cos b' \cos (l' - \odot') &= \rho' \cos (\lambda' - \odot') - R', \\
r' \cos b' \sin (l' - \odot') &= \rho' \sin (\lambda' - \odot'), \\
r' \sin b' &= \rho' \tan \beta'; \\
r'' \cos b'' \cos (l'' - \odot'') &= \rho'' \cos (\lambda'' - \odot'') - R'', \\
r'' \cos b'' \sin (l'' - \odot'') &= \rho'' \sin (\lambda'' - \odot''), \\
r'' \sin b'' &= \rho'' \tan \beta'',
\end{aligned}$$

which give

$$\begin{aligned}
l &= 5^\circ 14' 39''.53, & \log \tan b &= 8.4615572, & \log r &= 0.3040994, \\
l' &= 7 \ 45 \ 11.28, & \log \tan b' &= 8.4107555, & \log r' &= 0.3025673, \\
l'' &= 10 \ 21 \ 34.57, & \log \tan b'' &= 8.3497911, & \log r'' &= 0.3011010.
\end{aligned}$$

The agreement of the value of $\log r'$ thus obtained with that already found, is a proof of part of the calculation. Then, from

$$\tan i \sin \left(\frac{1}{2}(l'' + l) - \Omega \right) = \frac{\tan b'' + \tan b}{2 \cos \frac{1}{2}(l'' - l)},$$

$$\tan i \cos \left(\frac{1}{2}(l'' + l) - \Omega \right) = \frac{\tan b'' - \tan b}{2 \sin \frac{1}{2}(l'' - l)},$$

$$\tan u = \frac{\tan(l - \Omega)}{\cos i}, \quad \tan u' = \frac{\tan(l' - \Omega)}{\cos i}, \quad \tan u'' = \frac{\tan(l'' - \Omega)}{\cos i},$$

we get

$$\begin{aligned}
\Omega &= 207^\circ 2' 38''.16, & i &= 4^\circ 27' 23''.84, \\
u &= 158^\circ 8' 25''.78, & u' &= 160^\circ 39' 18''.13, & u'' &= 163^\circ 16' 4''.42.
\end{aligned}$$

The equation

$$\tan b' = \tan i \sin (l' - \Omega)$$

gives $\log \tan b' = 8.4107514$, which differs 0.0000041 from the value already found directly from ρ' . This difference, however, amounts to only $0''.05$ in the value of the heliocentric latitude, and is due to errors of calculation. If we compute n and n'' from the equations

$$n = \frac{r'r'' \sin(u'' - u')}{rr'' \sin(u'' - u)}, \quad n'' = \frac{rr' \sin(u' - u)}{rr'' \sin(u'' - u)},$$

the results should agree with the values of these quantities previously computed directly from P and Q . Using the values of u , u' , and u'' just found, we obtain

$$\log n = 9.7061158, \quad \log n'' = 9.6924683,$$

which differ in the last decimal places from the values used in finding ρ and ρ'' . According to the equations

$$\begin{aligned}d \log n &= -21.055 \cot(u'' - u') du', \\d \log n'' &= +21.055 \cot(u' - u) du',\end{aligned}$$

the differences of $\log n$ and $\log n''$ being expressed in units of the seventh decimal place, the correction to u' necessary to make the two values of $\log n$ agree is $-0''.15$; but for the agreement of the two values of $\log n''$, u' must be diminished by $0''.26$, so that it appears that this proof is not complete, although near enough for the first approximation. It should be observed, however, that a great circle passing through the extreme observed places of the planet passes very nearly through the third place of the sun, and hence the values of ρ and ρ'' as determined by means of the last two of equations (18) are somewhat uncertain. In this case it would be advisable to compute ρ and ρ'' , as soon as ρ' has been found, by means of the equations (22) and (23). Thus, from these equations we obtain

$$\log \rho = 0.0254918, \quad \log \rho'' = 0.0028874,$$

and hence

$$\begin{aligned}l &= 5^\circ 14' 40''.05, & \log \tan b &= 8.4615619, & \log r &= 0.3041042, \\l'' &= 10^\circ 21' 34''.19, & \log \tan b'' &= 8.3497919, & \log r'' &= 0.3011017, \\ \Omega &= 207^\circ 2' 32''.97, & i &= 4^\circ 27' 25''.13, \\u &= 158^\circ 8' 31''.47, & u' &= 160^\circ 39' 23''.31, & u'' &= 163^\circ 16' 9''.22.\end{aligned}$$

The value of $\log \tan b'$ derived from l' and these values of Ω and i , is 8.4107555, agreeing exactly with that derived from ρ' directly. The values of n and n'' given by these last results for u , u' and u'' , are

$$\log n = 9.7061144, \quad \log n'' = 9.6924640;$$

and this proof will be complete if we apply the correction $du' = -0''.18$ to the value of u' , so that we have

$$u'' - u' = 2^\circ 36' 46''.09, \quad u' - u = 2^\circ 30' 51''.66.$$

The results which have thus been obtained enable us to proceed to a second approximation to the correct values of P and Q , and we may also correct the times of observation for the time of aberration by means of the formulæ

$$t = t_0 - C\rho \sec \beta, \quad t' = t_0' - C\rho' \sec \beta', \quad t'' = t_0'' - C\rho'' \sec \beta'',$$

wherein $\log C = 7.760523$, expressed in parts of a day. Thus we get

$$t = 257.67467, \quad t' = 264.41976, \quad t'' = 271.38044,$$

and hence

$$\log \tau = 9.0782331, \quad \log \tau' = 9.3724848, \quad \log \tau'' = 9.0645692.$$

Then, to find the ratios denoted by s and s'' , we have

$$\begin{aligned} \tan \chi &= \sqrt{\frac{\tau''}{\tau}}, \\ \sin \gamma \cos G &= \sin \frac{1}{2} (u'' - u'), \\ \sin \gamma \sin G &= \cos \frac{1}{2} (u'' - u') \cos 2\chi, \\ \cos \gamma &= \cos \frac{1}{2} (u'' - u') \sin 2\chi; \\ \tan \chi'' &= \sqrt{\frac{\tau'}{\tau}}, \\ \sin \gamma'' \cos G'' &= \sin \frac{1}{2} (u' - u), \\ \sin \gamma'' \sin G'' &= \cos \frac{1}{2} (u' - u) \cos 2\chi'', \\ \cos \gamma'' &= \cos \frac{1}{2} (u' - u) \sin 2\chi''; \\ m &= \frac{\tau^2}{(\tau' + \tau'')^3 \cos^3 \gamma}, & j &= \frac{\sin^2 \frac{1}{2} \gamma}{\cos \gamma}, \\ m'' &= \frac{\tau'^2}{(r + r')^3 \cos^3 \gamma''}, & j'' &= \frac{\sin^2 \frac{1}{2} \gamma''}{\cos \gamma''}, \end{aligned}$$

from which we obtain

$$\begin{aligned} \chi &= 44^\circ 57' 6''.00, & \chi'' &= 44^\circ 56' 57''.50, \\ \gamma &= 1 \ 18 \ 35 \ .90, & \gamma'' &= 1 \ 15 \ 40 \ .69, \\ \log m &= 6.3482114, & \log m'' &= 6.3163548, \\ \log j &= 6.1163135, & \log j'' &= 6.0834230. \end{aligned}$$

From these, by means of the equations

$$\begin{aligned} \eta &= \frac{m}{\frac{5}{6} + j + \xi}, & x &= \frac{m}{s^2} - j, \\ \eta'' &= \frac{m''}{\frac{5}{6} + j'' + \xi''}, & x'' &= \frac{m''}{s''^2} - j'', \end{aligned}$$

using Tables XIII. and XIV., we compute s and s'' . First, in the case of s , we assume

$$\eta = \frac{m}{\frac{5}{6} + j} = 0.0002675,$$

and, with this as the argument, Table XIII. gives $\log s^2 = 0.0002581$. Hence we obtain $x' = 0.000092$, and, with this as the argument, Table XIV. gives $\xi = 0.00000001$; and, therefore, it appears that a repetition of the calculation is unnecessary. Thus we obtain

$$\log s = 0.0001290, \quad \log s'' = 0.0001200.$$

When the intervals are small, it is not necessary to use the formulæ

in the complete form here given, since these ratios may then be found by a simpler process, as will appear in the sequel. Then, from

$$P = \frac{\tau''}{\tau} \cdot \frac{s}{s''},$$

$$Q = \frac{\tau\tau''}{ss''} \cdot \frac{r'^2}{rr'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u)},$$

we find

$$\log P = 9.9863451,$$

$$\log Q = 8.1431341,$$

with which the second approximation may be completed. We now compute c_0 , k_0 , l_0 , z' , &c. precisely as in the first approximation; but we shall prefer, for the reason already stated, the values of ρ and ρ'' computed by means of the equations (22) and (23) instead of those obtained from the last two of the formulæ (18). The results thus derived are as follows:—

$$\log c_0 = 2.2298499,$$

$$\log k_0 = 0.0714280,$$

$$\log l_0 = 0.0719540,$$

$$\log \eta_0 = 0.3332233,$$

$$\zeta = 8^\circ 24' 12''.48,$$

$$\log m_0 = 1.2447277,$$

$$z' = 9^\circ 0' 30''.84,$$

$$\log r' = 0.3032587,$$

$$\log \rho' = 0.0137621,$$

$$\log n = 9.7061153,$$

$$\log n'' = 9.6924604,$$

$$\log \rho = 0.0269143,$$

$$\log \rho'' = 0.0041748,$$

$$l = 5^\circ 15' 57''.26, \quad \log \tan b = 8.4622524, \quad \log r = 0.3048368,$$

$$l' = 7^\circ 46' 2''.76, \quad \log \tan b' = 8.4114276, \quad \log r' = 0.3032587,$$

$$l'' = 10^\circ 22' 0''.91, \quad \log \tan b'' = 8.3504332, \quad \log r'' = 0.3017481,$$

$$\Omega = 207^\circ 0' 0''.72,$$

$$i = 4^\circ 28' 35''.20,$$

$$u = 158^\circ 12' 19''.54, \quad u' = 160^\circ 42' 45''.82, \quad u'' = 163^\circ 19' 7''.14.$$

The agreement of the two values of $\log r'$ is complete, and the value of $\log \tan b'$ computed from

$$\tan b' = \tan i \sin (l' - \Omega),$$

is $\log \tan b' = 8.4114279$, agreeing with the result derived directly from ρ' . The values of n and n'' obtained from the equations (54) are

$$\log n = 9.7061156,$$

$$\log n'' = 9.6924603,$$

which agree with the values already used in computing ρ and ρ'' , and the proof of the calculation is complete. We have, therefore,

$$u'' - u' = 2^\circ 36' 21''.32, \quad u' - u = 2^\circ 30' 26''.28, \quad u'' - u = 5^\circ 6' 47''.60.$$

From these values of $u'' - u'$ and $u' - u$, we obtain

$$\log s = 0.0001284,$$

$$\log s'' = 0.0001193,$$

and, recomputing P and Q , we get

$$\log P = 9.9863452, \quad \log Q = 8.1431359,$$

which differ so little from the preceding values of these quantities that another approximation is unnecessary. We may, therefore, from the results already derived, complete the determination of the elements of the orbit.

The equations

$$\begin{aligned} \tan \chi' &= \sqrt{\frac{r''}{r}}, \\ \sin \gamma' \cos G' &= \sin \frac{1}{2} (u'' - u), \\ \sin \gamma' \sin G' &= \cos \frac{1}{2} (u'' - u) \cos 2\chi', \\ \cos \gamma' &= \cos \frac{1}{2} (u'' - u) \sin 2\chi', \\ m' &= \frac{\tau^2}{(r + r'')^3 \cos^3 \gamma'}, \quad j' = \frac{\sin^2 \frac{1}{2} \gamma'}{\cos \gamma'}, \end{aligned}$$

give

$$\begin{aligned} \chi' &= 44^\circ 53' 53''.25, & \gamma' &= 2^\circ 33' 52''.97, & \log \tan G' &= 8.9011435 \\ \log m' &= 6.9332999, & \log j' &= 6.7001345. \end{aligned}$$

From these, by means of the formulæ

$$\eta' = \frac{m'}{\frac{5}{6} + j' + \xi'}, \quad x' = \frac{m'}{\xi'^2} - j',$$

and Tables XIII. and XIV., we obtain

$$\log s'^2 = 0.0009908, \quad \log x' = 6.5494116.$$

Then from

$$p = \left(\frac{s' r' r'' \sin (u'' - u)}{\tau'} \right)^2,$$

we get

$$\log p = 0.3691818.$$

The values of $\log p$ given by

$$p = \left(\frac{s' r' r'' \sin (u'' - u)}{\tau} \right)^2 = \left(\frac{s' r' r'' \sin (u' - u)}{\tau''} \right)^2$$

are 0.3691824 and 0.3691814, the mean of which agrees with the result obtained from $u'' - u$, and the differences between the separate results are so small that the approximation to P and Q is sufficient.

The equations

$$\begin{aligned} \sin \frac{1}{4} (E'' - E) &= \sqrt{x'}, \\ a \cos \varphi &= \frac{\sin \frac{1}{2} (u'' - u)}{\sin \frac{1}{2} (E'' - E)} \sqrt{r r''}, \\ \cos \varphi &= \frac{p}{a \cos \varphi}, \end{aligned}$$

give

$$\frac{1}{4}(E'' - E) = 1^\circ 4' 42''.903, \quad \log(a \cos \varphi) = 0.3770315, \\ \log \cos \varphi = 9.9921503.$$

Next, from

$$e \sin(\omega - \frac{1}{2}(u'' + u)) = \frac{p}{\cos \gamma' \sqrt{rr''}} \tan G', \\ e \cos(\omega - \frac{1}{2}(u'' + u)) = \frac{p}{\cos \gamma' \sqrt{rr''}} - \sec \frac{1}{2}(u'' - u),$$

we obtain

$$\omega = 190^\circ 15' 39''.57, \quad \log e = \log \sin \varphi = 9.2751434, \\ \varphi = 10 \quad 51 \quad 39 \quad .62, \quad \pi = \omega + \Omega = 37^\circ 15' 40''.29.$$

This value of φ gives $\log \cos \varphi = 9.9921501$, agreeing with the result already found.

To find a and μ , we have

$$a = \frac{p}{\cos^2 \varphi}, \quad \mu = \frac{k}{a^2},$$

the value of k expressed in seconds of arc being $\log k = 3.5500066$, from which the results are

$$\log a = 0.3848816, \quad \log \mu = 2.9726842.$$

The true anomalies are given by

$$v = u - \omega, \quad v' = u' - \omega, \quad v'' = u'' - \omega,$$

according to which we have

$$v = 327^\circ 56' 39''.97, \quad v' = 330^\circ 27' 6''.25, \quad v'' = 333^\circ 3' 27''.57.$$

If we compute r , r' , and r'' from these values by means of the polar equation of the ellipse, we get

$$\log r = 0.3048367, \quad \log r' = 0.3032586, \quad \log r'' = 0.3017481,$$

and the agreement of these results with those derived directly from ρ , ρ' , and ρ'' is a further proof of the calculation.

The equations

$$\tan \frac{1}{2}E = \tan(45^\circ - \frac{1}{2}\varphi) \tan \frac{1}{2}v, \\ \tan \frac{1}{2}E' = \tan(45^\circ - \frac{1}{2}\varphi) \tan \frac{1}{2}v', \\ \tan \frac{1}{2}E'' = \tan(45^\circ - \frac{1}{2}\varphi) \tan \frac{1}{2}v''$$

give

$$E = 333^\circ 17' 28''.18, \quad E' = 335^\circ 24' 38''.00, \quad E'' = 337^\circ 36' 19''.78.$$

The value of $\frac{1}{4}(E'' - E)$ thus obtained differs only $0''.003$ from that computed directly from x' .

Finally, for the mean anomalies we have

$$M = E - e \sin E, \quad M' = E' - e \sin E', \quad M'' = E'' - e \sin E'',$$

from which we get

$$M = 338^\circ 8' 36''.71, \quad M' = 339^\circ 54' 10''.61, \quad M'' = 341^\circ 43' 6''.97;$$

and if M_0 denotes the mean anomaly for the date $T=1863$ Sept. 21.5 Washington mean time, from the formulæ

$$\begin{aligned} M &= M - \mu(t - T) \\ &= M' - \mu(t' - T) \\ &= M'' - \mu(t'' - T), \end{aligned}$$

we obtain the three values $339^\circ 55' 25''.97$, $339^\circ 55' 25''.96$, and $339^\circ 55' 25''.96$, the mean of which gives

$$M_0 = 339^\circ 55' 25''.96.$$

The agreement of the three results for M_0 is a final proof of the accuracy of the entire calculation of the elements.

Collecting together the separate results obtained, we have the following elements:

Epoch = 1863 Sept. 21.5 Washington mean time.

$$M = 339^\circ 55' 25''.96$$

$$\pi = 37 \ 15 \ 40 \ .29$$

$$\Omega = 207 \ 0 \ 0 \ .72$$

$$i = 4 \ 28 \ 35 \ .20$$

$$\varphi = 10 \ 51 \ 39 \ .62$$

$$\log a = 0.3848816$$

$$\log \mu = 2.9726842$$

$$\mu = 939''.04022.$$

Ecliptic and Mean
Equinox 1863.0.

If we compute the geocentric right ascension and declination of the planet directly from these elements for the dates of the observations, as corrected for the time of aberration, and then reduce the observations to the centre of the earth by applying the corrections for parallax, the comparison of the results thus obtained will show how closely the elements represent the places on which they are based. Thus, we compute first the auxiliary constants for the equator, using the mean obliquity of the ecliptic,

$$\epsilon = 23^\circ 27' 24''.96,$$

and the following expressions for the heliocentric co-ordinates of the planet are obtained :

$$\begin{aligned}x &= r [9.9997272] \sin (296^\circ 55' 46''.05 + u), \\y &= r [9.9744699] \sin (206 \quad 12 \quad 42 \quad .79 + u), \\z &= r [9.5249539] \sin (212 \quad 39 \quad 14 \quad .62 + u).\end{aligned}$$

The numbers enclosed in the brackets are the logarithms of $\sin a$, $\sin b$, and $\sin c$, respectively; and these equations give the co-ordinates referred to the mean equinox and equator of 1863.0.

The places of the sun for the corrected times of observation, and referred to the mean equinox of 1863.0, are

| True Longitude. | Latitude. | Log R . |
|-----------------|-----------|------------|
| 172° 0' 29''.5 | — 0''.07 | 0.0022146, |
| 178 36 4 .5 | + 0 .77 | 0.0013864, |
| 185 25 42 .0 | + 0 .67 | 0.0005182. |

If we compute from these values, by means of the equations (104), the co-ordinates of the sun, and combine them with the corresponding heliocentric co-ordinates of the planet, we obtain the following geocentric places of the planet :

$$\begin{array}{lll}\alpha = 15^\circ 10' 29''.06, & \delta = + 9^\circ 53' 16''.72, & \log A = 0.02726, \\ \alpha' = 14 \quad 15 \quad 0 \quad .22, & \delta' = \quad 9 \quad 12 \quad 51 \quad .29, & \log A' = 0.01410, \\ \alpha'' = 13 \quad 3 \quad 49 \quad .47, & \delta'' = + 8 \quad 21 \quad 54 \quad .46, & \log A'' = 0.00433.\end{array}$$

To reduce these places to the apparent equinox of the date of observation, the corrections

$$+ 48''.14, \quad + 48''.54, \quad + 48''.91,$$

must be applied to the right ascensions, respectively, and

$$+ 18''.55, \quad + 18''.92, \quad + 19''.31,$$

to the declinations. Thus we obtain :

| Washington M. T. | Comp. α . | Comp. δ . |
|---------------------|---|------------------|
| 1863 Sept. 14.67467 | 1 ^h 0 ^m 45 ^s .15 | + 9° 53' 35''.3, |
| 21.41976 | 0 57 3.25 | 9 13 10 .2, |
| 28.38044 | 0 52 18.56 | + 8 22 13 .8. |

The corrections to be applied to the respective observations, in order to reduce them to the centre of the earth, are + 0°.24, — 0°.31, — 0°.34 in right ascension, and + 4''.5, + 4''.8, + 5''.1 in declination, so that we have, for the same dates,

| Observed α . | Observed δ . |
|---|---------------------|
| 1 ^h 0 ^m 45 ^s .15 | + 9° 53' 35".3, |
| 0 57 3.26 | 9 13 10.3, |
| 0 52 18.56 | + 8 22 13.8. |

The comparison of these with the computed values shows that the extreme places are exactly represented, while the difference in the middle place amounts to only 0'.01 in right ascension, and to 0".1 in declination. It appears, therefore, that the observations are completely satisfied by the elements obtained, and that the preliminary corrections for aberration and parallax, as determined by the equations (1) and (4), have been correctly computed.

It cannot be expected that a system of elements derived from observations including an interval of only fourteen days, will be so exact as the results which are obtained from a series of observations or from those including a much longer interval of time; and although the elements which have been derived completely represent the data, yet, on account of the smallness of $\beta' - \beta_0$, this difference being only 31".893, the slight errors of observation have considerable influence in the final results.

When approximate elements are already known, so that the correction for parallax may be applied directly to the observations, in order to take into account the latitude of the sun, the observed places of the body must be reduced, by means of equation (6), to the point in which a perpendicular let fall from the centre of the earth to the plane of the ecliptic cuts that plane. The times of observation must also be corrected for the time of aberration, and the corresponding places of both the planet and the sun must be reduced to the ecliptic and mean equinox of a fixed epoch; and further, the reduction to the fixed ecliptic should precede the application of equation (6).

If the intervals between the times of observation are considerable, it may become necessary to make three or more approximations to the values of P and Q , and in this case the equations (82) may be applied. But when approximate elements are already known, it will be advantageous to compute the first assumed values of P and Q directly from these elements by means of the equations (44) or by means of (48) and (51); and the ratios s and s'' may be found directly from the equations (46). In the case of very eccentric orbits this is indispensable, if it be desired to avoid prolixity in the numerical calculation, since otherwise the successive approximations to P and Q will slowly approach the limits required.

The various modifications of the formulæ for certain special cases, as well as the formulæ which must be used in the case of parabolic and hyperbolic orbits, and of those differing but little from the parabola, have been given in a form such that they require no further illustration.

94. In the determination of an unknown orbit, if the intervals are considerably unequal, it will be advantageous to correct the first assumed value of P before completing the first approximation in the manner already illustrated. The assumption of

$$Q = \tau\tau''$$

is correct to terms of the fourth order with respect to the time, and for the same degree of approximation to P we must, according to equation (28)₃, use the expression

$$P = \frac{\tau''}{\tau} \left(1 + \frac{1}{6} \frac{\tau^2 - \tau''^2}{r'^3} \right),$$

which becomes equal to $\frac{\tau''}{\tau}$ only when the intervals are equal. The first assumed values

$$P = \frac{\tau''}{\tau}, \quad Q = \tau\tau'',$$

furnish, with very little labor, an approximate value of r' ; and then, with the values of P and Q , derived from

$$P = \frac{\tau''}{\tau} \left(1 + \frac{1}{6} \frac{\tau^2 - \tau''^2}{r'^3} \right), \quad Q = \tau\tau'', \quad (98)$$

the entire calculation should be completed precisely as in the example given. Thus, in this example, the first assumed values give

$$\log r' = 0.30257,$$

and, recomputing P by means of the first of these equations, we get

$$\log P = 9.9863404, \quad \log Q = 8.1427822,$$

with which, if the first approximation to the elements be completed, the results will differ but little from those obtained, without this correction, from the second hypothesis. If the times had been already corrected for the time of aberration, the agreement would be still closer.

The comparison of equations (46) with (25)₃ gives, to terms of the fourth order,

$$s = 1 + \frac{1}{6} \frac{\tau^2}{r'^3}, \quad s' = 1 + \frac{1}{6} \frac{\tau'^2}{r''^3}, \quad s'' = 1 + \frac{1}{6} \frac{\tau''^2}{r'''^3},$$

and, if the intervals are equal, this value of s' is correct to terms of the fifth order. Since

$$\log_e s = \log_e (1 + (s - 1)) = s - 1 - \frac{1}{2}(s - 1)^2 + \&c.,$$

we have, neglecting terms of the fourth order,

$$\log s = \frac{\lambda_0}{6} \cdot \frac{\tau^2}{r'^3}, \quad (99)$$

in which $\log \frac{1}{6} \lambda_0 = 8.8596330$. We have, also, to the same degree of approximation,

$$\log s' = \frac{\lambda_0}{6} \cdot \frac{\tau'^2}{r''^3}, \quad \log s'' = \frac{\lambda_0}{6} \cdot \frac{\tau''^2}{r'''^3}. \quad (100)$$

For the values

$$\begin{aligned} \log \tau &= 9.0782331, & \log \tau' &= 9.3724848, & \log \tau'' &= 9.0645692, \\ \log r' &= 0.3032587, & & & & \end{aligned}$$

these formulæ give

$$\log s = 0.0001277, \quad \log s' = 0.0004953, \quad \log s'' = 0.0001199,$$

which differ but little from the correct values 0.0001284, 0.0004954, and 0.0001193 previously obtained.

Since

$$\sec^3 \gamma' = 1 + 6 \sin^2 \frac{1}{2} \gamma' + \&c.,$$

the second of equations (65) gives

$$m' = \frac{\tau'^2}{(r + r'')^3} + \frac{6\tau'^2}{(r + r'')^3} \sin^2 \frac{1}{2} \gamma' + \&c.$$

Substituting this value in the first of equations (66), we get

$$s'^2 (s' - 1) = \frac{\tau'^2}{y' (r + r'')^3} + \frac{6\tau'^2}{y' (r + r'')^3} \sin^2 \frac{1}{2} \gamma' + \&c.$$

If we neglect terms of the fourth order with respect to the time, it will be sufficient in this equation to put $y' = \frac{2}{3}$, according to (71), and hence we have

$$s'^2 (s' - 1) = \frac{4}{3} \frac{\tau'^2}{(r + r'')^3};$$

and, since $s' - 1$ is of the second order with respect to τ' , we have, to terms of the fourth order,

$$s'^2 (s' - 1) = \log_e s'.$$

Therefore,

$$\log s' = \frac{4}{3} \lambda_c \frac{\tau'^2}{(r + r'')^3}, \quad (101)$$

which, when the intervals are small, may be used to find s' from r and r'' . In the same manner, we obtain

$$\log s = \frac{4}{3} \lambda_0 \frac{\tau^2}{(r' + r'')^3}, \quad \log s'' = \frac{4}{3} \lambda_0 \frac{\tau''^2}{(r + r')^3}. \quad (102)$$

For logarithmic calculation, when addition and subtraction logarithms are not used, it is more convenient to introduce the auxiliary angles χ , χ' , and χ'' , by means of which these formulæ become

$$\log s = \frac{4}{3} \lambda_0 \frac{\tau^2 \cos^6 \chi}{r'^3}, \quad \log s' = \frac{4}{3} \lambda_0 \frac{\tau'^2 \cos^6 \chi'}{r^3}, \quad \log s'' = \frac{4}{3} \lambda_0 \frac{\tau''^2 \cos^6 \chi''}{r^3}, \quad (103)$$

in which $\log \frac{4}{3} \lambda_0 = 9.7627230$. For the first approximation these equations will be sufficient, even when the intervals are considerable, to determine the values of s and s'' required in correcting P and Q .

The values of τ , τ' , τ'' , and r'' above given, in connection with

$$\log r = 0.3048368, \quad \log r'' = 0.3017481,$$

give

$$\log s = 0.0001284, \quad \log s' = 0.0004951, \quad \log s'' = 0.0001193.$$

These results for $\log s$ and $\log s''$ are correct, and that for $\log s'$ differs only 3 in the seventh decimal place from the correct value.

CHAPTER V.

DETERMINATION OF THE ORBIT OF A HEAVENLY BODY FROM FOUR OBSERVATIONS,
OF WHICH THE SECOND AND THIRD MUST BE COMPLETE.

95. THE formulæ given in the preceding chapter are not sufficient to determine the elements of the orbit of a heavenly body when its apparent path is in the plane of the ecliptic. In this case, however, the position of the plane of the orbit being known, only four elements remain to be determined, and four observed longitudes will furnish the necessary equations. There is no instance of an orbit whose inclination is zero; but, although no such case may occur, it may happen that the inclination is very small, and that the elements derived from three observations will on this account be uncertain, and especially so, if the observations are not very exact. The difficulty thus encountered may be remedied by using for the data in the determination of the elements one or more additional observations, and neglecting those latitudes which are regarded as most uncertain. The formulæ, however, are most convenient, and lead most expeditiously to a knowledge of the elements of an orbit wholly unknown, when they are made to depend on four observations, the second and third of which must be complete; but of the extreme observations only the longitudes are absolutely required.

The preliminary reductions to be applied to the data are derived precisely as explained in the preceding chapter, preparatory to a determination of the elements of the orbit from three observations.

Let t, t', t'', t''' be the times of observation, r, r', r'', r''' the radii-vectores of the body, u, u', u'', u''' the corresponding arguments of the latitude, R, R', R'', R''' the distances of the earth from the sun, and $\odot, \odot', \odot'', \odot'''$ the longitudes of the sun corresponding to these times. Let us also put

$$\begin{aligned} [r'r'''] &= r'r''' \sin(u''' - u'), \\ [r''r'''] &= r''r''' \sin(u''' - u''), \end{aligned}$$

and

$$n' = \frac{[r''r''']}{[r'r''']}, \quad n'' = \frac{[r'r'']}{[r'r''']}. \quad (1)$$

Then, according to the equations (5)₃, we shall have

$$\begin{aligned} nx - x' + n'x'' &= 0, \\ ny - y' + n'y'' &= 0, \\ n'x' - x'' + n''x''' &= 0, \\ n'y' - y'' + n''y''' &= 0. \end{aligned} \quad (2)$$

Let $\lambda, \lambda', \lambda'', \lambda'''$ be the observed longitudes, $\beta, \beta', \beta'', \beta'''$ the observed latitudes corresponding to the times t, t', t'', t''' , respectively, and $\Delta, \Delta', \Delta'', \Delta'''$ the distances of the body from the earth. Further, let

$$\Delta''' \cos \beta''' = \rho''',$$

and for the last place we have

$$\begin{aligned} x''' &= \rho''' \cos \lambda''' - R''' \cos \odot''', \\ y''' &= \rho''' \sin \lambda''' - R''' \sin \odot'''. \end{aligned}$$

Introducing these values of x''' and y''' , and the corresponding values of x, x', x'', y, y', y'' into the equations (2), they become

$$\begin{aligned} 0 &= n(\rho \cos \lambda - R \cos \odot) - (\rho' \cos \lambda' - R' \cos \odot') \\ &\quad + n''(\rho'' \cos \lambda'' - R'' \cos \odot''), \\ 0 &= n(\rho \sin \lambda - R \sin \odot) - (\rho' \sin \lambda' - R' \sin \odot') \\ &\quad + n''(\rho'' \sin \lambda'' - R'' \sin \odot''), \\ 0 &= n'(\rho' \cos \lambda' - R' \cos \odot') - (\rho'' \cos \lambda'' - R'' \cos \odot'') \\ &\quad + n'''(\rho''' \cos \lambda''' - R''' \cos \odot'''), \\ 0 &= n'(\rho' \sin \lambda' - R' \sin \odot') - (\rho'' \sin \lambda'' - R'' \sin \odot'') \\ &\quad + n'''(\rho''' \sin \lambda''' - R''' \sin \odot'''). \end{aligned} \quad (3)$$

If we multiply the first of these equations by $\sin \lambda$, and the second by $-\cos \lambda$, and add the products, we get

$$\begin{aligned} 0 &= nR \sin(\lambda - \odot) - (\rho' \sin(\lambda' - \lambda) + R' \sin(\lambda - \odot')) \\ &\quad + n''(\rho'' \sin(\lambda'' - \lambda) + R'' \sin(\lambda - \odot'')); \end{aligned} \quad (4)$$

and in a similar manner, from the third and fourth equations, we find

$$\begin{aligned} 0 &= n'(\rho' \sin(\lambda''' - \lambda') - R' \sin(\lambda''' - \odot')) \\ &\quad - (\rho'' \sin(\lambda''' - \lambda'') - R'' \sin(\lambda''' - \odot'')) - n'''R''' \sin(\lambda''' - \odot'''). \end{aligned} \quad (5)$$

Whenever the values of $n, n', n'',$ and n''' are known, or may be determined in functions of the time so as to satisfy the conditions of motion in a conic section, these equations become distinct or independent of each other; and, since only two unknown quantities ρ'

and ρ'' are involved in them, they will enable us to determine these curtate distances.

Let us now put

$$\begin{aligned}\cos \beta' \sin (\lambda' - \lambda) &= A, & \cos \beta'' \sin (\lambda'' - \lambda) &= B, \\ \cos \beta' \sin (\lambda''' - \lambda'') &= C, & \cos \beta' \sin (\lambda''' - \lambda') &= D,\end{aligned}\quad (6)$$

and the preceding equations give

$$\begin{aligned}A\rho' \sec \beta' - Bn''\rho'' \sec \beta'' &= nR \sin (\lambda - \odot) - R' \sin (\lambda - \odot') \\ &\quad + n''R'' \sin (\lambda - \odot''), \\ Dn'\rho' \sec \beta' - C\rho'' \sec \beta'' &= n'R' \sin (\lambda''' - \odot') - R'' \sin (\lambda''' - \odot'') \\ &\quad + n'''R''' \sin (\lambda''' - \odot''').\end{aligned}\quad (7)$$

If we assume for n and n'' their values in the case of the orbit of the earth, which is equivalent to neglecting terms of the second order in the equations (26)₃, the second member of the first of these equations reduces rigorously to zero; and in the same manner it can be shown that when similar terms of the second order in the corresponding expressions for n' and n'' are neglected, the second member of the last equation reduces to zero. Hence the second member of each of these equations will generally differ from zero by a quantity which is of at least the second order with respect to the intervals of time between the observations. The coefficients of ρ' and ρ'' are of the first order, and it is easily seen that if we eliminate ρ'' from these equations, the resulting equation for ρ' is such that an error of the second order in the values of n and n'' may produce an error of the order zero in the result for ρ' , so that it will not be even an approximation to the correct value; and the same is true in the case of ρ'' . It is necessary, therefore, to retain terms of the second order in the first assumed values for n , n' , n'' , and n''' ; and, since the terms of the second order involve r' and r'' , we thus introduce two additional unknown quantities. Hence two additional equations involving r' , r'' , ρ' , ρ'' and quantities derived from observation, must be obtained, so that by elimination the values of the quantities sought may be found.

From equation (34)₄ we have

$$\rho' \sec \beta' = R' \cos \Psi' \pm \sqrt{r'^2 - R'^2 \sin^2 \Psi'}, \quad (8)$$

which is one of the equations required; and similarly we find, for the other equation,

$$\rho'' \sec \beta'' = R'' \cos \Psi'' \pm \sqrt{r''^2 - R''^2 \sin^2 \Psi''}. \quad (9)$$

Introducing these values into the equations (7), and putting

$$\begin{aligned} x' &= \pm \sqrt{r'^2 - R'^2 \sin^2 \Psi'}, \\ x'' &= \pm \sqrt{r''^2 - R''^2 \sin^2 \Psi''}, \end{aligned} \quad (10)$$

we get

$$\begin{aligned} Ax' - Bn''x'' &= nR \sin(\lambda - \odot) - R' \sin(\lambda - \odot') \\ &\quad + n''R'' \sin(\lambda - \odot'') - AR' \cos \Psi' + n''BR'' \cos \Psi'', \\ Dn'x' - Cx'' &= n'R' \sin(\lambda''' - \odot') - R'' \sin(\lambda''' - \odot'') \\ &\quad + n'''R''' \sin(\lambda''' - \odot''') - n'DR' \cos \Psi' + CR'' \cos \Psi''. \end{aligned}$$

Let us now put

$$\frac{B}{A} = h', \quad \frac{D}{C} = h'',$$

or

$$\begin{aligned} h' &= \frac{\cos \beta'' \sin(\lambda'' - \lambda)}{\cos \beta' \sin(\lambda' - \lambda)}, & h'' &= \frac{\cos \beta' \sin(\lambda''' - \lambda')}{\cos \beta'' \sin(\lambda''' - \lambda'')}, \\ R' \cos \Psi' + \frac{R' \sin(\lambda - \odot')}{A} &= a', \\ R'' \cos \Psi'' - \frac{R'' \sin(\lambda''' - \odot'')}{C} &= a'', \\ h'R'' \cos \Psi'' + \frac{R'' \sin(\lambda - \odot'')}{A} &= c', \\ h''R' \cos \Psi' - \frac{R' \sin(\lambda''' - \odot')}{C} &= c'', \\ \frac{R \sin(\lambda - \odot)}{A} &= d', & \frac{R''' \sin(\lambda''' - \odot''')}{C} &= -d'', \end{aligned} \quad (11)$$

and we have

$$\begin{aligned} x' &= h'n''x'' + nd' - a' + n''c', \\ x'' &= h''n'x' + n'''d'' - a'' + n'c''. \end{aligned} \quad (12)$$

These equations will serve to determine x' and x'' , and hence r' and r'' , as soon as the values of n , n' , n'' , and n''' are known.

96. In order to include terms of the second order in the values of n and n'' , we have, from the equations (26)₃,

$$n = \frac{\tau}{\tau'} \left(1 + \frac{1}{6} \frac{\tau'(\tau' + \tau)}{r'^3} \right), \quad n'' = \frac{\tau''}{\tau'} \left(1 + \frac{1}{6} \frac{\tau(\tau' + \tau'')}{r'^3} \right),$$

and, putting

$$P' = \frac{n}{n''}, \quad Q' = (n + n'' - 1) r'^3, \quad (13)$$

these give

$$P' = \frac{\tau}{\tau''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau'^2}{r'^3} \right), \quad (14)$$

$$Q' = \frac{1}{2} \tau \tau''.$$

Let us now put

$$\tau''' = k(t''' - t'), \quad \tau_0' = k(t''' - t'), \quad (15)$$

and, making the necessary changes in the notation in equations (26), we obtain

$$n''' = \frac{\tau}{\tau_0'} \left(1 + \frac{1}{6} \frac{\tau'''(\tau_0' + \tau)}{r'''^3} - \frac{1}{4} \frac{\tau'''(\tau'''^2 + \tau''' \tau - \tau^2)}{kr'''^4} \cdot \frac{dr''}{dt} \dots \right), \quad (16)$$

$$n' = \frac{\tau'''}{\tau_0'} \left(1 + \frac{1}{6} \frac{\tau(\tau_0' + \tau''')}{r'''^3} + \frac{1}{4} \frac{\tau(\tau^2 + \tau \tau''' - \tau'''^2)}{kr'''^4} \cdot \frac{dr''}{dt} \dots \right).$$

From these we get, including terms of the second order,

$$n''' = \frac{\tau}{\tau_0'} \left(1 + \frac{1}{6} \frac{\tau'''(\tau_0' + \tau)}{r'''^3} \right), \quad n' = \frac{\tau'''}{\tau_0'} \left(1 + \frac{1}{6} \frac{\tau(\tau_0' + \tau''')}{r'''^3} \right),$$

and hence, if we put

$$P'' = \frac{n'''}{n'}, \quad Q'' = (n' + n''' - 1) r'''^3, \quad (17)$$

we shall have, since $\tau_0' = \tau + \tau'''$,

$$P'' = \frac{\tau}{\tau'''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau'''^2}{r'''^3} \right), \quad (18)$$

$$Q'' = \frac{1}{2} \tau \tau'''.$$

When the intervals are equal, we have

$$P' = \frac{\tau}{\tau''}, \quad P'' = \frac{\tau}{\tau'''}.$$

and these expressions may be used, in the case of an unknown orbit, for the first approximation to the values of these quantities.

The equations (13) and (17) give

$$n'' = \frac{1}{1 + P'} \left(1 + \frac{Q'}{r'^3} \right),$$

$$n = n'' P';$$

$$n' = \frac{1}{1 + P''} \left(1 + \frac{Q''}{r'''^3} \right), \quad (19)$$

$$n''' = n' P'';$$

and, introducing these values, the equations (12) become

$$\begin{aligned}x &= \frac{1}{1 + P'} \left(1 + \frac{Q'}{r'^3} \right) (h'x'' + P'd' + c') - a', \\x'' &= \frac{1}{1 + P''} \left(1 + \frac{Q''}{r''^3} \right) (h''x' + P''d'' + c'') - a''.\end{aligned}\quad (20)$$

Let us now put

$$\begin{aligned}\frac{P'd' + c'}{1 + P'} &= c'_0, & \frac{h'}{1 + P'} &= f', \\ \frac{P''d'' + c''}{1 + P''} &= c''_0, & \frac{h''}{1 + P''} &= f'',\end{aligned}\quad (21)$$

and we shall have

$$\begin{aligned}x' &= \left(1 + \frac{Q'}{r'^3} \right) (f'x'' + c'_0) - a', \\ x'' &= \left(1 + \frac{Q''}{r''^3} \right) (f''x' + c''_0) - a''.\end{aligned}\quad (22)$$

We have, further, from equations (10),

$$\begin{aligned}r'^3 &= (x'^2 + R'^2 \sin^2 \Psi')^{\frac{3}{2}}, \\ r''^3 &= (x''^2 + R''^2 \sin^2 \Psi'')^{\frac{3}{2}},\end{aligned}\quad (23)$$

If we substitute these values of r'^3 and r''^3 in equations (22), the two resulting equations will contain only two unknown quantities x' and x'' , when P' , P'' , Q' , and Q'' are known, and hence they will be sufficient to solve the problem. But if we effect the elimination of either of the unknown quantities directly, the resulting equation becomes of a high order. It is necessary, therefore, in the numerical application, to solve the equations (22) by successive trials, which may be readily effected.

If z' represents the angle at the planet between the sun and the earth at the time of the second observation, and z'' the same angle at the time of the third observation, we shall have

$$\begin{aligned}r' &= \frac{R' \sin \Psi'}{\sin z'}, \\ r'' &= \frac{R'' \sin \Psi''}{\sin z''}.\end{aligned}\quad (24)$$

Substituting these values of r' and r'' in equations (10), we get

$$\begin{aligned}x' &= r' \cos z', \\ x'' &= r'' \cos z'',\end{aligned}\quad (25)$$

and hence

$$\begin{aligned}\tan z' &= \frac{R' \sin \psi'}{x'}, \\ \tan z'' &= \frac{R'' \sin \psi''}{x''}\end{aligned}\quad (26)$$

by means of which we may find z' and z'' as soon as x' and x'' shall have been determined; and then r' and r'' are obtained from (24) or (25). The last equations show that when x' is negative, z' must be greater than 90° , and hence that in this case r' is less than R' .

In the numerical application of equations (22), for a first approximation to the values of x' and x'' , since Q' and Q'' are quantities of the second order with respect to τ or τ''' , we may generally put

$$Q' = 0, \quad Q'' = 0;$$

and we have

$$\begin{aligned}x' &= f'x'' + c'_0 - a', \\ x'' &= f''x' + c''_0 - a'',\end{aligned}$$

or, by elimination,

$$\begin{aligned}x' &= \frac{c'_0 + f'c''_0 - f'a'' - a'}{1 - f'f''}, \\ x'' &= \frac{c''_0 + f''c'_0 - f''a' - a''}{1 - f'f''}.\end{aligned}$$

With the approximate values of x' and x'' derived from these equations, we compute first r' and r'' from the equations (26) and (24), and then new values of x' and x'' from (22), the operation being repeated until the true values are obtained. To facilitate these approximations, the equations (22) give

$$\begin{aligned}x'' &= \frac{x' + a'}{f' \left(1 + \frac{Q'}{r'^3}\right)} - \frac{c'_0}{f'}, \\ x' &= \frac{x'' + a''}{f'' \left(1 + \frac{Q''}{r''^3}\right)} - \frac{c''_0}{f''}.\end{aligned}\quad (27)$$

Let an approximate value of x' be designated by x'_0 , and let the value of x'' derived from this by means of the first of equations (27) be designated by x''_0 . With the value of x''_0 for x'' we derive a new value of x' from the second of these equations, which we denote by x'_1 . Then, recomputing x'' and x' , we obtain a third approximate value of the latter quantity, which may be designated by x'_2 ; and, if we put

$$x'_1 - x'_0 = \alpha_0, \quad x'_2 - x'_1 = \alpha'_0,$$

we shall have, according to the equation (67)₃, the necessary changes being made in the notation,

$$x' = x_1' - \frac{\alpha_0 \alpha_0'}{\alpha_0' - \alpha_0} = x_2' - \frac{\alpha_0'^2}{\alpha_0' - \alpha_0}. \quad (28)$$

The value of x' thus obtained will give, by means of the first of equations (27), a new value of x'' , and the substitution of this in the last of these equations will show whether the correct result has been found. If a repetition of the calculation be found necessary, the three values of x' which approximate nearest to the true value will, by means of (28), give the correct result. In the same manner, if we assume for x'' the value derived by putting $Q' = 0$ and $Q'' = 0$, and compute x' , three successive approximate results for x'' will enable us to interpolate the correct value.

When the elements of the orbit are already approximately known, the first assumed value of x' should be derived from

$$x' = \sqrt{r'^2 - R'^2 \sin^2 \psi'}$$

instead of by putting Q' and Q'' equal to zero.

97. It should be observed that when $\lambda' = \lambda$ or $\lambda''' = \lambda''$, the equations (22) are inapplicable, but that the original equations (7) give, in this case, either ρ'' or ρ' directly in terms of n and n'' or of n' and n''' and the data furnished by observation. If we divide the first of equations (22) by h' , we have

$$\frac{x'}{h'} = \left(1 + \frac{Q'}{r'^3}\right) \left(\frac{f'}{h'} x'' + \frac{c_0'}{h'}\right) - \frac{a'}{h'}.$$

The equations (21) give

$$\frac{f'}{h'} = \frac{1}{1 + P'}, \quad \frac{c_0'}{h'} = \frac{P' \frac{d'}{h'} + \frac{c'}{h'}}{1 + P'},$$

and from (11) we get

$$\begin{aligned} \frac{a'}{h'} &= \frac{R' \cos \psi'}{h'} + \frac{R' \sin (\lambda - \odot')}{B}, \\ \frac{c'}{h'} &= R'' \cos \psi'' + \frac{R'' \sin (\lambda - \odot'')}{B}, \\ \frac{d'}{h'} &= \frac{R \sin (\lambda - \odot)}{B}. \end{aligned} \quad (29)$$

Then, if we put

$$C_0' = P' \frac{d'}{h'} + \frac{c'}{h'},$$

its value may be found from the results for $\frac{c'}{h'}$ and $\frac{d'}{h'}$ derived by means of these equations, and we shall have

$$\frac{x'}{h'} = \frac{1}{1+P'} \left(1 + \frac{Q'}{r'^3} \right) (x'' + C'_0) - \frac{a'}{h'}. \quad (30)$$

When $\lambda' = \lambda$, we have $h' = \infty$, and this formula becomes

$$0 = \left(1 + \frac{Q'}{r'^3} \right) (x'' + C'_0) - \frac{a'}{h'} (1 + P'),$$

the value of $\frac{a'}{h'}$ being given by the first of equations (29). This equation and the second of equations (22) are sufficient to determine x' and x'' in the special case under consideration.

The second of equations (22) may be treated in precisely the same manner, so that when $\lambda''' = \lambda''$, it becomes

$$0 = \left(1 + \frac{Q''}{r''^3} \right) (x' + C''_0) - \frac{a''}{h''} (1 + P''),$$

and this must be solved in connection with the first of these equations in order to find x' and x'' .

98. As soon as the numerical values of x' and x'' have been derived, those of r' and r'' may be found by means of the equations (26) and (24). Then, according to (41)₄, we have

$$\begin{aligned} \rho' &= \frac{R' \sin(z' + \psi')}{\sin z'} \cos \beta', \\ \rho'' &= \frac{R'' \sin(z'' + \psi'')}{\sin z''} \cos \beta''. \end{aligned} \quad (31)$$

The heliocentric places are then found from ρ' and ρ'' by means of the equations (71)₃, and the values of r' and r'' thus obtained should agree with those already derived. From these places we compute the position of the plane of the orbit, and thence the arguments of the latitude for the times t' and t'' .

The values of r' , r'' , u' , u'' , n , n'' , n' , and n''' enable us to determine r , r''' , u , and u''' . Thus, we have

$$[r'r''] = r'r'' \sin(u'' - u'),$$

and, from the equations (1) and (3)₃,

$$\begin{aligned}
[rr'] &= \frac{n''}{n} [r'r''], \\
[rr''] &= \frac{1}{n} [r'r'''], \\
[r'r'''] &= \frac{n'}{n'''} [r'r''], \\
r'r''' &= \frac{1}{n'''} [r'r''].
\end{aligned}$$

Therefore,

$$\begin{aligned}
r \sin(u' - u) &= \frac{n''}{n} r'' \sin(u'' - u'), \\
r \sin(u'' - u) &= \frac{1}{n} r' \sin(u'' - u'), \\
r''' \sin(u''' - u'') &= \frac{n'}{n'''} r' \sin(u'' - u'), \\
r''' \sin(u''' - u') &= \frac{1}{n'''} r'' \sin(u'' - u').
\end{aligned} \tag{32}$$

From the first and second of these equations, by addition and subtraction, we get

$$\begin{aligned}
r \sin((u' - u) + \tfrac{1}{2}(u'' - u')) &= \frac{r' + n''r''}{n} \sin \tfrac{1}{2}(u'' - u'), \\
r \cos((u' - u) + \tfrac{1}{2}(u'' - u')) &= \frac{r' - n''r''}{n} \cos \tfrac{1}{2}(u'' - u'),
\end{aligned} \tag{33}$$

from which we may find r , $u' - u$, and $u = u' - (u' - u)$.

In a similar manner, from the third and fourth of equations (32), we obtain

$$\begin{aligned}
r''' \sin((u''' - u'') + \tfrac{1}{2}(u'' - u')) &= \frac{r'' + n'r'''}{n'''} \sin \tfrac{1}{2}(u'' - u'), \\
r''' \cos((u''' - u'') + \tfrac{1}{2}(u'' - u')) &= \frac{r'' - n'r'''}{n'''} \cos \tfrac{1}{2}(u'' - u'),
\end{aligned} \tag{34}$$

from which to find r''' and u''' .

When the approximate values of r , r' , r'' , r''' , and u , u' , u'' , u''' have been found, by means of the preceding equations, from the assumed values of P' , P'' , Q' , and Q'' , the second approximation to the elements may be commenced. But, in the case of an unknown orbit, it will be expedient to derive, first, approximate values of and r'' , using

$$P' = \frac{\tau}{\tau'}, \quad P'' = \frac{\tau}{\tau''},$$

and then recompute P' and P'' by means of the equations (14) and

(18), before finding u' and u'' . The terms of the second order will thus be completely taken into account in the first approximation.

99. If the times of observation have not been corrected for the time of aberration, as in the case of an orbit wholly unknown, this correction may be applied before the second approximation to the elements is effected, or at least before the final approximation is commenced. For this purpose, the distances of the body from the earth for the four observations must be determined; and, since the curtate distances ρ' and ρ'' are already given, there remain only ρ and ρ''' to be found. If we eliminate ρ' from the first two of equations (3), the result is

$$\rho = \rho'' \frac{n'' \sin (\lambda'' - \lambda')}{n \sin (\lambda' - \lambda)} + \frac{nR \sin (\lambda' - \odot) - R' \sin (\lambda' - \odot') + n'' R'' \sin (\lambda' - \odot'')}{n \sin (\lambda' - \lambda)}; \quad (35)$$

and, by eliminating ρ'' from the last two of these equations, we also obtain

$$\rho''' = \rho' \frac{n' \sin (\lambda'' - \lambda')}{n''' \sin (\lambda''' - \lambda'')} - \frac{n' R' \sin (\lambda'' - \odot') - R'' \sin (\lambda'' - \odot'') + n''' R''' \sin (\lambda'' - \odot''')}{n''' \sin (\lambda''' - \lambda'')}, \quad (36)$$

by means of which ρ and ρ''' may be found. The combination of the first and second of equations (3) gives

$$\rho = \frac{\rho'}{n} \cos (\lambda' - \lambda) - \frac{n'' \rho''}{n} \cos (\lambda'' - \lambda) + \frac{nR \cos (\lambda - \odot) - R' \cos (\lambda - \odot') + n'' R'' \cos (\lambda - \odot'')}{n}, \quad (37)$$

and from the third and fourth we get

$$\rho''' = \frac{\rho''}{n'''} \cos (\lambda''' - \lambda'') - \frac{n' \rho'}{n'''} \cos (\lambda''' - \lambda') + \frac{n' R' \cos (\lambda''' - \odot') - R'' \cos (\lambda''' - \odot'') + n''' R''' \cos (\lambda''' - \odot''')}{n'''}. \quad (38)$$

Further, instead of these, any of the various formulæ which have been given for finding the ratio of two curtate distances, may be employed; but, if the latitudes β , β' , &c. are very small, the values of ρ and ρ''' which depend on the differences of the observed longitudes of the body must be preferred.

The values of ρ' and ρ''' may also be derived by computing the heliocentric places of the body for the times t and t''' by means of the equations (82)₁, and then finding the geocentric places, or those which belong to the points to which the observations have been reduced, by means of (90)₁, writing ρ in place of $\Delta \cos \beta$. This process affords a verification of the numerical calculation, namely, the values of λ and λ''' thus found should agree with those furnished by observation, and the agreement of the computed latitudes β and β''' with those observed, in case the latter are given, will show how nearly the position of the plane of the orbit as derived from the second and third observations represents the extreme latitudes. If it were not desirable to compute λ and λ''' in order to check the calculation, even when β and β''' are given by observation, we might derive ρ and ρ''' from the equations

$$\begin{aligned}\rho &= r \sin u \sin i \cot \beta, \\ \rho''' &= r''' \sin u''' \sin i \cot \beta''',\end{aligned}\tag{39}$$

when the latitudes are not very small.

In the final approximation to the elements, and especially when the position of the plane of the orbit cannot be obtained with the required precision from the second and third observations, it will be advantageous, provided that the data furnish the extreme latitudes β and β''' , to compute ρ and ρ''' as soon as ρ' and ρ'' have been found, and then find l , l''' , b , and b''' directly from these by means of the formulæ (71)₃. The values of Ω and i may thus be obtained from the extreme places, or, the heliocentric places for the times t' and t''' being also computed directly from ρ' and ρ'' , from those which are best suited to this purpose. But, since the data will be more than sufficient for the solution of the problem, when the extreme latitudes are used, if we compute the heliocentric latitudes b' and b''' from the equations

$$\begin{aligned}\tan b' &= \tan i \sin (l' - \Omega), \\ \tan b''' &= \tan i \sin (l''' - \Omega),\end{aligned}$$

they will not agree exactly with the results obtained directly from ρ' and ρ'' , unless the four observations are completely satisfied by the elements obtained. The values of r' and r'' , however, computed directly from ρ' and ρ'' by means of (71)₃, must agree with those derived from x' and x'' .

The corrections to be applied to the times of observation on account

of aberration may now be found. Thus, if t_0, t'_0, t''_0 , and t'''_0 are the uncorrected times of observation, the corrected values will be

$$\begin{aligned} t &= t_0 - C\rho \sec \beta, \\ t' &= t'_0 - C\rho' \sec \beta', \\ t'' &= t''_0 - C\rho'' \sec \beta'', \\ t''' &= t'''_0 - C\rho''' \sec \beta''', \end{aligned} \quad (40)$$

wherein $\log C = 7.760523$, and from these we derive the corrected values of $\tau, \tau', \tau'', \tau'''$, and τ_0' .

100. To find the values of $P', P'', Q',$ and Q'' , which will be exact when r, r', r'', r''' , and u, u', u'', u''' are accurately known, we have, according to the equations (47)₄ and (51)₄, since $Q' = \frac{1}{2}Q$,

$$\begin{aligned} P' &= \frac{\tau}{\tau''} \cdot \frac{s''}{s}, \\ Q' &= \frac{1}{2} \frac{\tau\tau''}{ss''} \cdot \frac{r'^2}{r'r'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u)}. \end{aligned} \quad (41)$$

In a similar manner, if we designate by s''' the ratio of the sector formed by the radii-vectores r'' and r''' to the triangle formed by the same radii-vectores and the chord joining their extremities, we find

$$\begin{aligned} P'' &= \frac{\tau}{\tau'''} \cdot \frac{s'''}{s}, \\ Q'' &= \frac{1}{2} \frac{\tau\tau'''}{ss'''} \cdot \frac{r''^2}{r'r''' \cos \frac{1}{2}(u''' - u'') \cos \frac{1}{2}(u''' - u') \cos \frac{1}{2}(u'' - u')}. \end{aligned} \quad (42)$$

The formulæ for finding the value of s''' are obtained from those for s by writing $\chi''', \gamma''', G''',$ &c. in place of $\chi, \gamma, G,$ &c., and using $r'', r''', u''' - u''$ instead of $r', r'',$ and $u'' - u'$, respectively.

By means of the results obtained from the first approximation to the values of $P', P'', Q',$ and Q'' , we may, from equations (41) and (42), derive new and more nearly accurate values of these quantities, and, by repeating the calculation, the approximations to the exact values may be carried to any extent which may be desirable. When three approximate values of P' and Q' , and of P'' and Q'' , have been derived, the next approximation will be facilitated by the use of the formulæ (82)₄, as already explained.

When the values of $P', P'', Q',$ and Q'' have been derived with sufficient accuracy, we proceed from these to find the elements of the orbit. After $\Omega, i, r, r', r'', r''', u, u', u'',$ and u''' have been found, the remaining elements may be derived from any two radii-vectores

and the corresponding arguments of the latitude. It will be most accurate, however, to derive the elements from $r, r''', u,$ and u''' . If the values of $P', P'', Q',$ and Q'' have been obtained with great accuracy, the results derived from any two places will agree with those obtained from the extreme places.

In the first place, from

$$\begin{aligned}\tan \chi_0 &= \sqrt{\frac{r'''}{r}}, \\ \sin \gamma_0 \cos G_0 &= \sin \frac{1}{2} (u''' - u), \\ \sin \gamma_0 \sin G_0 &= \cos \frac{1}{2} (u''' - u) \cos 2\chi_0, \\ \cos \gamma_0 &= \cos \frac{1}{2} (u''' - u) \sin 2\chi_0,\end{aligned}\quad (43)$$

we find γ_0 and G_0 . Then we have

$$\begin{aligned}\tau_0 &= k(t''' - t), \\ m_0 &= \frac{\tau_0^2}{(r + r''')^3 \cos^3 \gamma_0}, & j_0 &= \frac{\sin \frac{1}{2} \gamma_0}{\cos \gamma_0}, \\ \eta_0 &= \frac{m_0}{s_0^2 + j_0 + \xi_0}, & x_0 &= \frac{m_0}{s_0^2} - j_0,\end{aligned}\quad (44)$$

from which, by means of Tables XIII. and XIV., to find s_0 and x_0 .

We have, further,

$$p = \left(\frac{s_0 r r''' \sin (u''' - u)}{\tau_0} \right)^2,$$

and the agreement of the value of p thus found with the separate results for the same quantity obtained from the combination of any two of the four places, will show the extent to which the approximation to $P', P'', Q',$ and Q'' has been carried. The elements are now to be computed from the extreme places precisely as explained in the preceding chapter, using r''' in the place of r'' in the formulæ there given and introducing the necessary modifications in the notation, which have been already suggested and which will be indicated at once.

101. EXAMPLE.—For the purpose of illustrating the application of the formulæ for the calculation of an orbit from four observations, let us take the following normal places of *Eurynome* ② derived by comparing a series of observations with an ephemeris computed from approximate elements.

| Greenwich M. T. | α | δ |
|-----------------|---------------|-----------------|
| 1863 Sept. 20.0 | 14° 30' 35".6 | + 9° 23' 49".7, |
| Dec. 9.0 | 9 54 17 .0 | 2 53 41 .8, |
| 1864 Feb. 2.0 | 28 41 34 .1 | 9 6 2 .8, |
| April 30.0 | 74 29 58 .9 | + 19 35 41 .5. |

These normals give the geocentric places of the planet referred to the mean equinox and equator of 1864.0, and free from aberration. For the mean obliquity of the ecliptic of 1864.0, the *American Nautical Almanac* gives

$$\varepsilon = 23^{\circ} 27' 24''.49,$$

and, by means of this, converting the observed right ascensions and declinations, as given by the normal places, into longitudes and latitudes, we get

| Greenwich M. T. | λ | β |
|-----------------|-------------------------|----------------------------|
| 1863 Sept. 20.0 | $16^{\circ} 59' 9''.42$ | $+ 2^{\circ} 56' 44''.58,$ |
| Dec. 9.0 | $10 14 17 .57$ | $- 1 15 48 .82,$ |
| 1864 Feb. 2.0 | $29 53 21 .99$ | $2 29 57 .38,$ |
| April 30.0 | $75 23 46 .90$ | $- 3 4 44 .49.$ |

These places are referred to the ecliptic and mean equinox of 1864.0, and, for the same dates, the geocentric latitudes of the sun referred also to the ecliptic of 1864.0 are

$$+ 0''.60, \quad + 0''.53, \quad + 0''.36, \quad + 0''.19.$$

For the reduction of the geocentric latitudes of the planet to the point in which a perpendicular let fall from the centre of the earth to the plane of the ecliptic cuts that plane, the equation (6)₄ gives the corrections $- 0''.57$, $- 0''.38$, $- 0''.18$, and $- 0''.07$ to be applied to these latitudes respectively, the logarithms of the approximate distances of the planet from the earth being

$$0.02618, \quad 0.13355, \quad 0.29033, \quad 0.44990.$$

Thus we obtain

$$\begin{array}{lll} t = 0.0, & \lambda = 16^{\circ} 59' 9''.42, & \beta = + 2^{\circ} 56' 44''.01, \\ t' = 80.0, & \lambda' = 10 14 17 .57, & \beta' = - 1 15 49 .20, \\ t'' = 135.0, & \lambda'' = 29 53 21 .99, & \beta'' = - 2 29 57 .56, \\ t''' = 223.0, & \lambda''' = 75 23 46 .90, & \beta''' = - 3 4 44 .56; \end{array}$$

and, for the same times, the true places of the sun referred to the mean equinox of 1864.0 are

$$\begin{array}{ll} \odot = 177^{\circ} 0' 58''.6, & \log R = 0.0015899. \\ \odot' = 256 58 35 .9, & \log R' = 9.9932638, \\ \odot'' = 312 57 49 .8, & \log R'' = 9.9937748, \\ \odot''' = 40 21 26 .8, & \log R''' = 0.0035149, \end{array}$$

From the equations

$$\begin{aligned}\tan w' &= \frac{\tan \beta'}{\sin(\lambda' - \odot')}, & \tan \psi' &= \frac{\tan(\lambda' - \odot')}{\cos w'}, \\ \tan w'' &= \frac{\tan \beta''}{\sin(\lambda'' - \odot'')}, & \tan \psi'' &= \frac{\tan(\lambda'' - \odot'')}{\cos w''},\end{aligned}$$

we obtain

$$\begin{aligned}\psi' &= 113^\circ 15' 20''.10, & \log(R' \cos \psi') &= 9.5896777, \\ \psi'' &= 76 \quad 56 \quad 17 \quad .75, & \log(R' \sin \psi') &= 9.9564624, \\ & & \log(R'' \cos \psi'') &= 9.3478848, \\ & & \log(R'' \sin \psi'') &= 9.9823904.\end{aligned}$$

The quadrant in which ψ' must be taken, is indicated by the condition that $\cos \psi'$ and $\cos(\lambda' - \odot')$ must have the same sign. The same condition exists in the case of ψ'' . Then, the formulæ

$$\begin{aligned}A &= \cos \beta' \sin(\lambda' - \lambda), & B &= \cos \beta'' \sin(\lambda'' - \lambda), \\ C &= \cos \beta'' \sin(\lambda''' - \lambda''), & D &= \cos \beta' \sin(\lambda''' - \lambda'), \\ \frac{B}{A} &= h', & \frac{D}{C} &= h'', \\ a' &= R' \cos \psi' + \frac{R' \sin(\lambda - \odot')}{A}, \\ a'' &= R'' \cos \psi'' - \frac{R'' \sin(\lambda''' - \odot'')}{C}, \\ c' &= h' R'' \cos \psi'' + \frac{R'' \sin(\lambda - \odot'')}{A}, \\ c'' &= h'' R' \cos \psi' - \frac{R' \sin(\lambda''' - \odot')}{C}, \\ d' &= \frac{R \sin(\lambda - \odot)}{A}, & d'' &= -\frac{R''' \sin(\lambda''' - \odot''')}{C},\end{aligned}$$

give the following results:—

$$\begin{aligned}\log A &= 9.0699254_n, & \log C &= 9.8528803, \\ \log B &= 9.3484939, & \log D &= 9.9577271, \\ \log h' &= 0.2785685_n, & \log h'' &= 0.1048468, \\ \log a' &= 0.8834880_n, & \log a'' &= 9.9752915_n, \\ \log c' &= 0.9012910_n, & \log c'' &= 9.7267348_n, \\ \log d' &= 0.4650841, & \log d'' &= 9.9096469_n.\end{aligned}$$

We are now prepared to make the first hypothesis in regard to the values of P' , Q' , P'' , and Q'' . If the elements were entirely unknown, it would be necessary, in the first instance, to assume for these quantities the values given by the expressions

$$\begin{aligned} P' &= \frac{\tau}{\tau''}, & Q' &= \frac{1}{2}\tau\tau'', \\ P'' &= \frac{\tau}{\tau'''}, & Q'' &= \frac{1}{2}\tau\tau'''; \end{aligned}$$

then approximate values of r' and r'' are readily obtained by means of the equations (27), (26), and (24) or (25). The first assumed value of x' to be used in the second member of the first of equations (27), is obtained from the expression which results from (22) by putting $Q' = 0$ and $Q'' = 0$, namely,

$$x' = \frac{c'_0 + f'c''_0 - f'a'' - a'}{1 - f'f''};$$

after which the values of x' and x'' will be obtained by trial from (27). It should be remarked, further, that in the first determination of an orbit entirely unknown, the intervals of time between the observations will generally be small, and hence the value of x' derived from the assumption of $Q' = 0$ and $Q'' = 0$ will be sufficiently approximate to facilitate the solution of equations (27).

As soon as the approximate values of r' and r'' have thus been found, those of P' and P'' must be recomputed from the expressions

$$P' = \frac{\tau}{\tau''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau''^2}{r'^3} \right), \quad P'' = \frac{\tau}{\tau'''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau'''^2}{r'^3} \right).$$

With the results thus derived for P' and P'' , and with the values of Q' and Q'' already obtained, the first approximation to the elements must be completed.

When the elements are already approximately known, the first assumed values of P' , P'' , Q' , and Q'' should be computed by means of these elements. Thus, from

$$\begin{aligned} n &= \frac{r'r'' \sin(v' - v')}{rr'' \sin(v'' - v)}, & n'' &= \frac{rr' \sin(v' - v)}{rr'' \sin(v'' - v)}, \\ n' &= \frac{r'r'' \sin(v'' - v')}{r'r''' \sin(v''' - v')}, & n''' &= \frac{r'r'' \sin(v'' - v')}{r'r''' \sin(v''' - v')}, \end{aligned}$$

we find n , n' , n'' , and n''' . The approximate elements of *Eurynome* give

$$\begin{aligned} v &= 322^\circ 55' 9''.3, & \log r &= 0.308327, \\ v' &= 353 \quad 19 \quad 26 \quad .3, & \log r' &= 0.294225, \\ v'' &= 14 \quad 45 \quad 8 \quad .5, & \log r'' &= 0.296088, \\ v''' &= 47 \quad 23 \quad 32 \quad .8, & \log r''' &= 0.317278, \end{aligned}$$

and hence we obtain

$$\begin{aligned}\log n &= 9.653052, & \log n'' &= 9.806836, \\ \log n' &= 9.825408, & \log n''' &= 9.633171.\end{aligned}$$

Then, from

$$\begin{aligned}P' &= \frac{n}{n''}, & Q' &= (n + n'' - 1) r'^3, \\ P'' &= \frac{n'''}{n'}, & Q'' &= (n' + n''' - 1) r''^3,\end{aligned}$$

we get

$$\begin{aligned}\log P' &= 9.846216, & \log Q' &= 9.840771, \\ \log P'' &= 9.807763, & \log Q'' &= 9.882480.\end{aligned}$$

The values of these quantities may also be computed by means of the equations (41) and (42).

Next, from

$$\begin{aligned}c'_0 &= \frac{P'd' + c'}{1 + P'}, & f' &= \frac{h'}{1 + P'}, \\ c''_0 &= \frac{P''d'' + c''}{1 + P''}, & f'' &= \frac{h''}{1 + P''},\end{aligned}$$

we find

$$\begin{aligned}\log c'_0 &= 0.541344, & \log f' &= 0.047658, \\ \log c''_0 &= 9.807665, & \log f'' &= 9.889385.\end{aligned}$$

Then we have

$$\begin{aligned}x'' &= \frac{x' + a'}{f' \left(1 + \frac{Q'}{r'^3}\right)} - \frac{c'_0}{f'}, \\ x' &= \frac{x'' + a''}{f'' \left(1 + \frac{Q''}{r''^3}\right)} - \frac{c''_0}{f''}, \\ \tan z' &= \frac{R' \sin \Psi'}{x'}, & \tan z'' &= \frac{R'' \sin \Psi''}{x''}, \\ r' &= \frac{R' \sin \Psi'}{\sin z'} = \frac{x'}{\cos z'}, & r'' &= \frac{R'' \sin \Psi''}{\sin z''} = \frac{x''}{\cos z''}\end{aligned}$$

from which to find r' and r'' . In the first place, from

$$x' = \sqrt{r'^2 - R'^2 \sin^2 \Psi'},$$

we obtain the approximate value

$$\log x' = 0.242737.$$

Then the first of the preceding equations gives

$$\log x'' = 0.237687.$$

From this we get

$$z'' = 29^\circ 3' 11''.7, \quad \log r'' = 0.296092;$$

and then the equation for x' gives

$$\log x' = 0.242768.$$

Hence we have

$$z' = 27^\circ 20' 59''.6, \quad \log r' = 0.294249;$$

and, repeating the operation, using these results for x' and r' , we get

$$\log x'' = 0.237678, \quad \log x' = 0.242757.$$

The correct value of $\log x'$ may now be found by means of equation (28). Thus, in units of the sixth decimal place, we have

$$a_0 = 242768 - 242737 = +31, \quad a'_0 = 242757 - 242768 = -11,$$

and for the correction to be applied to the last value of $\log x'$, in units of the sixth decimal place,

$$\Delta \log x' = -\frac{a_0'^2}{a'_0 - a_0} = +3.$$

Therefore, the corrected value is

$$\log x' = 0.242760,$$

and from this we derive

$$\log x'' = 0.237681.$$

These results satisfy the equations for x' and x'' , and give

$$\begin{aligned} z' &= 27^\circ 21' 1''.2, & \log r' &= 0.294242, \\ z'' &= 29^\circ 3' 12''.9, & \log r'' &= 0.296087. \end{aligned}$$

To find the curtate distances for the first and second observations, the formulæ are

$$\rho' = \frac{R' \sin(z' + \psi')}{\sin z'} \cos \beta', \quad \rho'' = \frac{R'' \sin(z'' + \psi'')}{\sin z''} \cos \beta'',$$

which give

$$\log \rho' = 0.133474, \quad \log \rho'' = 0.289918.$$

Then, by means of the equations

$$\begin{aligned} r' \cos b' \cos (\ell' - \odot') &= \rho' \cos (\lambda' - \odot') - R', \\ r' \cos b' \sin (\ell' - \odot') &= \rho' \sin (\lambda' - \odot'), \\ r' \sin b' &= \rho' \tan \beta', \end{aligned}$$

$$\begin{aligned} r'' \cos b'' \cos (\ell'' - \odot'') &= \rho'' \cos (\lambda'' - \odot'') - R'', \\ r'' \cos b'' \sin (\ell'' - \odot'') &= \rho'' \sin (\lambda'' - \odot''), \\ r'' \sin b'' &= \rho'' \tan \beta'', \end{aligned}$$

we find the following heliocentric places :

$$\begin{array}{lll} \ell' = 37^\circ 35' 26''.4, & \log \tan b' = 8.182861_n, & \log r' = 0.294243, \\ \ell'' = 58 \ 58 \ 15 \ .3, & \log \tan b'' = 8.634209_n, & \log r'' = 0.296087. \end{array}$$

The agreement of these values of $\log r'$ and $\log r''$ with those obtained directly from x' and x'' is a partial proof of the numerical calculation.

From the equations

$$\begin{aligned} \tan i \sin \left(\frac{1}{2} (\ell'' + \ell') - \Omega \right) &= \frac{1}{2} (\tan b'' + \tan b') \sec \frac{1}{2} (\ell'' - \ell'), \\ \tan i \cos \left(\frac{1}{2} (\ell'' + \ell') - \Omega \right) &= \frac{1}{2} (\tan b'' - \tan b') \operatorname{cosec} \frac{1}{2} (\ell'' - \ell'), \\ \tan u &= \frac{\tan (\ell' - \Omega)}{\cos i}, & \tan u'' &= \frac{\tan (\ell'' - \Omega)}{\cos i}, \end{aligned}$$

we obtain

$$\begin{array}{ll} \Omega = 206^\circ 42' 24''.0, & i = 4^\circ 36' 47''.2, \\ u' = 190 \ 55 \ 6 \ .6 & u'' = 212 \ 20 \ 53 \ .5. \end{array}$$

Then, from

$$\begin{aligned} n'' &= \frac{1}{1 + P'} \left(1 + \frac{Q'}{r'^3} \right), & n &= n' P', \\ n' &= \frac{1}{1 + P''} \left(1 + \frac{Q''}{r''^3} \right), & n''' &= n' P'', \end{aligned}$$

we get

$$\begin{array}{ll} \log n'' = 9.806832, & \log n = 9.653048, \\ \log n' = 9.825408, & \log n''' = 9.633171, \end{array}$$

and the equations

$$\begin{aligned} r \sin ((u' - u) + \frac{1}{2} (u'' - u')) &= \frac{r' + n'' r''}{n} \sin \frac{1}{2} (u'' - u'), \\ r \cos ((u' - u) + \frac{1}{2} (u'' - u')) &= \frac{r' - n'' r''}{n} \cos \frac{1}{2} (u'' - u'), \\ r''' \sin ((u''' - u'') + \frac{1}{2} (u'' - u')) &= \frac{r'' + n' r'}{n'''} \sin \frac{1}{2} (u'' - u'), \\ r''' \cos ((u''' - u'') + \frac{1}{2} (u'' - u')) &= \frac{r'' - n' r'}{n'''} \cos \frac{1}{2} (u'' - u'), \end{aligned}$$

give

$$\begin{aligned}\log r &= 0.308379, & u &= 160^\circ 30' 57''.6, \\ \log r''' &= 0.317273, & u''' &= 244 \ 59 \ 32 \ .5.\end{aligned}$$

Next, by means of the formulæ

$$\begin{aligned}\tan(l - \Omega) &= \cos i \tan u, & \tan b &= \tan i \sin(l - \Omega), \\ \tan(l''' - \Omega) &= \cos i \tan u''', & \tan b''' &= \tan i \sin(l''' - \Omega), \\ \rho \cos(\lambda - \odot) &= r \cos b \cos(l - \odot) + R, \\ \rho \sin(\lambda - \odot) &= r \cos b \sin(l - \odot), \\ \rho \tan \beta &= r \sin b; \\ \rho''' \cos(\lambda''' - \odot''') &= r''' \cos b''' \cos(l''' - \odot''') + R''', \\ \rho''' \sin(\lambda''' - \odot''') &= r''' \cos b''' \sin(l''' - \odot'''), \\ \rho''' \tan \beta''' &= r''' \sin b''',\end{aligned}$$

we obtain

$$\begin{aligned}l &= 7^\circ 16' 51''.8, & l''' &= 91^\circ 37' 40''.0, \\ b &= + 1 \ 32 \ 14 \ .4, & b''' &= - 4 \ 10 \ 47 \ .4, \\ \lambda &= 16 \ 59 \ 9 \ .0, & \lambda''' &= 75 \ 23 \ 46 \ .9, \\ \beta &= + 2 \ 56 \ 40 \ .1, & \beta''' &= - 3 \ 4 \ 43 \ .4, \\ \log \rho &= 0.025707, & \log \rho''' &= 0.449258.\end{aligned}$$

The value of λ''' thus obtained agrees exactly with that given by observation, but λ differs $0''.4$ from the observed value. This difference does not exceed what may be attributed to the unavoidable errors of calculation with logarithms of six decimal places. The differences between the computed and the observed values of β and β''' show that the position of the plane of the orbit, as determined by means of the second and third places, will not completely satisfy the extreme places.

The four curtate distances which are thus obtained enable us, in the case of an orbit entirely unknown, to complete the correction for aberration according to the equations (40).

The calculation of the quantities which are independent of P' , P'' , Q' , and Q'' , and which are therefore the same in the successive hypotheses, should be performed as accurately as possible. The value of $\frac{c'_0}{f'}$ required in finding x'' from x' , may be computed directly from

$$\frac{c'_0}{f'} = P' \frac{d'}{h'} + \frac{c'}{h'}$$

the values of $\frac{d'}{h'}$ and $\frac{c'}{h'}$ being found by means of the equations (29);

and a similar method may be adopted in the case of $\frac{c''}{f''}$. Further, in the computation of x' and x'' , it may in some cases be advisable to employ one or both of the equations (22) for the final trial. Thus, in the present case, x'' is found from the first of equations (27) by means of the difference of two larger numbers, and an error in the last decimal place of the logarithm of either of these numbers affects in a greater degree the result obtained. But as soon as r'' is known so nearly that the logarithm of the factor $1 + \frac{Q''}{r'^3}$ remains unchanged, the second of equations (22) gives the value of x'' by means of the sum of two smaller numbers. In general, when two or more formulæ for finding the same quantity are given, of those which are otherwise equally accurate and convenient for logarithmic calculation, that in which the number sought is obtained from the sum of smaller numbers should be preferred instead of that in which it is obtained by taking the difference of larger numbers.

The values of r, r', r'', r''' , and u, u', u'', u''' , which result from the first hypothesis, suffice to correct the assumed values of $P', P'', Q',$ and Q'' . Thus, from

$$\begin{aligned} \tau &= k(t'' - t'), & \tau' &= k(t' - t), & \tau'' &= k(t'' - t'), \\ \tan \chi &= \sqrt{\frac{r''}{r'}}, & \tan \chi' &= \sqrt{\frac{r'}{r}}, & \tan \chi'' &= \sqrt{\frac{r'''}{r''}}, \end{aligned}$$

$$\begin{aligned} \sin \gamma \cos G &= \sin \frac{1}{2}(u'' - u'), & \sin \gamma'' \cos G'' &= \sin \frac{1}{2}(u' - u), \\ \sin \gamma \sin G &= \cos \frac{1}{2}(u'' - u') \cos 2\chi, & \sin \gamma'' \sin G'' &= \cos \frac{1}{2}(u' - u) \cos 2\chi'', \\ \cos \gamma &= \cos \frac{1}{2}(u'' - u') \sin 2\chi, & \cos \gamma'' &= \cos \frac{1}{2}(u' - u) \sin 2\chi'', \end{aligned}$$

$$\begin{aligned} \sin \gamma''' \cos G''' &= \sin \frac{1}{2}(u''' - u''), \\ \sin \gamma''' \sin G''' &= \cos \frac{1}{2}(u''' - u'') \cos 2\chi''' \\ \cos \gamma''' &= \cos \frac{1}{2}(u''' - u'') \sin 2\chi'''; \end{aligned}$$

$$\begin{aligned} m &= \frac{\tau^2 \cos^6 \chi}{r'^3 \cos^3 \gamma}, & m' &= \frac{\tau'^2 \cos^6 \chi'}{r'^3 \cos^3 \gamma'}, & m'' &= \frac{\tau''^2 \cos^6 \chi''}{r''^3 \cos^3 \gamma''}, \\ j &= \frac{\sin^2 \frac{1}{2} \gamma}{\cos \gamma}, & j' &= \frac{\sin^2 \frac{1}{2} \gamma'}{\cos \gamma'}, & j'' &= \frac{\sin^2 \frac{1}{2} \gamma''}{\cos \gamma''}, \\ \eta &= \frac{m}{\frac{5}{6} + j + \xi}, & \eta' &= \frac{m'}{\frac{5}{6} + j' + \xi'}, & \eta'' &= \frac{m''}{\frac{5}{6} + j'' + \xi''}, \\ x &= \frac{m}{s^2} - j, & x' &= \frac{m'}{s'^2} - j', & x'' &= \frac{m''}{s''^2} - j'', \end{aligned}$$

in connection with Tables XIII. and XIV. we find $s, s',$ and s'' . The results are

$$\begin{array}{lll}
\log \tau = 9.9759441, & \log \tau'' = 0.1386714, & \log \tau''' = 0.1800641, \\
\chi = 45^\circ 3' 39''.1, & \chi'' = 44^\circ 32' 1''.4, & \chi''' = 45^\circ 41' 55''.2, \\
\gamma = .10 42 55 .9, & \gamma'' = 15 13 45 .0, & \gamma''' = 16 22 48 .5, \\
\log m = 8.186217, & \log m'' = 8.516727, & \log m''' = 8.590596, \\
\log j = 7.948097, & \log j'' = 8.260013, & \log j''' = 8.325365, \\
\log s = 0.0085248, & \log s'' = 0.0174621, & \log s''' = 0.0204063.
\end{array}$$

Then, by means of the formulæ

$$\begin{aligned}
P' &= \frac{\tau}{\tau''} \cdot \frac{s''}{s}, \\
Q' &= \frac{1}{2} \frac{\tau \tau''}{s s''} \cdot \frac{r'^2}{r' r'' \cos \frac{1}{2}(u'' - u') \cos \frac{1}{2}(u'' - u) \cos \frac{1}{2}(u' - u)}, \\
P'' &= \frac{\tau}{\tau'''} \cdot \frac{s'''}{s}, \\
Q'' &= \frac{1}{2} \frac{\tau \tau'''}{s s'''} \cdot \frac{r''^2}{r' r''' \cos \frac{1}{2}(u''' - u'') \cos \frac{1}{2}(u''' - u') \cos \frac{1}{2}(u'' - u')}.
\end{aligned}$$

we obtain

$$\begin{array}{ll}
\log P' = 9.8462100, & \log Q' = 9.8407536, \\
\log P'' = 9.8077615, & \log Q'' = 9.8824728,
\end{array}$$

with which the next approximation may be completed.

We now recompute $c_0', c_0'', f', f'', x', x'',$ &c. precisely as already illustrated; and the results are

$$\begin{array}{ll}
\log c_0' = 0.5413485_n, & \log c_0'' = 9.8076649_n, \\
\log f' = 0.0476614_n, & \log f'' = 9.8893851_n, \\
\log x' = 0.2427528, & \log x'' = 0.2376752, \\
z' = 27^\circ 21' 2''.71, & z'' = 29^\circ 3' 14''.09, \\
\log r' = 0.2942369, & \log r'' = 0.2960826, \\
\log \rho' = 0.1334635, & \log \rho'' = 0.2899124, \\
\log n = 9.6530445, & \log n'' = 9.8068345, \\
\log n' = 9.8254092, & \log n''' = 9.6331707.
\end{array}$$

Then we obtain

$$\begin{array}{lll}
l' = 37^\circ 35' 27''.88, & \log \tan b' = 8.1828572_n, & \log r' = 0.2942369, \\
l'' = 58 58 16 .48, & \log \tan b'' = 8.6342073_n, & \log r'' = 0.2960827.
\end{array}$$

These results for $\log r'$ and $\log r''$ agree with those obtained directly from z' and z'' , thus checking the calculation of ψ' and ψ'' and of the heliocentric places.

Next, we derive

$$\begin{array}{ll}
\Omega = 206^\circ 42' 25''.89, & i = 4^\circ 36' 47''.20, \\
u' = 190 55 6 .27, & u'' = 212 20 52 .96,
\end{array}$$

and from $u'' - u', r', r'', n, n'', n',$ and n''' , we obtain

$$\begin{aligned}\log r &= 0.3083734, & u &= 160^\circ 30' 55''.45, \\ \log r''' &= 0.3172674, & u''' &= 244 \quad 59 \quad 31 \quad .98.\end{aligned}$$

For the purpose of proving the accuracy of the numerical results, we compute also, as in the first approximation,

$$\begin{aligned}l &= 7^\circ 16' 51''.54, & l''' &= 91^\circ 37' 41''.20, \\ b &= + 1 \quad 32 \quad 14 \quad .07, & b''' &= - 4 \quad 10 \quad 47 \quad .36, \\ \lambda &= 16 \quad 59 \quad 9 \quad .38, & \lambda''' &= 75 \quad 23 \quad 46 \quad .99, \\ \beta &= + 2 \quad 56 \quad 39 \quad .54, & \beta''' &= - 3 \quad 4 \quad 43 \quad .33, \\ \log \rho &= 0.0256960, & \log \rho''' &= 0.4492539.\end{aligned}$$

The values of λ and λ''' thus found differ, respectively, only $0''.04$ and $0''.09$ from those given by the normal places, and hence the accuracy of the entire calculation, both of the quantities which are independent of $P', P'', Q',$ and Q'' , and of those which depend on the successive hypotheses, is completely proved. This condition, however, must always be satisfied whatever may be the assumed values of $P', P'', Q',$ and Q'' .

From $r, r', u, u',$ &c., we derive

$$\log s = 0.0085254, \quad \log s'' = 0.0174637, \quad \log s''' = 0.0204076,$$

and hence the corrected values of $P', P'', Q',$ and Q'' become

$$\begin{aligned}\log P' &= 9.8462110, & \log Q' &= 9.8407524, \\ \log P'' &= 9.8077622, & \log Q'' &= 9.8824726.\end{aligned}$$

These values differ so little from those for the second approximation, the intervals of time between the observations being very large, that a further repetition of the calculation is unnecessary, since the results which would thus be obtained can differ but slightly from those which have been derived. We shall, therefore, complete the determination of the elements of the orbit, using the extreme places. Thus, from

$$\begin{aligned}\tau_0 &= k(t''' - t), & \tan \chi_0 &= \sqrt{\frac{r'''}{r}}, \\ \sin \gamma_0 \cos G_0 &= \sin \frac{1}{2}(u''' - u), \\ \sin \gamma_0 \sin G_0 &= \cos \frac{1}{2}(u''' - u) \cos 2\chi_0, \\ \cos \gamma_0 &= \cos \frac{1}{2}(u''' - u) \sin 2\chi_0, \\ m_0 &= \frac{\tau_0^2}{(r + r''')^3 \cos^3 \gamma_0}, & j_0 &= \frac{\sin^2 \frac{1}{2} \gamma_0}{\cos \gamma_0}, \\ \eta_0 &= \frac{m_0}{\frac{5}{6} + j_0 + \zeta_0}, & x_0 &= \frac{m_0}{s_0^3} - j_0.\end{aligned}$$

we get

$$\begin{aligned}\log \tau_0 &= 0.5838863, & \log \tan G_0 &= 8.0521953, \\ \gamma_0 &= 42^\circ 14' 30''.17, & \log m_0 &= 9.7179026, \\ \log s_0^2 &= 0.2917731, & \log x_0 &= 8.9608397.\end{aligned}$$

The formula

$$p = \left(\frac{s_0 r r''' \sin(u''' - u)}{\tau_0} \right)^2$$

gives

$$\log p = 0.3712401;$$

and if we compute the same quantity by means of

$$p = \left(\frac{s r' r'' \sin(u'' - u')}{\tau} \right)^2 = \left(\frac{s'' r r' \sin(u' - u)}{\tau''} \right)^2 = \left(\frac{s''' r'' r''' \sin(u''' - u'')}{\tau'''} \right)^2,$$

the separate results are, respectively, 0.3712397, 0.3712418, and 0.3712414. The differences between these results are very small, and arise both from the unavoidable errors of calculation and from the deviation of the adopted values of P' , P'' , Q' , and Q'' from the limit of accuracy attainable with logarithms of seven decimal places. A variation of only $0''.2$ in the values of $u' - u$ and $u''' - u''$ will produce an entire accordance of the particular results.

From the equations

$$\begin{aligned}\sin \frac{1}{4}(E''' - E) &= \sqrt{x_0}, \\ a \cos \varphi &= \frac{\sin \frac{1}{2}(u''' - u)}{\sin \frac{1}{2}(E''' - E)} \sqrt{r r''}, \\ \cos \varphi &= \frac{p}{a \cos \varphi},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{4}(E''' - E) &= 17^\circ 35' 42''.12, & \log(a \cos \varphi) &= 0.3796883, \\ \log \cos \varphi &= 9.9915518.\end{aligned}$$

The formulæ

$$\begin{aligned}e \sin(\omega - \frac{1}{2}(u''' + u)) &= \frac{p}{\cos \gamma_0 \sqrt{r r''}} \tan G_0, \\ e \cos(\omega - \frac{1}{2}(u''' + u)) &= \frac{p}{\cos \gamma_0 \sqrt{r r''}} - \sec \frac{1}{2}(u''' - u),\end{aligned}$$

give

$$\begin{aligned}\omega &= 197^\circ 38' 8''.48, & \log e &= \log \sin \varphi = 9.2907881, \\ \varphi &= 11^\circ 15' 52''.22, & \pi = \omega + \Omega &= 44^\circ 20' 34''.37.\end{aligned}$$

This result for φ gives $\log \cos \varphi = 9.9915521$, which differs only 3 in the last decimal place from the value found from p and $a \cos \varphi$. Then, from

$$a = \frac{p}{\cos^2 \varphi}, \quad \mu = \frac{k}{a^{\frac{3}{2}}},$$

the value of k being expressed in seconds of arc, or $\log k = 3.5500066$, we get

$$\log a = 0.3881359, \quad \log \mu = 2.9678027.$$

For the eccentric anomalies we have

$$\begin{aligned} \tan \frac{1}{2}E &= \tan \frac{1}{2}(u - \omega) \tan (45^\circ - \frac{1}{2}\varphi), \\ \tan \frac{1}{2}E' &= \tan \frac{1}{2}(u' - \omega) \tan (45^\circ - \frac{1}{2}\varphi), \\ \tan \frac{1}{2}E'' &= \tan \frac{1}{2}(u'' - \omega) \tan (45^\circ - \frac{1}{2}\varphi), \\ \tan \frac{1}{2}E''' &= \tan \frac{1}{2}(u''' - \omega) \tan (45^\circ - \frac{1}{2}\varphi). \end{aligned}$$

from which the results are

$$\begin{aligned} E &= 329^\circ 11' 46''.01, & E'' &= 12^\circ 5' 33''.63, \\ E' &= 354 \quad 29 \quad 11 \quad .84, & E''' &= 39 \quad 34 \quad 34 \quad .65. \end{aligned}$$

The value of $\frac{1}{4}(E''' - E)$ thus derived differs only $0''.03$ from that obtained directly from x_0 .

For the mean anomalies, we have

$$\begin{aligned} M &= E - e \sin E, & M'' &= E'' - e \sin E'', \\ M' &= E' - e \sin E', & M''' &= E''' - e \sin E''', \end{aligned}$$

which give

$$\begin{aligned} M &= 334^\circ 55' 39''.32, & M'' &= 9^\circ 44' 52''.82, \\ M' &= 355 \quad 33 \quad 42 \quad .97, & M''' &= 32 \quad 26 \quad 44 \quad .74. \end{aligned}$$

Finally, if M_0 denotes the mean anomaly for the epoch $T = 1864$ Jan. 1.0 mean time at Greenwich, from

$$\begin{aligned} M_0 &= M - \mu(t - T) = M' - \mu(t' - T) \\ &= M'' - \mu(t'' - T) = M''' - \mu(t''' - T), \end{aligned}$$

we obtain the four values

$$\begin{aligned} M_0 &= 1^\circ 29' 39''.40 \\ &\quad 39 \quad .49 \\ &\quad 39 \quad .40 \\ &\quad 39 \quad .40, \end{aligned}$$

the agreement of which completely proves the entire calculation of the elements from the data. Collecting together the several results, we have the following elements:

Epoch = 1864 Jan. 1.0 Greenwich mean time.

$$M = 1^{\circ} 29' 39''.42$$

$$\left. \begin{array}{l} \pi = 44 \quad 20 \quad 34 \quad .37 \\ \Omega = 206 \quad 42 \quad 25 \quad .89 \\ i = 4 \quad 36 \quad 47 \quad .20 \end{array} \right\} \begin{array}{l} \text{Ecliptic and Mean} \\ \text{Equinox 1864.0.} \end{array}$$

$$\varphi = 11 \quad 15 \quad 52 \quad .22$$

$$\log a = 0.3881359$$

$$\log \mu = 2.9678027$$

$$\mu = 928''.54447.$$

102. The elements thus derived completely represent the four observed longitudes and the latitudes for the second and third places, which are the actual data of the problem; but for the extreme latitudes the residuals are, computation minus observation,

$$\Delta\beta = -4''.47, \quad \Delta\beta''' = +1''.23.$$

These remaining errors arise chiefly from the circumstance that the position of the plane of the orbit cannot be determined from the second and third places with the same degree of precision as from the extreme places. It would be advisable, therefore, in the final approximation, as soon as ρ' , ρ'' , n , n'' , n' , and n''' are obtained, to compute from these and the data furnished directly by observation the curtate distances for the extreme places. The corresponding heliocentric places may then be found, and hence the position of the plane of the orbit as determined by the first and fourth observations. Thus, by means of the equations (37) and (38), we obtain

$$\log \rho = 0.0256953, \quad \log \rho''' = 0.4492542.$$

With these values of ρ and ρ''' , the following heliocentric places are obtained:

$$\begin{array}{lll} l = 7^{\circ} 16' 51''.54, & \log \tan b = 8.4289064, & \log r = 0.3083732, \\ l''' = 91 \quad 37 \quad 40 \quad .96, & \log \tan b''' = 8.8638549, & \log r''' = 0.3172678. \end{array}$$

Then from

$$\begin{array}{l} \tan i \sin \left(\frac{1}{2} (l''' + l) - \Omega \right) = \frac{1}{2} (\tan b''' + \tan b) \sec \frac{1}{2} (l''' - l), \\ \tan i \cos \left(\frac{1}{2} (l''' + l) - \Omega \right) = \frac{1}{2} (\tan b''' - \tan b) \operatorname{cosec} \frac{1}{2} (l''' - l), \end{array}$$

we get

$$\Omega = 206^{\circ} 42' 45''.23, \quad i = 4^{\circ} 36' 49''.76.$$

For the arguments of the latitude the results are

$$u = 160^{\circ} 30' 35''.99, \quad u''' = 244^{\circ} 59' 12''.53.$$

The equations

$$\begin{aligned}\tan b' &= \tan i \sin (l' - \Omega), \\ \tan b'' &= \tan i \sin (l'' - \Omega),\end{aligned}$$

give

$$\log \tan b' = 8.1827129_n, \quad \log \tan b'' = 8.6342104_n,$$

and the comparison of these results with those derived directly from ρ' and ρ'' exhibits a difference of $+1''.04$ in b' , and of $-0''.06$ in b'' . Hence, the position of the plane of the orbit as determined from the extreme places very nearly satisfies the intermediate latitudes.

If we compute the remaining elements by means of these values of r , r''' , and u , u''' , the separate results are:

$$\begin{aligned}\log \tan G_0 &= 8.0522282_n, & \log m_0 &= 9.7179026, \\ \log s_0^2 &= 0.2917731, & \log x_0 &= 8.9608397, \\ \log p &= 0.3712405, & \frac{1}{4}(E''' - E) &= 17^\circ 35' 42''.12, \\ \log(a \cos \varphi) &= 0.3796884, & \log \cos \varphi &= 9.9915521, \\ \omega &= 197^\circ 37' 47''.72, & \log e &= 9.2907906, \\ \varphi &= 11 \ 15 \ 52 \ .46, & \log \cos \varphi &= 9.9915520, \\ \log a &= 0.3881365, & \log \mu &= 2.9678019, \\ E &= 329^\circ 11' 47''.24, & E''' &= 39^\circ 34' 35''.70, \\ M &= 334 \ 55 \ 40 \ .46, & M''' &= 32 \ 26 \ 45 \ .49, \\ M_0 &= 1 \ 29 \ 40 \ .36, & M_0 &= 1 \ 29 \ 40 \ .37.\end{aligned}$$

Hence, the elements are as follows:

$$\begin{aligned}\text{Epoch} &= 1864 \text{ Jan. } 1.0 \text{ Greenwich mean time.} \\ M &= 1^\circ 29' 40''.36 \\ \pi &= 44 \ 20 \ 32 \ .95 \\ \Omega &= 206 \ 42 \ 45 \ .23 \\ i &= 4 \ 36 \ 49 \ .76 \\ \varphi &= 11 \ 15 \ 52 \ .46 \\ \log a &= 0.3881365 \\ \mu &= 928''.5427.\end{aligned} \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Ecliptic and Mean} \\ \text{Equinox } 1864.0. \end{array}$$

It appears, therefore, that the principal effect of neglecting the extreme latitudes in the determination of an orbit from four observations is on the inclination of the orbit and on the longitude of the ascending node, the other elements being very slightly changed. The elements thus derived represent the extreme places exactly, and if we compute the second and third places directly from these elements, we obtain

$$\begin{aligned}M' &= 355^\circ 33' 43''.88, & M'' &= 9^\circ 44' 53''.73, \\ E' &= 354 \ 29 \ 12 \ .93, & E'' &= 12 \ 5 \ 34 \ .81, \\ v' &= 353 \ 16 \ 59 \ .07, & v'' &= 14 \ 42 \ 45 \ .96.\end{aligned}$$

| | |
|-----------------------------------|------------------------------------|
| $\log r' = 0.2942366,$ | $\log r'' = 0.2960826,$ |
| $u' = 190^\circ 54' 46''.79,$ | $u'' = 212^\circ 20' 33''.68,$ |
| $l' = 37 \ 35 \ 27 \ .75,$ | $l'' = 58 \ 58 \ 16 \ .50,$ |
| $b' = - \ 0 \ 52 \ 21 \ .25,$ | $b'' = - \ 2 \ 27 \ 59 \ .06,$ |
| $\lambda' = 10 \ 14 \ 17 \ .35,$ | $\lambda'' = 29 \ 53 \ 21 \ .99,$ |
| $\beta' = - \ 1 \ 15 \ 47 \ .67,$ | $\beta'' = - \ 2 \ 29 \ 57 \ .62,$ |
| $\log \rho' = 0.1334634,$ | $\log \rho'' = 0.2899122.$ |

Hence, the residuals for the second and third places of the planet are—

Comp. — Obs.

| | |
|------------------------------|-----------------------------|
| $\Delta\lambda' = -0''.22,$ | $\Delta\beta' = +1''.53,$ |
| $\Delta\lambda'' = 0 \ .00,$ | $\Delta\beta'' = -0 \ .06;$ |

and the elements very nearly represent the four normal places. Since the interval between the extreme places is 223 days, these elements must represent, within the limits of the errors of observation, the entire series of observations on which the normals are based. It may be observed, also, that the successive approximations, in the case of intervals which are very large, do not converge with the same degree of rapidity as when the intervals are small, and that in such cases the numerical calculation is very much abbreviated by the determination, in the first instance, of the assumed values of P' , P'' , Q' , and Q'' by means of approximate elements already known. For the first determination of an unknown orbit, the intervals will generally be so small that the first assumed values of these quantities, as determined by the equations

$$P' = \frac{\tau}{\tau'} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau'^2}{\tau'^3} \right), \quad Q' = \frac{1}{2} \tau \tau',$$

$$P'' = \frac{\tau}{\tau''} \left(1 - \frac{1}{6} \frac{\tau^2 - \tau''^2}{\tau''^3} \right), \quad Q'' = \frac{1}{2} \tau \tau'',$$

will not differ much from the correct values, and two or three hypotheses, or even less, will be sufficient. But when the intervals are large, and especially if the eccentricity is also considerable, several hypotheses may be required, the last of which will be facilitated by using the equations (82).

The application of the formulæ for the determination of an orbit from four observations, is not confined to orbits whose inclination to the ecliptic is very small, corresponding to the cases in which the method of finding the elements by means of three observations fails,

or at least becomes very uncertain. On the contrary, these formulæ apply equally well in the case of orbits of any inclination whatever, and since the labor of computing an orbit from four observations does not much exceed that required when only three observed places are used, while the results must evidently be more approximate, it will be expedient, in very many cases, to use the formulæ given in this chapter both for the first approximation to an unknown orbit and for the subsequent determination from more complete data.

CHAPTER VI.

INVESTIGATION OF VARIOUS FORMULÆ FOR THE CORRECTION OF THE APPROXIMATE ELEMENTS OF THE ORBIT OF A HEAVENLY BODY.

103. IN the case of the discovery of a planet, it is often convenient, before sufficient data have been obtained for the determination of elliptic elements, to compute a system of circular elements, an ephemeris computed from these being sufficient to follow the planet for a brief period, and to identify the comparison stars used in differential observations. For this purpose, only two observed places are required, there being but four elements to be determined, namely, Ω , i , a , and, for any instant, the longitude in the orbit. As soon as a has been found, the geocentric distances of the planet for the instants of observation may be obtained by means of the formulæ

$$\begin{aligned}\Delta &= R \cos \psi + \sqrt{a^2 - R^2 \sin^2 \psi}, \\ \Delta'' &= R'' \cos \psi'' + \sqrt{a^2 - R''^2 \sin^2 \psi''},\end{aligned}\quad (1)$$

the values of ψ and ψ'' being computed from the equations (42)₃ and (43)₃. For convenient logarithmic calculation, we may first find z and z'' from

$$\sin z = \frac{R \sin \psi}{a}, \quad \sin z'' = \frac{R'' \sin \psi''}{a}, \quad (2)$$

since the formulæ will generally be required for cases such that these angles may be obtained with sufficient accuracy by means of their sines. Then we have

$$\rho = \frac{R \sin(z + \psi)}{\sin z} \cos \beta, \quad \rho'' = \frac{R'' \sin(z'' + \psi'')}{\sin z''} \cos \beta'', \quad (3)$$

from which to find ρ and ρ'' . These having been found, we have

$$\begin{aligned}\tan(l - \odot) &= \frac{\rho \sin(\lambda - \odot)}{\rho \cos(\lambda - \odot) - R}, \\ \sin b &= \frac{\rho \tan \beta}{a},\end{aligned}\quad (4)$$

for the determination of l and b , and similarly for l'' and b'' . The

inclination of the orbit and the longitude of the ascending node are then found by means of the formulæ (75)₃, and the arguments of the latitude by means of (77)₃. Since $u'' - u$ is the distance on the celestial sphere between two points of which the heliocentric spherical co-ordinates are l , b , and l'' , b'' , we have, also, the equations

$$\begin{aligned}\sin(u'' - u) \sin B &= \cos b'' \sin(l'' - l), \\ \sin(u'' - u) \cos B &= \cos b \sin b'' - \sin b \cos b'' \cos(l'' - l), \\ \cos(u'' - u) &= \sin b \sin b'' + \cos b \cos b'' \cos(l'' - l),\end{aligned}$$

for the determination of $u'' - u$, the angle opposite the side $90^\circ - b''$ of the spherical triangle being denoted by B . The solution of these equations is facilitated by the introduction of auxiliary angles, as already illustrated for similar cases.

In a circular orbit, the eccentricity being equal to zero, $u'' - u$ expresses the mean motion of the planet during the interval $t'' - t$, and we must also have

$$t'' - t = \frac{a^{\frac{3}{2}}}{k} (u'' - u), \quad (5)$$

the value of k being expressed in seconds of arc, or $\log k = 3.5500066$.

These formulæ will be applied only when the interval $t'' - t$ is small, and for the case of the asteroid planets we may first assume

$$a = 2.7,$$

which is about the average mean distance of the group. With this we compute ρ and ρ'' by means of the equations (2) and (3), and the corresponding heliocentric places by means of (4). If the inclination is small, $u'' - u$ will differ very little from $l'' - l$. Therefore, in the first approximation, when the heliocentric longitudes have been found, the corresponding value of $t'' - t$ may be obtained from equation (5), writing $l'' - l$ in place of $u'' - u$. If this comes out less than the actual interval between the times of observation, we infer that the assumed value of a is too small; but if it comes out greater, the assumed value of a is too large. The value to be used in a repetition of the calculation may be computed from the expression

$$\log a = \frac{2}{3} (\log(t'' - t) + \log k - \log(u'' - u)),$$

the difference $u'' - u$ being expressed in seconds of arc. With this we recompute ρ , ρ'' , l , and l'' , and find also b , b'' , Ω , i , u , and u'' . Then, if the value of a computed from the last result for $u'' - u$ differs from the last assumed value, a further repetition of the calcu-

lation becomes necessary. But when three successive approximate values of a have been found, the correct value may be readily interpolated according to the process already illustrated for similar cases.

As soon as the value of a has been obtained which completely satisfies equation (5), this result and the corresponding values of Ω , i , and the argument of the latitude for a fixed epoch, complete the system of circular elements which will exactly satisfy the two observed places. If we denote by u_0 the argument of the latitude for the epoch T , we shall have, for any instant t ,

$$u = u_0 + \mu(t - T),$$

u being the mean or actual daily motion computed from

$$\mu = \frac{k}{a^{\frac{3}{2}}}.$$

The value of u thus found, and $r = a$, substituted in the formulæ for computing the places of a heavenly body, will furnish the approximate ephemeris required.

The corrections for parallax and aberration are neglected in the first determination of circular elements; but as soon as these approximate elements have been derived, the geocentric distances may be computed to a degree of accuracy sufficient for applying these corrections directly to the observed places, preparatory to the determination of elliptic elements. The assumption of $r' = a$ will also be sufficient to take into account the term of the second order in the first assumed value of P , according to the first of equations (98)₄.

104. When approximate elements of the orbit of a heavenly body have been determined, and it is desired to correct them so as to satisfy as nearly as possible a series of observations including a much longer interval of time than in the case of the observations used in finding these approximate elements, a variety of methods may be applied. For a very long series of observations, the approximate elements being such that the squares of the corrections which must be applied to them may be neglected, the most complete method is to form the equations for the variations of any two spherical co-ordinates which fix the place of the body in terms of the variations of the six elements of the orbit; and the differences between the computed places for different dates and the corresponding observed places thus furnish equations of condition, the solution of which gives the corrections to be applied to the elements. But when the observations do not in-

clude a very long interval of time, instead of forming the equations for the variations of the geocentric places in terms of the variations of the elements of the orbit, it will be more convenient to form the equations for these variations in terms of quantities, less in number, from which the elements themselves are readily obtained. If no assumption is made in regard to the form of the orbit, the quantities which present the least difficulties in the numerical calculation are the geocentric distances of the body for the dates of the extreme observations, or at least for the dates of those which are best adapted to the determination of the elements. As soon as these distances are accurately known, the two corresponding complete observations are sufficient to determine all the elements of the orbit.

The approximate elements enable us to assume, for the dates t and t'' , the values of Δ and Δ'' ; and the elements computed from these by means of the data furnished by observation, will exactly represent the two observed places employed. Further, the elements may be supposed to be already known to such a degree of approximation that the squares and products of the corrections to be applied to the assumed values of Δ and Δ'' may be neglected, so that we shall have, for any date,

$$\begin{aligned}\cos \delta \Delta \alpha &= \cos \delta \frac{d\alpha}{d\Delta} \Delta \Delta + \cos \delta \frac{d\alpha}{d\Delta''} \Delta \Delta'', \\ \Delta \delta &= \frac{d\delta}{d\Delta} \Delta \Delta + \frac{d\delta}{d\Delta''} \Delta \Delta''.\end{aligned}\tag{6}$$

If, therefore, we compare the elements computed from Δ and Δ'' with any number of additional or intermediate observed places, each observed spherical co-ordinate will furnish an equation of condition for the correction of the assumed distances. But in order that the equations (6) may be applied, the numerical values of the partial differential coefficients of α and δ with respect to Δ and Δ'' must be found. Ordinarily, the best method of effecting the determination of these is to compute three systems of elements, the first from Δ and Δ'' , the second from $\Delta + D$ and Δ'' , and the third from Δ and $\Delta'' + D''$, D and D'' being small increments assigned to Δ and Δ'' respectively. If now, for any date t' , we compute α' and δ' from each system of elements thus obtained, we may find the values of the differential coefficients sought. Thus, let the spherical co-ordinates for the time t' computed from the first system be denoted by α' and δ' ; those computed from the second system of elements, by $\alpha' + a \sec \delta'$ and $\delta' + d$; and those from the third system, by $\alpha' + a' \sec \delta'$ and $\delta' + d''$. Then we shall have

$$\begin{aligned} \cos \delta' \frac{d\alpha'}{d\Delta} &= \frac{a}{D'} & \frac{d\delta}{d\Delta} &= \frac{d}{D'} \\ \cos \delta' \frac{d\alpha'}{d\Delta''} &= \frac{a''}{D''} & \frac{d\delta}{d\Delta''} &= \frac{d''}{D''}; \end{aligned} \quad (7)$$

and the equations (6) give

$$\begin{aligned} \cos \delta' \Delta \alpha' &= \frac{a}{D} \Delta \Delta + \frac{a''}{D''} \Delta \Delta'', \\ \Delta \delta' &= \frac{d}{D} \Delta \Delta + \frac{d''}{D''} \Delta \Delta''. \end{aligned} \quad (8)$$

In the same manner, computing the places for various dates, for which observed places are given, by means of each of the three systems of elements, the equations for the correction of Δ and Δ'' , as determined by each of the additional observations employed, may be formed.

105. For the purpose of illustrating the application of this method, let us suppose that three observed places are given, referred to the ecliptic as the fundamental plane, and that the corrections for parallax, aberration, precession, and nutation have all been duly applied. By means of the approximate elements already known, we compute the values of Δ and Δ'' for the extreme places, and from these the heliocentric places are obtained by means of the equations (71)₃ and (72)₃, writing $\Delta \cos \beta$ and $\Delta'' \cos \beta''$ in place of ρ and ρ'' . The values of Ω , i , u , and u'' will be obtained by means of the formulæ (76)₃ and (77)₃; and from r , r'' and $u'' - u$ the remaining elements of the orbit are determined as already illustrated. The first system of elements is thus obtained. Then we assign an increment to Δ , which we denote by D , and with the geocentric distances $\Delta + D$ and Δ'' we compute in precisely the same manner a second system of elements. Next, we assign to Δ'' an increment D'' , and from Δ and $\Delta'' + D''$ a third system of elements is derived. Let the geocentric longitude and latitude for the date of the middle observation computed from the first system of elements be designated, respectively, by λ_1' and β_1' ; from the second system of elements, by λ_2' and β_2' ; and from the third system, by λ_3' and β_3' . Then from

$$\begin{aligned} a &= (\lambda_2' - \lambda_1') \cos \beta_1', & d &= \beta_2' - \beta_1', \\ a'' &= (\lambda_3' - \lambda_1') \cos \beta_1', & d'' &= \beta_3' - \beta_1', \end{aligned} \quad (9)$$

we compute a , a'' , d , and d'' , and by means of these and the values of D and D'' we form the equations

$$\begin{aligned}\frac{a}{D} \Delta \mathcal{A} + \frac{a''}{D''} \Delta \mathcal{A}'' &= \cos \beta' \Delta \lambda', \\ \frac{d}{D} \Delta \mathcal{A} + \frac{d''}{D''} \Delta \mathcal{A}'' &= \Delta \beta',\end{aligned}\tag{10}$$

for the determination of the corrections to be applied to the first assumed values of \mathcal{A} and \mathcal{A}'' , by means of the differences between observation and computation. The observed longitude and latitude being denoted by λ' and β' , respectively, we shall have

$$\begin{aligned}\cos \beta' \Delta \lambda' &= (\lambda' - \lambda_1') \cos \beta', \\ \Delta \beta' &= \beta' - \beta_1',\end{aligned}\tag{11}$$

for finding the values of the second members of the equations (10), and then by elimination we obtain the values of the corrections $\Delta \mathcal{A}$ and $\Delta \mathcal{A}''$ to be applied to the assumed values of the distances. Finally, we compute a fourth system of elements corresponding to the geocentric distances $\mathcal{A} + \Delta \mathcal{A}$ and $\mathcal{A}'' + \Delta \mathcal{A}''$ either directly from these values, or by interpolation from the three systems of elements already obtained; and, if the first assumption is not considerably in error, these elements will exactly represent the middle place. It should be observed, however, that if the second system of elements represents the middle place better than the first system, λ_2' and β_2' should be used instead of λ_1' and β_1' in the equations (11), and, in this case, the final system of elements must be computed with the distances $\mathcal{A} + D + \Delta \mathcal{A}$ and $\mathcal{A}'' + \Delta \mathcal{A}''$. Similarly, if the middle place is best represented by the third system of elements, the corrections will be obtained for the distances used in the third hypothesis.

If the computation of the middle place by means of the final elements still exhibits residuals, on account of the neglected terms of the second order, a repetition of the calculation of the corrections $\Delta \mathcal{A}$ and $\Delta \mathcal{A}''$, using these residuals for the values of the second members of the equations (10), will furnish the values of the distances for the extreme places with all the precision desired. The increments D and D'' to be assigned successively to the first assumed values of \mathcal{A} and \mathcal{A}'' may, without difficulty, be so taken that the true elements shall differ but little from one of the three systems computed; and in all the formulæ it will be convenient to use, instead of the geocentric distances themselves, the logarithms of these distances, and to express the variations of these quantities in units of the last decimal place of the logarithms.

These formulæ will generally be applied for the correction of

approximate elements by means of several observed places, which may be either single observations or normal places, each derived from several observations, and the two places selected for the computation of the elements from Δ and Δ'' should not only be the most accurate possible, but they should also be such that the resulting elements are not too much affected by small errors in these geocentric places. They should moreover be as distant from each other as possible, the other considerations not being overlooked. When the three systems of elements have been computed, each of the remaining observed places will furnish two equations of condition, according to equations (10), for the determination of the corrections to be applied to the assumed values of the geocentric distances; and, since the number of equations will thus exceed the number of unknown quantities, the entire group must be combined according to the method of least squares. Thus, we multiply each equation by the coefficient of $\Delta\Delta$ in that equation, taken with its proper algebraic sign, and the sum of all the equations thus formed gives one of the final equations required. Then we multiply each equation by the coefficient of $\Delta\Delta''$ in that equation, taken also with its proper algebraic sign, and the sum of all these gives the second equation required. From these two final equations, by elimination, the most probable values of $\Delta\Delta$ and $\Delta\Delta''$ will be obtained; and a system of elements computed with the distances thus corrected will exactly represent the two fundamental places selected, while the sum of the squares of the residuals for the other places will be a minimum. The observations are thus supposed to be equally good; but if certain observed places are entitled to greater influence than the others, the relative precision of these places must be taken into account in the combination of the equations of condition, the process for which will be fully explained in the next chapter.

When a number of observed places are to be used for the correction of the approximate elements of the orbit of a planet or comet, it will be most convenient to adopt the equator as the fundamental plane. In this case the heliocentric places will be computed from the assumed values of Δ and Δ'' , and the corresponding geocentric right ascensions and declinations by means of the formulæ (106)₃ and (107)₃; and the position of the plane of the orbit as determined from these by means of the equations (76)₃ will be referred to the equator as the fundamental plane. The formation of the equations of condition for the corrections $\Delta\Delta$ and $\Delta\Delta''$ to be applied to the assumed values of the distances will then be effected precisely as in the case of λ and β , the

necessary changes being made in the notation. In a similar manner, the calculation may be effected for any other fundamental plane which may be adopted.

It should be observed, further, that when the ecliptic is taken as the fundamental plane, the geocentric latitudes should be corrected by means of the equation (6)₄, in order that the latitudes of the sun shall vanish, otherwise, for strict accuracy, the heliocentric places must be determined from Δ and Δ'' in accordance with the equations (89)₁.

106. The partial differential coefficients of the two spherical co-ordinates with respect to Δ and Δ'' may be computed directly by means of differential formulæ; but, except for special cases, the numerical calculation is less expeditious than in the case of the indirect method, while the liability of error is much greater. If we adopt the plane of the orbit as determined by the approximate values of Δ and Δ'' as the fundamental plane, and introduce χ as one of the elements of the orbit, as in the equations (72)₂, the variation of the geocentric longitude θ measured in this plane, neglecting terms of the second order, depends on only four elements; and in this case the differential formulæ may be applied with facility. Thus, if we express r and v in terms of the elements φ , M_0 , and μ , we shall have

$$\frac{dr}{d\Delta} = \frac{dr}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dr}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dr}{d\mu} \cdot \frac{d\mu}{d\Delta},$$

and

$$\frac{dv}{d\Delta} = \frac{dv}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dv}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dv}{d\mu} \cdot \frac{d\mu}{d\Delta},$$

or

$$\frac{d(v + \chi)}{d\Delta} = \frac{d\chi}{d\Delta} + \frac{dv}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dv}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dv}{d\mu} \cdot \frac{d\mu}{d\Delta},$$

In like manner, we have

$$\begin{aligned} \frac{dr''}{d\Delta} &= \frac{dr''}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dr''}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dr''}{d\mu} \cdot \frac{d\mu}{d\Delta}, \\ \frac{d(v'' + \chi)}{d\Delta} &= \frac{dv''}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dv''}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dv''}{d\mu} \cdot \frac{d\mu}{d\Delta} + \frac{d\chi}{d\Delta}. \end{aligned}$$

As soon as the values of $\frac{dr}{d\Delta}$, $\frac{d(v + \chi)}{d\Delta}$, $\frac{dr''}{d\Delta}$, and $\frac{d(v'' + \chi)}{d\Delta}$ are known, the equations necessary for finding the differential coefficients of the elements χ , φ , M_0 , and μ with respect to Δ are thus provided. In the case under consideration, when an increment is assigned to Δ ,

the value of Δ'' remaining unchanged, r'' and $v'' + \chi$ are not changed, and hence

$$\frac{dr''}{d\Delta} = 0, \quad \frac{d(v'' + \chi)}{d\Delta} = 0.$$

To find $\frac{dr}{d\Delta}$ and $\frac{d(v + \chi)}{d\Delta}$, from the equations

$$\begin{aligned} \Delta \cos \eta \cos \theta &= x + X, \\ \Delta \cos \eta \sin \theta &= y + Y, \end{aligned}$$

in which η is the geocentric latitude in reference to the plane of the orbit computed from Δ and Δ'' as the fundamental plane, and X , Y the geocentric co-ordinates of the sun referred to the same plane, we get

$$\begin{aligned} dx &= \cos \eta \cos \theta d\Delta, \\ dy &= \cos \eta \sin \theta d\Delta, \end{aligned}$$

or, substituting for dx and dy their values given by (73)₂,

$$\begin{aligned} \cos \eta \cos \theta d\Delta &= \cos u dr - r \sin u d(v + \chi), \\ \cos \eta \sin \theta d\Delta &= \sin u dr + r \cos u d(v + \chi). \end{aligned}$$

Eliminating, successively, $d(v + \chi)$ and dr , we get

$$\begin{aligned} \frac{dr}{d\Delta} &= \cos \eta \cos(\theta - u), \\ \frac{d(v + \chi)}{d\Delta} &= \frac{1}{r} \cos \eta \sin(\theta - u). \end{aligned} \tag{12}$$

Therefore, we shall have

$$\begin{aligned} \frac{d\chi}{d\Delta} + \frac{dv}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dv}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dv}{d\mu} \cdot \frac{d\mu}{d\Delta} &= \frac{1}{r} \cos \eta \sin(\theta - u), \\ \frac{dr}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dr}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dr}{d\mu} \cdot \frac{d\mu}{d\Delta} &= \cos \eta \cos(\theta - u), \\ \frac{d\chi}{d\Delta} + \frac{dv''}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dv''}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dv''}{d\mu} \cdot \frac{d\mu}{d\Delta} &= 0, \\ \frac{dr''}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \frac{dr''}{dM_0} \cdot \frac{dM_0}{d\Delta} + \frac{dr''}{d\mu} \cdot \frac{d\mu}{d\Delta} &= 0; \end{aligned} \tag{13}$$

and if we compute the numerical values of the differential coefficients of r , r'' , v , and v'' with respect to the elements φ , M_0 , and μ , these equations will furnish, by elimination, the values of the four unknown quantities $\frac{d\chi}{d\Delta}$, $\frac{d\varphi}{d\Delta}$, $\frac{dM_0}{d\Delta}$, and $\frac{d\mu}{d\Delta}$.

In precisely the same manner we derive the following equations

for the determination of the partial differential coefficients of these elements with respect to Δ'' :—

$$\begin{aligned}
 \frac{d\chi}{d\Delta''} + \frac{dv}{d\varphi} \cdot \frac{d\varphi}{d\Delta''} + \frac{dv}{dM_0} \cdot \frac{dM_0}{d\Delta''} + \frac{dv}{d\mu} \cdot \frac{d\mu}{d\Delta''} &= 0, \\
 \frac{dr}{d\varphi} \cdot \frac{d\varphi}{d\Delta''} + \frac{dr}{dM_0} \cdot \frac{dM_0}{d\Delta''} + \frac{dr}{d\mu} \cdot \frac{d\mu}{d\Delta''} &= 0, \\
 \frac{d\chi}{d\Delta''} + \frac{dv''}{d\varphi} \cdot \frac{d\varphi}{d\Delta''} + \frac{dv''}{dM_0} \cdot \frac{dM_0}{d\Delta''} + \frac{dv''}{d\mu} \cdot \frac{d\mu}{d\Delta''} &= \frac{1}{r''} \cos \gamma'' \sin (\theta'' - u''), \\
 \frac{dr''}{d\varphi} \cdot \frac{d\varphi}{d\Delta''} + \frac{dr''}{dM_0} \cdot \frac{dM_0}{d\Delta''} + \frac{dr''}{d\mu} \cdot \frac{d\mu}{d\Delta''} &= \cos \gamma'' \cos (\theta'' - u'').
 \end{aligned} \tag{14}$$

Since the geocentric latitude γ is affected chiefly by a change of the position of the plane of the orbit, while the variation of the longitude θ is independent of Ω and i when the squares and products of the variations of the elements are neglected, if we determine the elements which exactly represent the places to which Δ and Δ'' belong, as well as the longitudes for two additional places, or, if we determine those which satisfy the two fundamental places and the longitudes for any number of additional observed places, so that the sum of the squares of their residuals shall be a minimum, the results thus obtained will very nearly satisfy the several latitudes.

Let θ' denote the geocentric longitude of the body, referred to the plane of the orbit computed from Δ and Δ'' as the fundamental plane, for the date t' of any one of the observed places to be used for correcting these assumed distances. Then, to find the partial differential coefficients of θ' with respect to Δ and Δ'' , we have

$$\begin{aligned}
 \cos \gamma' \frac{d\theta'}{d\Delta} &= \cos \gamma' \frac{d\theta'}{d\chi} \cdot \frac{d\chi}{d\Delta} + \cos \gamma' \frac{d\theta'}{d\varphi} \cdot \frac{d\varphi}{d\Delta} + \cos \gamma' \frac{d\theta'}{dM_0} \cdot \frac{dM_0}{d\Delta} \\
 &\quad + \cos \gamma' \frac{d\theta'}{d\mu} \cdot \frac{d\mu}{d\Delta}, \\
 \cos \gamma' \frac{d\theta'}{d\Delta''} &= \cos \gamma' \frac{d\theta'}{d\chi} \cdot \frac{d\chi}{d\Delta''} + \cos \gamma' \frac{d\theta'}{d\varphi} \cdot \frac{d\varphi}{d\Delta''} + \cos \gamma' \frac{d\theta'}{dM_0} \cdot \frac{dM_0}{d\Delta''} \\
 &\quad + \cos \gamma' \frac{d\theta'}{d\mu} \cdot \frac{d\mu}{d\Delta''},
 \end{aligned} \tag{15}$$

and by means of the results thus derived, we form the equation

$$\cos \gamma' \Delta \theta' = \cos \gamma' \frac{d\theta'}{d\Delta} \Delta \Delta + \cos \gamma' \frac{d\theta'}{d\Delta''} \Delta \Delta''. \tag{16}$$

A fourth observed place will furnish, in the same manner, the additional equation required for finding $\Delta \Delta$ and $\Delta \Delta''$. If more than two

observations are used in addition to the fundamental places on which the assumed elements as derived from Δ and Δ'' are based, the several longitudes will furnish each an equation of condition, and the most probable values of $\Delta\Delta$ and $\Delta\Delta''$ will be obtained by combining the entire group of equations of condition according to the method of least squares.

107. In the actual application of these formulæ to the correction of the approximate elements, after all the preliminary corrections have been applied to the data, we select the proper observed places for determining the elements from the corresponding assumed distances Δ and Δ'' , according to the conditions which have already been stated, and from these we derive the six elements of the orbit. Since the data furnished directly by observation are the right ascensions and the declinations of the body, the elements will be derived in reference to the equator as the plane to which the inclination and the longitude of the ascending node belong. These elements will exactly represent the two fundamental places, and, if the assumed distances Δ and Δ'' are not much in error, they will also very nearly satisfy the remaining places.

We now adopt as the fundamental plane the plane of the approximate orbit thus determined, and by means of the equations (83)₂ and (85)₂, or by means of (87)₂, writing α , δ , Ω' , and i' in place of λ , β , Ω , and i , respectively, we compute the values of θ , η , and γ for the dates of the several places to be employed. Then the residuals for each of the observed places are found from the formulæ

$$\begin{aligned}\cos \eta \Delta \theta &= \sin \gamma \Delta \delta + \cos \gamma \cos \delta \Delta \alpha, \\ \Delta \eta &= \cos \gamma \Delta \delta - \sin \gamma \cos \delta \Delta \alpha,\end{aligned}\tag{17}$$

the values of $\Delta\alpha$ and $\Delta\delta$ for each place being found by subtracting from the observed right ascension and declination, respectively, the right ascension and declination computed by means of the elements derived from Δ and Δ'' . The values of θ , η , and γ being required only for finding $\cos \eta \Delta \theta$, $\Delta \eta$, and the differential coefficients of θ and η , with respect to the elements of the orbit, need not be determined with great accuracy.

Next, we compute $\frac{dr}{d\Delta}$ and $\frac{d(v + \chi')}{d\Delta}$ from equations (12), and from (16)₂ the values of $\frac{dr}{d\varphi}$, $\frac{dr''}{d\varphi}$, $\frac{dv}{d\varphi}$, $\frac{dv''}{d\varphi}$, $\frac{dr}{dM_0}$, &c., by means of which, using the value of u in reference to the equator, we form the equations (13). The accent is added to χ to indicate that it refers to the

equator as the plane for defining the elements. Thus we obtain four equations, from which, by elimination, the values of the differential coefficients of χ' , φ , M_0 , and μ with respect to Δ may be obtained. In the numerical solution, by subtracting the third equation from the first, the unknown quantity $\frac{d\chi'}{d\Delta}$ is immediately eliminated, so that we have three equations to find the three unknown quantities $\frac{d\varphi}{d\Delta}$, $\frac{dM_0}{d\Delta}$, and $\frac{d\mu}{d\Delta}$. These having been found, $\frac{d\chi'}{d\Delta}$ may be obtained from the first or from the third equation.

In the same manner we form the equations (14), and thence derive the values of $\frac{d\chi'}{d\Delta''}$, $\frac{d\varphi}{d\Delta''}$, $\frac{dM_0}{d\Delta''}$, and $\frac{d\mu}{d\Delta''}$. Then, by means of the formulæ (76)₂, (78)₂, and (79)₂, we compute for the date of each place to be employed in correcting the assumed distances the values of $\cos \gamma' \frac{d\theta'}{d\chi'}$, $\cos \gamma' \frac{d\theta'}{d\varphi}$, &c., and hence from (15) the values of $\cos \gamma' \frac{d\theta'}{d\Delta}$ and $\cos \gamma' \frac{d\theta'}{d\Delta''}$. The results thus obtained, together with the residuals computed by means of the equations (17), enable us to form, according to (16), the equations of condition for finding the values of the corrections $\Delta\Delta$ and $\Delta\Delta''$. The solution of all the equations thus formed, according to the method of least squares, will give the most probable values of these quantities, and the system of elements which corresponds to the distances thus corrected will very nearly satisfy the entire series of observations. Since the values of $\cos \gamma' \Delta\theta'$ are expressed in seconds of arc, the resulting values of $\Delta\Delta$ and $\Delta\Delta''$ will also be expressed in seconds of arc in a circle whose radius is equal to the mean distance of the earth from the sun. To express them in parts of the unit of space, we must divide their values in seconds of arc by 206264.8.

The corrections to be applied to the elements computed from Δ and Δ'' , in order to satisfy the corrected values $\Delta + \Delta\Delta$ and $\Delta'' + \Delta\Delta''$, may be computed by means of the partial differential coefficients already derived. Thus, in the case of χ' , we have

$$\Delta\chi' = \frac{d\chi'}{d\Delta} \Delta\Delta + \frac{d\chi'}{d\Delta''} \Delta\Delta'',$$

from which to find $\Delta\chi'$; and in a similar manner $\Delta\varphi$, ΔM_0 , and $\Delta\mu$ may be obtained. If, from the values of $\frac{d(v + \chi')}{d\Delta}$ and $\frac{d(v'' + \chi')}{d\Delta''}$ we compute

$$\Delta v = \frac{d(v + \chi')}{d\Delta} \Delta\Delta - \Delta\chi',$$

$$\Delta v'' = \frac{d(v'' + \chi')}{d\Delta''} \Delta\Delta'' - \Delta\chi',$$

and apply these corrections to the values of v and v'' found from Δ and Δ'' , we obtain the true anomalies corresponding to the distances $\Delta + \Delta\Delta$ and $\Delta'' + \Delta\Delta''$. The corrections to be applied to the values of r and r'' derived from Δ and Δ'' are given by

$$\Delta r = \frac{dr}{d\Delta} \Delta\Delta, \quad \Delta r'' = \frac{dr''}{d\Delta''} \Delta\Delta''.$$

If $\Delta\Delta$ and $\Delta\Delta''$ are expressed in seconds of arc, the corresponding values of Δr and $\Delta r''$ must be divided by 206264.8. The corrected results thus obtained should agree with the values of r and r'' computed directly from the corrected values of v , v'' , p , and e by means of the polar equation of the conic section. Finally, we have

$$dz = \sin \eta d\Delta,$$

and similarly for dz'' ; and the last of equations (73)₂ gives

$$\begin{aligned} r \sin u \Delta i' - r \cos u \sin i' \Delta \Omega' &= \sin \eta \Delta\Delta, \\ r'' \sin u'' \Delta i' - r'' \cos u'' \sin i' \Delta \Omega' &= \sin \eta'' \Delta\Delta'', \end{aligned} \quad (18)$$

from which to find $\Delta i'$ and $\Delta \Omega'$, u and u'' being the arguments of the latitude in reference to the equator. We have also, according to (72)₂,

$$\begin{aligned} \Delta \omega' &= \Delta \chi' - \cos i' \Delta \Omega', \\ \Delta \pi' &= \Delta \chi' + 2 \sin^2 \frac{1}{2} i' \Delta \Omega', \end{aligned}$$

from which to find the corrections to be applied to ω' and π' . The elements which refer to the equator may then be converted into those for the ecliptic by means of the formulæ which may be derived from (109)₁ by interchanging Ω and Ω' and $180^\circ - i'$ and i .

The final residuals of the longitudes may be obtained by substituting the adopted values of $\Delta\Delta$ and $\Delta\Delta''$ in the several equations of condition, or, which affords a complete proof of the accuracy of the entire calculation, by direct calculation from the corrected elements; and the determination of the remaining errors in the values of η will show how nearly the position of the plane of the orbit corresponding to the corrected distances satisfies the intermediate latitudes.

Instead of φ , M_0 , and μ , we may introduce any other elements which determine the form and magnitude of the orbit, the necessary

changes being made in the formulæ. Thus, if we use the elements T , q , and e , these must be written in place of M_0 , μ , and φ , respectively, in the equations (13), (14), and (15), and the partial differential coefficients of r , r'' , v , and v'' with respect to these elements must be computed by means of the various differential formulæ which have already been investigated. Further, in all these cases, the homogeneity of the formulæ must be carefully attended to.

108. The approximate elements of the orbit of a heavenly body may also be corrected by varying the elements which fix the position of the plane of the orbit. Thus, if the observed longitude and latitude and the values of Ω and i are given, the three equations (91)₁ will contain only three unknown quantities, namely, Δ , r , and u , and the values of these may be found by elimination. When the observed latitude β is corrected by means of the formula (6)₄, the latitudes of the sun disappear from these equations, and if we multiply the first by $\sin(\odot - \Omega) \sin \beta$, the second (using only the upper sign) by $-\cos(\odot - \Omega) \sin \beta$, and the third by $-\sin(\lambda - \odot) \cos \beta$, and add the products, we get

$$\tan u = \frac{\sin \beta \sin(\odot - \Omega)}{\cos i \sin \beta \cos(\odot - \Omega) - \sin i \cos \beta \sin(\lambda - \odot)}, \quad (19)$$

from which u may be found. If we multiply the second of these equations by $\sin \beta$, and the third by $-\cos \beta \sin(\lambda - \Omega)$, and add the products, we find

$$r = \frac{R \sin(\odot - \Omega)}{\sin u (\sin i \cot \beta \sin(\lambda - \Omega) - \cos i)}. \quad (20)$$

The expression for r in terms of the known quantities may also be found by combining the first and second, or by combining the first and third, of equations (91)₁. If we put

$$\begin{aligned} n \cos N &= \sin \beta \cos(\odot - \Omega), \\ n \sin N &= \cos \beta \sin(\lambda - \odot). \end{aligned}$$

the formula for u becomes

$$\tan u = \frac{\cos N}{\cos(N+i)} \tan(\odot - \Omega). \quad (21)$$

The last of equations (91)₁ shows that $\sin u$ and $\sin \beta$ must have the same sign, and thus the quadrant in which u must be taken is determined. Putting, also,

$$\begin{aligned} m \cos M &= \sin u, \\ m \sin M &= \sin u \cot \beta \sin(\lambda - \Omega), \end{aligned}$$

we have

$$r = -\frac{\cos M}{\cos(M+i)} \cdot \frac{R \sin(\odot - \Omega)}{\sin u}. \quad (22)$$

When any other plane is taken as the fundamental plane, the latitude of the sun (which will then refer to this plane) will be retained in the equations (91)₁ and in the resulting expressions for u and r .

The value of u may also be obtained by first computing w and ψ by means of the equations (42)₃, and then, if z denotes the angle at the planet or comet between the earth and sun, the values of u and z , as may be readily seen, will be determined by means of the relations of the parts of a spherical triangle of which the sides are $180^\circ - (z + \psi)$, $180^\circ + \odot - \Omega$, and u , the angle opposite to the side u being that which we designate by w , and the side $180^\circ + \odot - \Omega$ being included by this and the inclination i . Let $S = 180^\circ - (z + \psi)$, and, according to Napier's analogies, this spherical triangle gives

$$\begin{aligned} \tan \frac{1}{2}(S+u) &= \frac{\cos \frac{1}{2}(i-w)}{\cos \frac{1}{2}(i+w)} \cot \frac{1}{2}(\Omega - \odot), \\ \tan \frac{1}{2}(S-u) &= \frac{\sin \frac{1}{2}(i-w)}{\sin \frac{1}{2}(i+w)} \cot \frac{1}{2}(\Omega - \odot), \end{aligned} \quad (23)$$

from which S and u are readily found. Then we have

$$\begin{aligned} z &= 180^\circ - \psi - S, \\ r &= \frac{R \sin \psi}{\sin z}, \end{aligned} \quad (24)$$

to find r .

If we assume approximate values of Ω and i , as given by a system of elements already known, the equations here given enable us to find r , u , r'' , and u'' from λ , β and λ'' , β'' , corresponding to the dates t and t'' of the fundamental places selected, and from these results for two radii-vectores and arguments of the latitude, the remaining elements may be derived. From these the geocentric place of the body may be found for the date t' of any intermediate or additional observed place, and the difference between the computed and the observed place will indicate the degree of precision of the assumed values of Ω and i . Then we assign to Ω the increment $\delta\Omega$, i remaining unchanged, and compute a second system of elements, and from these the geocentric place for the time t' . We also compute a third system from Ω and $i + \delta i$, and by a process entirely analogous to that already indicated in the case of the variation of two geocentric

distances, we obtain the numerical values of the differential coefficients of λ' and β' with respect to Ω and i . Thus the equations

$$\begin{aligned}\cos \beta' \Delta \lambda' &= \cos \beta' \frac{d\lambda'}{d\Omega} \Delta \Omega + \cos \beta' \frac{d\lambda'}{di} \Delta i, \\ \Delta \beta' &= \frac{d\beta'}{d\Omega} \Delta \Omega + \frac{d\beta'}{di} \Delta i,\end{aligned}\tag{25}$$

for finding the corrections $\Delta \Omega$ and Δi to be applied to the assumed values of these elements, will be formed; and each additional observation or normal place will furnish two equations of condition for the determination of these corrections.

If the observed right ascensions and declinations are used directly instead of the longitudes and latitudes, the elements Ω and i must be referred to the equator as the fundamental plane, and the declinations of the sun will appear in the formulæ for u and r obtained from the equations (91)₁, thus rendering them more complex. Their derivation offers no difficulty, being similar in all respects to that of the equations (19) and (20), and since they will be rarely, if ever, required, it is not necessary to give the process here in detail. In general, the equations (23) and (24) will be most convenient for finding r and u from the geocentric spherical co-ordinates and the elements Ω and i , since w , ψ , w'' , and ψ'' remain unchanged for the three hypotheses.

When the equator is taken as the fundamental plane, ψ is the distance between two points on the celestial sphere for which the geocentric spherical co-ordinates are A , D and α , δ , those of the sun being denoted by A and D . Hence we shall have

$$\begin{aligned}\sin \psi \sin B &= \cos \delta \sin (\alpha - A), \\ \sin \psi \cos B &= \cos D \sin \delta - \sin D \cos \delta \cos (\alpha - A), \\ \cos \psi &= \sin D \sin \delta + \cos D \cos \delta \cos (\alpha - A),\end{aligned}\tag{26}$$

from which to find ψ and B , the angle opposite to the side $90^\circ - \delta$ of the spherical triangle being denoted by B . Let K denote the right ascension of the ascending node on the equator of a great circle passing through the places of the sun and comet or planet for the time t , and let w_0 denote its inclination to the equator; then we shall have

$$\begin{aligned}\sin w_0 \cos (A - K) &= \cos B, \\ \sin w_0 \sin (A - K) &= \sin B \sin D, \\ \cos w_0 &= \sin B \cos D,\end{aligned}\tag{27}$$

from which to find w_0 and K . In a similar manner, we may com-

pute the values of $u'' - u$, Ω , and i from the heliocentric spherical co-ordinates l , b and l'' , b'' .

From the equations

$$\begin{aligned}\tan \frac{1}{2} (S_0 + u) &= \frac{\cos \frac{1}{2} (i' - w_0)}{\cos \frac{1}{2} (i'' + w_0)} \cot \frac{1}{2} (\Omega' - K), \\ \tan \frac{1}{2} (S_0 - u) &= \frac{\sin \frac{1}{2} (i' - w_0)}{\sin \frac{1}{2} (i'' + w_0)} \cot \frac{1}{2} (\Omega' - K),\end{aligned}\quad (28)$$

the accents being added to distinguish the elements in reference to the equator from those with respect to the ecliptic, the values of S_0 and u (in reference to the equator) may be found. Let s_0 denote the angular distance between the place of the sun and that point of the equator for which the right ascension is K , and the equation

$$\cot s_0 = \cos w_0 \cot (K - A) \quad (29)$$

gives the value of s_0 , the quadrant in which it is situated being determined by the condition that $\cos s_0$ and $\cos (K - A)$ shall have the same sign. Then we have $S = S_0 - s_0$, and

$$\begin{aligned}z &= 180^\circ - \psi - S_0 + s_0, \\ r &= \frac{R \sin \psi}{\sin z},\end{aligned}\quad (30)$$

from which to find r .

109. In both the method of the variation of two geocentric distances and that of the variation of Ω and i , instead of using the geocentric spherical co-ordinates given by an intermediate observation, in forming the equations for the corrections to be applied to the assumed quantities, we may use any other two quantities which may be readily found from the data furnished by observation. Thus, if we compute r' and u' for the date of a third observation directly from each of the three systems of elements, the differences between the successive results will furnish the numerical values of the partial differential coefficients of r' and u' with respect to Δ and Δ'' , or with respect to Ω and i , as the case may be. Then, computing the values of r' and u' from the observed geocentric spherical co-ordinates by means of the values of Ω and i for the system of elements to be corrected, the differences between the results thus derived and those obtained directly from the elements enable us to form the equations

$$\begin{aligned}\frac{du'}{d\Delta} \Delta\Delta + \frac{du'}{d\Delta''} \Delta\Delta'' &= \Delta u', \\ \frac{dr'}{d\Delta} \Delta\Delta + \frac{dr'}{d\Delta''} \Delta\Delta'' &= \Delta r',\end{aligned}\quad (31)$$

or the corresponding expressions in the case of the variation of Ω and i , by means of which the corrections to be applied to the assumed values will be determined. In the numerical application of these equations, $\Delta u'$ being expressed in seconds of arc, $\Delta r'$ should also be expressed in seconds, and the resulting values of $\Delta \Delta$ and $\Delta \Delta''$ will be converted into those expressed in parts of the unit of space by dividing them by 206264.8.

When only three observed places are to be used for correcting an approximate orbit, from the values of r, r', r'' and u, u', u'' obtained by means of the formulæ which have been given, we may find p and a or $\frac{1}{a}$ —the latter in the case of very eccentric orbits—from the first and second places, and also from the first and third places. If these results agree, the elements do not require any correction; but if a difference is found to exist, by computing the differences, in the case of each of these two elements, for three hypotheses in regard to Δ and Δ'' or in regard to Ω and i , the equations may be formed by means of which the corrections to be applied to the assumed values of the two geocentric distances, or to those of Ω and i , will be obtained.

110. The formulæ which have thus far been given for the correction of an approximate orbit by varying the geocentric distances, depend on two of these distances when no assumption is made in regard to the form of the orbit, and these formulæ apply with equal facility whether three or more than three observed places are used. But when a series of places can be made available, the problem may be successfully treated in a manner such that it will only be necessary to vary one geocentric distance. Thus, let x, y, z be the rectangular heliocentric co-ordinates, and r the radius-vector of the body at the time t , and let X, Y, Z be the geocentric co-ordinates of the sun at the same instant. Let the geocentric co-ordinates of the body be designated by x_0, y_0, z_0 , and let the plane of the equator be taken as the fundamental plane, the positive axis of x being directed to the vernal equinox. Further, let ρ denote the projection of the radius-vector of the body on the plane of the equator, or the curtate distance with respect to the equator; then we shall have

$$x_0 = \rho \cos \alpha, \quad y_0 = \rho \sin \alpha, \quad z_0 = \rho \tan \delta. \quad (32)$$

If we represent the right ascension of the sun by A , and its declination by D , we also have

$$X = R \cos D \cos A, \quad Y = R \cos D \sin A, \quad Z = R \sin D. \quad (33)$$

The fundamental equations for the undisturbed motion of the planet or comet, neglecting its mass in comparison with that of the sun, are

$$\frac{d^2x}{dt^2} + \frac{k^2x}{r^3} = 0, \quad \frac{d^2y}{dt^2} + \frac{k^2y}{r^3} = 0, \quad \frac{d^2z}{dt^2} + \frac{k^2z}{r^3} = 0;$$

but since

$$x = x_0 - X, \quad y = y_0 - Y, \quad z = z_0 - Z,$$

and, neglecting also the mass of the earth,

$$\frac{d^2X}{dt^2} + \frac{k^2X}{R^3} = 0, \quad \frac{d^2Y}{dt^2} + \frac{k^2Y}{R^3} = 0, \quad \frac{d^2Z}{dt^2} + \frac{k^2Z}{R^3} = 0,$$

these become

$$\begin{aligned} \frac{d^2x_0}{dt^2} + \frac{k^2x_0}{r^3} + k^2X \left(\frac{1}{R^3} - \frac{1}{r^3} \right) &= 0, \\ \frac{d^2y_0}{dt^2} + \frac{k^2y_0}{r^3} + k^2Y \left(\frac{1}{R^3} - \frac{1}{r^3} \right) &= 0, \\ \frac{d^2z_0}{dt^2} + \frac{k^2z_0}{r^3} + k^2Z \left(\frac{1}{R^3} - \frac{1}{r^3} \right) &= 0. \end{aligned} \quad (34)$$

Substituting for x_0 , y_0 , and z_0 their values in terms of α and δ , and putting

$$k^2X \left(\frac{1}{R^3} - \frac{1}{r^3} \right) = \xi, \quad k^2Y \left(\frac{1}{R^3} - \frac{1}{r^3} \right) = \eta, \quad k^2Z \left(\frac{1}{R^3} - \frac{1}{r^3} \right) = \zeta, \quad (35)$$

we get

$$\begin{aligned} \frac{d^2x_0}{dt^2} + \frac{k^2\rho}{r^3} \cos \alpha + \xi &= 0, \\ \frac{d^2y_0}{dt^2} + \frac{k^2\rho}{r^3} \sin \alpha + \eta &= 0, \\ \frac{d^2z_0}{dt^2} + \frac{k^2\rho}{r^3} \tan \delta + \zeta &= 0. \end{aligned} \quad (36)$$

Differentiating the equations (32) with respect to t , we find

$$\begin{aligned} \frac{dx_0}{dt} &= \cos \alpha \frac{d\rho}{dt} - \rho \sin \alpha \frac{d\alpha}{dt}, \\ \frac{dy_0}{dt} &= \sin \alpha \frac{d\rho}{dt} + \rho \cos \alpha \frac{d\alpha}{dt}, \\ \frac{dz_0}{dt} &= \tan \delta \frac{d\rho}{dt} + \rho \sec^2 \delta \frac{d\delta}{dt}. \end{aligned} \quad (37)$$

Differentiating again with respect to t , and substituting in the equations (36) the values thus found, the results are

$$\begin{aligned} \left(\frac{k^2 \rho}{r^3} + \frac{d^2 \rho}{dt^2} - \rho \frac{d^2 \alpha}{dt^2} \right) \cos \alpha - \left(\rho \frac{d^2 \alpha}{dt^2} + 2 \frac{d\rho}{dt} \cdot \frac{d\alpha}{dt} \right) \sin \alpha + \xi &= 0, \\ \left(\frac{k^2 \rho}{r^3} + \frac{d^2 \rho}{dt^2} - \rho \frac{d^2 \alpha}{dt^2} \right) \sin \alpha + \left(\rho \frac{d^2 \alpha}{dt^2} + 2 \frac{d\rho}{dt} \cdot \frac{d\alpha}{dt} \right) \cos \alpha + \eta &= 0, \\ \left(\frac{k^2 \rho}{r^3} + \frac{d^2 \rho}{dt^2} \right) \tan \delta + 2 \sec^2 \delta \frac{d\delta}{dt} \cdot \frac{d\rho}{dt} + 2\rho \sec^2 \delta \tan \delta \frac{d\delta^2}{dt^2} + \rho \sec^2 \delta \frac{d^2 \delta}{dt^2} + \zeta &= 0. \end{aligned} \quad (38)$$

If we multiply the first of these equations by $\sin \alpha$, and the second by $-\cos \alpha$, and add the products, we obtain

$$\frac{d\rho}{dt} = \frac{1}{2} \frac{\xi \sin \alpha - \eta \cos \alpha - \rho \frac{d^2 \alpha}{dt^2}}{\frac{d\alpha}{dt}}.$$

Now, from (35) we get

$$\xi \sin \alpha - \eta \cos \alpha = k^2 \left(\frac{1}{R^3} - \frac{1}{r^3} \right) R \cos D \sin (\alpha - A),$$

and the preceding equation becomes

$$\frac{d\rho}{dt} = \frac{1}{2} \frac{k^2 \left(\frac{1}{R^3} - \frac{1}{r^3} \right) R \cos D \cos (\alpha - A) - \rho \frac{d^2 \alpha}{dt^2}}{\frac{d\alpha}{dt}}. \quad (39)$$

The value of $\frac{d\rho}{dt}$ thus found is independent of the differential coefficients of δ with respect to t . To find another value of $\frac{d\rho}{dt}$, using all three of equations (38), we multiply the first of these equations by $\sin A \tan \delta$, the second by $-\cos A \tan \delta$, and the third by $-\sin (\alpha - A)$. Then, adding the products, since $\xi \sin A = \eta \cos A$, the result is

$$\begin{aligned} 2 \frac{d\rho}{dt} \left(\cot (\alpha - A) \frac{d\alpha}{dt} - \cot \delta \sec^2 \delta \frac{d\delta}{dt} \right) = \\ \rho \left(\frac{d\alpha^2}{dt^2} - \cot (\alpha - A) \frac{d^2 \alpha}{dt^2} + \sec^2 \delta \left(2 \frac{d\delta^2}{dt^2} + \cot \delta \frac{d^2 \delta}{dt^2} \right) \right) + \zeta \cot \delta \end{aligned}$$

from which we get

$$\frac{d\rho}{dt} = \frac{1}{2} \rho \frac{\frac{d\alpha^2}{dt^2} - \cot (\alpha - A) \frac{d^2 \alpha}{dt^2} + \sec^2 \delta \left(2 \frac{d\delta^2}{dt^2} + \cot \delta \frac{d^2 \delta}{dt^2} \right) + \frac{\zeta}{\rho} \cot \delta}{\cot (\alpha - A) \frac{d\alpha}{dt} - \cot \delta \sec^2 \delta \frac{d\delta}{dt}}. \quad (40)$$

When the ecliptic is taken as the fundamental plane, the last term of the numerator of the second member of this equation vanishes, and the equation may be written

$$\frac{d\rho}{dt} = C\rho, \quad (41)$$

the coefficient C being independent of ρ .

111. When the value of ρ is given, that of $\frac{d\rho}{dt}$ will be determined in terms of the data furnished directly by observation and of the differential coefficients of α and δ with respect to t from equation (39), or from (40), the latter being preferred when the motion of the body in right ascension is very slow. The value of $\frac{d\rho}{dt}$ having been found, we may compute the velocities of the body in directions parallel to the co-ordinate axes. Thus, since

$$x_0 = x + X, \quad y_0 = y + Y, \quad z_0 = z + Z,$$

the equations (37) give

$$\begin{aligned} \frac{dx}{dt} &= \cos \alpha \frac{d\rho}{dt} - \rho \sin \alpha \frac{d\alpha}{dt} - \frac{dX}{dt}, \\ \frac{dy}{dt} &= \sin \alpha \frac{d\rho}{dt} + \rho \cos \alpha \frac{d\alpha}{dt} - \frac{dY}{dt}, \\ \frac{dz}{dt} &= \tan \delta \frac{d\rho}{dt} + \rho \sec^2 \delta \frac{d\delta}{dt} - \frac{dZ}{dt}, \end{aligned} \quad (42)$$

by means of which $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ may be determined.

To find the values of $\frac{dX}{dt}$, $\frac{dY}{dt}$, and $\frac{dZ}{dt}$, the equations

$$\begin{aligned} X &= R \cos \odot, \\ Y &= R \sin \odot \cos \varepsilon, \\ Z &= R \sin \odot \sin \varepsilon, \end{aligned}$$

give, by differentiation,

$$\begin{aligned} \frac{dX}{dt} &= \cos \odot \frac{dR}{dt} - R \sin \odot \frac{d\odot}{dt}, \\ \frac{dY}{dt} &= \sin \odot \cos \varepsilon \frac{dR}{dt} + R \cos \odot \cos \varepsilon \frac{d\odot}{dt}, \\ \frac{dZ}{dt} &= \sin \odot \sin \varepsilon \frac{dR}{dt} + R \cos \odot \sin \varepsilon \frac{d\odot}{dt}. \end{aligned} \quad (43)$$

Now, according to equation (52)₁, we have

$$\frac{d\odot}{dt} = \frac{k\sqrt{(1-e_0^2)(1+m_0)}}{R^2}, \quad (44)$$

m_0 denoting the mass of the earth, and e_0 the eccentricity of its orbit. The polar equation of the conic section gives

$$\frac{dr}{dt} = \frac{r^2 e \sin v}{p} \cdot \frac{dv}{dt}.$$

Let Γ denote the longitude of the sun's perigee, and this equation gives

$$\frac{dR}{dt} = \frac{R^2 e_0 \sin(\odot - \Gamma)}{1 - e_0^2} \cdot \frac{d\odot}{dt} = \frac{k\sqrt{1+m_0}}{\sqrt{1-e_0^2}} e_0 \sin(\odot - \Gamma). \quad (45)$$

If we neglect the square of the eccentricity of the earth's orbit, we have simply

$$\frac{d\odot}{dt} = \frac{k\sqrt{1+m_0}}{R^2}, \quad \frac{dR}{dt} = k\sqrt{1+m_0} e_0 \sin(\odot - \Gamma). \quad (46)$$

The values of $\frac{d\odot}{dt}$ and $\frac{dR}{dt}$ having been found by means of these formulæ, the equations (43) give the required results for $\frac{dX}{dt}$, $\frac{dY}{dt}$, and $\frac{dZ}{dt}$, and hence, by means of (42), we obtain the velocities of the comet or planet in directions parallel to the co-ordinate axes.

112. The values of x , y , and z may be derived by means of the equations

$$\begin{aligned} x &= \Delta \cos \delta \cos \alpha - X, \\ y &= \Delta \cos \delta \sin \alpha - Y, \\ z &= \Delta \sin \delta - Z, \end{aligned}$$

and from these, in connection with the corresponding velocities, the elements of the orbit may be found. The equations (32)₁ give immediately the values of the inclination, the semi-parameter, and the right ascension of the ascending node on the equator. Then, the position of the plane of the orbit being known, we may compute r and u directly from the geocentric right ascension and declination by means of the equations (28) and (30). But if we use the values of the heliocentric co-ordinates directly, multiplying the first of equations (93)₁ by $\cos \Omega$, and the second by $\sin \Omega$, and adding the products, we have

$$\begin{aligned} r \sin u &= z \operatorname{cosec} i, \\ r \cos u &= x \cos \Omega + y \sin \Omega, \end{aligned} \quad (47)$$

from which r and u may be found, the argument of the latitude u being referred to the plane of xy as the fundamental plane. The equation

$$r^2 = x^2 + y^2 + z^2$$

gives

$$\frac{dr}{dt} = \frac{x}{r} \cdot \frac{dx}{dt} + \frac{y}{r} \cdot \frac{dy}{dt} + \frac{z}{r} \cdot \frac{dz}{dt}, \quad (48)$$

and, since

$$\frac{dr}{dt} = \frac{r^2 e \sin v}{p} \cdot \frac{dv}{dt}, \quad \frac{dv}{dt} = \frac{k \sqrt{p}}{r^2},$$

we shall have

$$\begin{aligned} e \sin v &= \frac{\sqrt{p}}{k} \cdot \frac{dr}{dt}, \\ e \cos v &= \frac{p}{r} - 1, \end{aligned} \quad (49)$$

from which to find e and v . Then the distance between the perihelion and the ascending node is given by

$$\omega = u - v.$$

The semi-transverse axis is obtained from p and e by means of the relation

$$a = \frac{p}{1 - e^2}.$$

Finally, from the value of v the eccentric anomaly and thence the mean anomaly may be found, and the latter may then be referred to any epoch by means of the mean motion determined from a .

In the case of very eccentric orbits, the perihelion distance will be given by

$$q = \frac{p}{1 + e};$$

and the time of perihelion passage may be found from v and e by means of Table IX. or Table X., as already illustrated.

The equation (21)₁ gives, if we substitute for f its value in terms of p , denote by V the linear velocity of the planet or comet, and neglect the mass,

$$V^2 r^3 - r^2 \frac{dr^2}{dt^2} = k^2 p.$$

Let ψ_0 denote the angle which the tangent to the orbit at the extremity of the radius-vector makes with the prolongation of this radius-vector, and we shall have

$$r V \cos \psi_0 = r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt},$$

so that the preceding equation gives

$$k^2 p = V^2 r^2 \sin^2 \psi_0.$$

Hence we derive the equations

$$\begin{aligned} Vr \sin \psi_0 &= k \sqrt{p}, \\ Vr \cos \psi_0 &= x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt}, \end{aligned} \quad (50)$$

from which Vr and ψ_0 may be found. Then, since

$$V^2 = k^2 \left(\frac{2}{r} - \frac{1}{a} \right),$$

we shall have

$$\frac{k^2}{a} = \frac{2k^2}{r} - V^2, \quad (51)$$

by means of which a may be determined, and then e may be found by means of this and the value of p .

The equations (49) and (50) give

$$\begin{aligned} e \sin(u - \omega) &= \frac{V^2}{k^2} r \sin \psi_0 \cos \psi_0, \\ e \cos(u - \omega) &= \frac{V^2}{k^2} r \sin^2 \psi_0 - 1, \end{aligned}$$

and, since

$$\frac{V^2}{k^2} = \frac{2}{r} - \frac{1}{a},$$

these are easily transformed into

$$\begin{aligned} 2ae \sin(u - \omega) &= (2a - r) \sin 2\psi_0, \\ 2ae \cos(u - \omega) &= -(2a - r) \cos 2\psi_0 - r. \end{aligned}$$

If we multiply the first of these equations by $-\cos u$ and the second by $\sin u$, and add the products; then multiply the first by $\sin u$ and the second by $\cos u$, and add, we obtain

$$\begin{aligned} 2ae \sin \omega &= -(2a - r) \sin(2\psi_0 + u) - r \sin u, \\ 2ae \cos \omega &= -(2a - r) \cos(2\psi_0 + u) - r \cos u, \end{aligned} \quad (52)$$

These equations give the values of ω and e .

113. We have thus derived all the formulæ necessary for finding the elements of the orbit of a heavenly body from one geocentric distance, provided that the first and second differential coefficients of α and δ with respect to the time are accurately known. It remains,

therefore, to devise the means by which these differential coefficients may be determined with accuracy from the data furnished by observation. The approximate elements derived from three or from a small number of observations will enable us to correct the entire series of observations for parallax and aberration, and to form the normal places which shall represent the series of observed places. We may now assume that the deviation of the spherical co-ordinates computed by means of the approximate elements from those which would be obtained if the true elements were used, may be exactly represented by the formula

$$\Delta\theta = A + Bh + Ch^2, \quad (53)$$

h denoting the interval between the time at which the deviation is expressed by A and the time for which this difference is $\Delta\theta$. The differences between the normal places and those computed with the approximate elements to be corrected, will then suffice to form equations of condition by means of which the values of the coefficients A , B , and C may be determined. The epoch for which $h = 0$ may be chosen arbitrarily, but it will generally be advantageous to fix it at or near the date of the middle observed place. If three observed places are given, the difference between the observed and the computed value of each right ascension will give an equation of condition, according to (53), and the three equations thus formed will furnish the numerical values of A , B , and C . These having been determined, the equation (53) will give the correction to be applied to the computed right ascension for any date within the limits of the extreme observations of the series. When more than three normal places are determined, the resulting equations of condition may be reduced by the method of least squares to three final equations, from which, by elimination, the most probable values of A , B , and C will be derived. In like manner, the corrections to be applied to the computed latitudes may be determined. These corrections being applied, the ephemeris thus obtained may be assumed to represent the apparent path of the body with great precision, and may be employed as an auxiliary in determining the values of the differential coefficients of α and δ with respect to t .

Let $f(a)$ denote the right ascension of the body at the middle epoch or that for which $h = 0$, and let $f(a \pm n\omega)$ denote the value of α for any other date separated by the interval $n\omega$, in which ω is the interval between the successive dates of the ephemeris. Then, if we put n successively equal to 1, 2, 3, &c., we shall have

| Function. | I. Diff. | II. Diff. | III. Diff. | IV. Diff. | V. Diff. |
|------------------|-----------------------------|--------------------|-------------------------------|-----------------------|-------------------------------|
| $f(a - 3\omega)$ | $f'(a - \frac{5}{2}\omega)$ | $f''(a - 2\omega)$ | $f'''(a - \frac{3}{2}\omega)$ | $f^{iv}(a - \omega)$ | $f^v(a - \frac{1}{2}\omega)$ |
| $f(a - 2\omega)$ | $f'(a - \frac{3}{2}\omega)$ | $f''(a - \omega)$ | $f'''(a - \frac{1}{2}\omega)$ | $f^{iv}(a)$ | $f^v(a + \frac{1}{2}\omega)$ |
| $f(a - \omega)$ | $f'(a - \frac{1}{2}\omega)$ | $f''(a)$ | $f'''(a + \frac{1}{2}\omega)$ | $f^{iv}(a + \omega)$ | $f^v(a + \frac{3}{2}\omega)$ |
| $f(a)$ | $f'(a + \frac{1}{2}\omega)$ | $f''(a + \omega)$ | $f'''(a + \frac{3}{2}\omega)$ | $f^{iv}(a + 2\omega)$ | $f^v(a + \frac{5}{2}\omega)$ |
| $f(a + \omega)$ | $f'(a + \frac{3}{2}\omega)$ | $f''(a + 2\omega)$ | $f'''(a + \frac{5}{2}\omega)$ | $f^{iv}(a + 3\omega)$ | $f^v(a + \frac{7}{2}\omega)$ |
| $f(a + 2\omega)$ | $f'(a + \frac{5}{2}\omega)$ | $f''(a + 3\omega)$ | $f'''(a + \frac{7}{2}\omega)$ | $f^{iv}(a + 4\omega)$ | $f^v(a + \frac{9}{2}\omega)$ |
| $f(a + 3\omega)$ | $f'(a + \frac{7}{2}\omega)$ | $f''(a + 4\omega)$ | $f'''(a + \frac{9}{2}\omega)$ | $f^{iv}(a + 5\omega)$ | $f^v(a + \frac{11}{2}\omega)$ |

The series of functions and differences may be extended in the same manner in either direction. If we expand $f(a + n\omega)$ into a series, the result is

$$f(a + n\omega) = a + \frac{d\mathbf{a}}{dt} n\omega + \frac{1}{2} \frac{d^2\mathbf{a}}{dt^2} n^2\omega^2 + \frac{1}{6} \frac{d^3\mathbf{a}}{dt^3} n^3\omega^3 + \frac{1}{24} \frac{d^4\mathbf{a}}{dt^4} n^4\omega^4 + \&c.,$$

or, putting for brevity $A = \frac{d\mathbf{a}}{dt} \omega$, $B = \frac{1}{2} \frac{d^2\mathbf{a}}{dt^2} \omega^2$, &c.,

$$f(a + n\omega) = a + An + Bn^2 + Cn^3 + Dn^4 + \&c.$$

If we now put n successively equal to $-4, -3, -2, -1, -0, +1$, &c., we obtain the values of $f(a - 4\omega), f(a - 3\omega), \dots, f(a + 4\omega)$ in terms of A, B, C , &c. Then, taking the successive orders of differences and symbolizing them as indicated above, we obtain a series of equations by means of which A, B, C , &c. will be determined in terms of the successive orders of differences. Finally, replacing A, B, C , &c. by the quantities which they represent, and putting

$$\begin{aligned} \frac{1}{2}f'(a - \frac{1}{2}\omega) + \frac{1}{2}f'(a + \frac{1}{2}\omega) &= f'(a), \\ \frac{1}{2}f'''(a - \frac{1}{2}\omega) + \frac{1}{2}f'''(a + \frac{1}{2}\omega) &= f'''(a), \&c., \end{aligned}$$

we obtain

$$\begin{aligned} \frac{d\mathbf{a}}{dt} &= \frac{1}{\omega} (f'(a) - \frac{1}{6}f'''(a) + \frac{1}{30}f^v(a) - \frac{1}{140}f^{vii}(a) + \&c.), \\ \frac{d^2\mathbf{a}}{dt^2} &= \frac{1}{\omega^2} (f''(a) - \frac{1}{12}f^{iv}(a) + \frac{1}{90}f^{vi}(a) - \frac{1}{560}f^{viii}(a) + \&c.), \\ \frac{d^3\mathbf{a}}{dt^3} &= \frac{1}{\omega^3} (f'''(a) - \frac{1}{4}f^v(a) + \frac{7}{120}f^{vii}(a) - \&c.), \\ \frac{d^4\mathbf{a}}{dt^4} &= \frac{1}{\omega^4} (f^{iv}(a) - \frac{1}{6}f^{vi}(a) + \frac{7}{240}f^{viii}(a) - \&c.), \\ \frac{d^5\mathbf{a}}{dt^5} &= \frac{1}{\omega^5} (f^v(a) - \frac{1}{3}f^{vii}(a) + \&c.), \\ \frac{d^6\mathbf{a}}{dt^6} &= \frac{1}{\omega^6} (f^{vi}(a) - \frac{1}{4}f^{viii}(a) + \&c.), \\ \frac{d^7\mathbf{a}}{dt^7} &= \frac{1}{\omega^7} (f^{vii}(a) - \&c.), \quad \frac{d^8\mathbf{a}}{dt^8} = \frac{1}{\omega^8} (f^{viii}(a) - \&c.), \end{aligned} \tag{54}$$

by means of which the successive differential coefficients of α with respect to t may be determined. The derivation of these coefficients in the case of δ is entirely analogous to the process here indicated for α . Since the successive differences will be expressed in seconds of arc, the resulting values of the differential coefficients of α and δ with respect to t will also be expressed in seconds, and must be divided by 206264.8 in order to express them abstractly.

We may adopt directly the values of $\frac{d\alpha}{dt}$, $\frac{d^2\alpha}{dt^2}$, $\frac{d\delta}{dt}$, and $\frac{d^2\delta}{dt^2}$ determined by means of the corrected ephemeris, or, if the observed places do not include a very long interval, we may determine only the values of $\frac{d^2\alpha}{dt^2}$, $\frac{d^4\alpha}{dt^4}$, &c. by means of the ephemeris, and then find $\frac{d\alpha}{dt}$ and $\frac{d^2\alpha}{dt^2}$ directly from the normal places or observations. Thus, let α , α' , α'' be three observed right ascensions corresponding to the times t , t' , t'' , and we shall have

$$\begin{aligned}\alpha &= \alpha' - \frac{d\alpha'}{dt}(t'-t) + \frac{1}{2}\frac{d^2\alpha'}{dt^2}(t'-t)^2 - \frac{1}{6}\frac{d^3\alpha'}{dt^3}(t'-t)^3 + \frac{1}{24}\frac{d^4\alpha'}{dt^4}(t'-t)^4 - \&c., \\ \alpha'' &= \alpha' + \frac{d\alpha'}{dt}(t''-t') + \frac{1}{2}\frac{d^2\alpha'}{dt^2}(t''-t')^2 + \frac{1}{6}\frac{d^3\alpha'}{dt^3}(t''-t')^3 + \frac{1}{24}\frac{d^4\alpha'}{dt^4}(t''-t')^4 + \&c.,\end{aligned}$$

which give

$$\begin{aligned}\frac{d\alpha'}{dt} - \frac{1}{2}(t'-t)\frac{d^2\alpha'}{dt^2} &= \frac{\alpha' - \alpha}{t' - t} - \frac{1}{6}(t'-t)^2\frac{d^3\alpha'}{dt^3} + \frac{1}{24}(t'-t)^3\frac{d^4\alpha'}{dt^4} - \&c., \\ \frac{d\alpha}{dt} + \frac{1}{2}(t''-t')\frac{d^2\alpha'}{dt^2} &= \frac{\alpha'' - \alpha}{t'' - t'} - \frac{1}{6}(t''-t')^2\frac{d^3\alpha'}{dt^3} - \frac{1}{24}(t''-t')^3\frac{d^4\alpha'}{dt^4} - \&c.\end{aligned}\quad (55)$$

These equations, being solved numerically, will give the values of $\frac{d\alpha}{dt}$ and $\frac{d^2\alpha}{dt^2}$, and we may thus by triple combinations of the observed places, using always the same middle place, form equations of condition for the determination of the most probable values of these differential coefficients by the solution of the equations according to the method of least squares.

In a similar manner the values of $\frac{d\delta}{dt}$ and $\frac{d^2\delta}{dt^2}$ may be derived.

114. In applying these formulæ to the calculation of an orbit, after the normal places have been derived, an ephemeris should be computed at intervals of four or eight days, arranging it so that one of the dates shall correspond to that of the middle observation or normal place. This ephemeris should be computed with the utmost

care, since it is to be employed as an auxiliary in determining quantities on which depends the accuracy of the final results. The comparison of the ephemeris with the observed places will furnish, by means of equations of the form

$$\begin{aligned} A + Bh + Ch^2 &= \Delta\alpha', \\ A' + B'h + C'h^2 &= \Delta\delta', \end{aligned}$$

h being the interval between the middle date t' and that of the place used, the values of A , B , C , A' , &c.; and the corrections to be applied to the ephemeris will be determined by

$$\begin{aligned} A + Bn\omega + Cn^2\omega^2 &= \Delta\alpha, \\ A' + B'n\omega + C'n^2\omega^2 &= \Delta\delta. \end{aligned}$$

The unit of h may be ten days, or any other convenient interval, observing, however, that $n\omega$ in the last equations must be expressed in parts of the same unit. With the ephemeris thus corrected, we compute the values of $\frac{d\alpha}{dt}$, $\frac{d^2\alpha}{dt^2}$, $\frac{d\delta}{dt}$, and $\frac{d^2\delta}{dt^2}$ as already explained. These differential coefficients should be determined with great care, since it is on their accuracy that the subsequent calculation principally depends. We compute, also, the velocities $\frac{dX}{dt}$, $\frac{dY}{dt}$, and $\frac{dZ}{dt}$ by means of the formulæ (43), $\frac{d\odot}{dt}$ and $\frac{dR}{dt}$ being computed from (46). The quantities thus far derived remain unchanged in the two hypotheses with regard to Δ .

Then we assume an approximate value of Δ , and compute

$$\rho = \Delta \cos \delta;$$

and by means of the equation (40) or (39) we compute the value of $\frac{d\rho}{dt}$. It will be observed that if we put the equation (40) in the form

$$\frac{d\rho}{dt} = \frac{P}{Q}\rho + \frac{\zeta}{Q}\cot\delta,$$

the coefficient $\frac{P}{Q}$ remains the same in the two hypotheses. The three equations (38) may be so combined that the resulting value of $\frac{d\rho}{dt}$ will not contain $\frac{d^2\alpha}{dt^2}$. This transformation is easily effected, and may be advantageous in special cases for which the value of $\frac{d^2\alpha}{dt^2}$ is very uncertain.

The heliocentric spherical co-ordinates will be obtained from the

assumed value of Δ by means of the equations (106)₃, and the rectangular co-ordinates from

$$\begin{aligned}x &= r \cos b \cos l, \\y &= r \cos b \sin l, \\z &= r \sin b.\end{aligned}$$

The velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ will be given by (42), and from these and the co-ordinates x, y, z the elements of the orbit will be computed by means of the equations (32)₁, (47), (49), &c. With the elements thus derived we compute the geocentric places for the dates of the normals, and find the differences between computation and observation. Then a second system of elements is computed from $\Delta + \delta\Delta$, and compared with the observed places. Let the difference between computation and observation for either of the two spherical co-ordinates be denoted by n for the first system of elements, and by n' for the second system. The final correction to be applied to Δ , in order that the observed place may be exactly represented, will be determined by

$$\frac{\Delta\Delta}{d\Delta} (n' - n) + n = 0. \quad (56)$$

Each observed right ascension and each observed declination will thus furnish an equation of condition for the determination of $\Delta\Delta$, observing that the residuals in right ascension should in each case be multiplied by $\cos \delta$. Finally, the elements which correspond to the geocentric distance $\Delta + \Delta\Delta$ will be determined either directly or by interpolation, and these must represent the entire series of observed places.

115. The equations (52)₃ enable us to find two radii-vectores when the ratio of the corresponding curtate distances is known, provided that an additional equation involving r, r'', κ , and known quantities is given. For the special case of parabolic motion, this additional equation involves only the interval of time, the two radii-vectores, and the chord joining their extremities. The corresponding equation for the general conic section involves also the semi-transverse axis of the orbit, and hence, if the ratio M of the curtate distances is known, this equation will, in connection with the equations (52)₃, enable us to find the values of r and r'' corresponding to a given value of α . To derive this expression, let us resume the equations

$$\begin{aligned}\frac{\tau'}{a^{\frac{3}{2}}} &= E'' - E - 2e \sin \frac{1}{2}(E'' - E) \cos \frac{1}{2}(E'' + E), \\ r + r'' &= 2a - 2ae \cos \frac{1}{2}(E'' - E) \cos \frac{1}{2}(E'' + E).\end{aligned}\quad (57)$$

For the chord x we have

$$x^2 = (r + r'')^2 - 4rr'' \cos^2 \frac{1}{2}(u'' - u),$$

which, by means of (58)₄, gives

$$x^2 = (r + r'')^2 - 4a^2 (\cos^2 \frac{1}{2}(E'' - E) - 2e \cos \frac{1}{2}(E'' - E) \cos \frac{1}{2}(E'' + E) + e^2 \cos^2 \frac{1}{2}(E'' + E));$$

and, substituting for $r + r''$ its value given by the last of equations (57), we get

$$x^2 = 4a^2 \sin^2 \frac{1}{2}(E'' - E) (1 - e^2 \cos^2 \frac{1}{2}(E'' + E)). \quad (58)$$

Let us now introduce an auxiliary angle h , such that

$$\cos h = e \cos \frac{1}{2}(E'' + E),$$

the condition being imposed that h shall be less than 180° , and put

$$g = \frac{1}{2}(E'' - E);$$

then the equations (57) and (58) become

$$\begin{aligned}\frac{\tau'}{a^{\frac{3}{2}}} &= 2g - 2 \sin g \cos h, \\ r + r'' &= 2a (1 - \cos g \cos h), \\ x &= 2a \sin g \sin h.\end{aligned}\quad (59)$$

Further, let us put

$$h - g = \delta, \quad h + g = \epsilon,$$

and the last two of equations (59) give

$$\begin{aligned}r + r'' + x &= 4a \sin^2 \frac{1}{2}\epsilon, \\ r + r'' - x &= 4a \sin^2 \frac{1}{2}\delta.\end{aligned}\quad (60)$$

Introducing δ and ϵ into the first of equations (59), it becomes

$$\frac{\tau'}{a^{\frac{3}{2}}} = (\epsilon - \sin \epsilon) - (\delta - \sin \delta). \quad (61)$$

The formulæ (60) enable us to determine ϵ and δ from $r + r''$, x , and a , and then the time $\tau' = k(t'' - t)$ may be determined from (61). Since, according to (58)₄,

$$\sqrt{rr''} \cos \frac{1}{2}(u'' - u) = a(\cos g - \cos h) = 2 \sin \frac{1}{2}\epsilon \sin \frac{1}{2}\delta,$$

and since $\sin \frac{1}{2}\varepsilon$ is necessarily positive, it appears that when $u'' - \kappa$ exceeds 180° , the value of $\sin \frac{1}{2}\delta$ must be negative, and when $u'' - \kappa = 180^\circ$, we have $\delta = 0$; and thus the quadrant in which δ must be taken is determined. It will be observed that the value of $\frac{1}{2}\varepsilon$, as given by the first of equations (60), may be either in the first or the second quadrant; but, in the actual application of the formulæ, the ambiguity is easily removed by means of the known circumstances in regard to the motion of the body during the interval $t'' - t$.

In the application of the equations (52)₃, by means of an approximate value of κ we compute d , and thence r and r'' . Then we compute ε and δ corresponding to the given value of a , and from (61) we derive the value of

$$t'' - t = \frac{\tau'}{k}.$$

If this agrees with the observed interval $t'' - t$, the assumed value of κ is correct; but if a difference exists, by varying κ we may readily find, by a few trials, the value which will exactly satisfy the equations. The formulæ (70)₃ will then enable us to determine the curtate distances ρ and ρ'' , and from these and the observed spherical co-ordinates the elements of the orbit may be found.

As soon as the values of u and u'' have been computed, since $\varepsilon - \delta = E'' - E$, we have, according to equation (85)₄,

$$\cos \varphi = \frac{\sin \frac{1}{2}(u'' - u)}{a \sin \frac{1}{2}(\varepsilon - \delta)} \sqrt{rr''},$$

which may be used to determine φ when the orbit is very eccentric. To find p and q , we have

$$p = a \cos^2 \varphi, \quad q = 2a \sin^2(45^\circ - \frac{1}{2}\varphi);$$

and the value of ω may be found by means of the equations (87)₄ or (88)₄.

116. The process here indicated will be applied chiefly in the determination of the orbits of comets, and generally for cases in which a is large. In such cases the angles ε and δ will be small, so that the slightest errors will have considerable influence in vitiating the value of $t'' - t$ as determined by equation (61); but if we transform this equation so as to eliminate the divisor $a^{\frac{3}{2}}$ in the first member, the uncertainty of the solution may be overcome. The difference $\varepsilon - \sin \varepsilon$

may be expressed by a series which converges rapidly when ε is small. Thus, let us put

$$\varepsilon - \sin \varepsilon = y \sin^3 \frac{1}{2} \varepsilon, \quad x = \sin^2 \frac{1}{4} \varepsilon,$$

and we have

$$\frac{dy}{d\varepsilon} = 2 \operatorname{cosec} \frac{1}{2} \varepsilon - \frac{3}{2} y \cot \frac{1}{2} \varepsilon,$$

$$\frac{d\varepsilon}{dx} = 4 \operatorname{cosec} \frac{1}{2} \varepsilon.$$

Therefore

$$\frac{dy}{dx} = \frac{8 - 6y \cos \frac{1}{2} \varepsilon}{\sin^2 \frac{1}{2} \varepsilon} = \frac{4 - 3y(1 - 2x)}{2x(1 - x)}.$$

If we suppose y to be expanded into a series of the form

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.,$$

we get, by differentiation,

$$\frac{dy}{dx} = \beta + 2\gamma x + 3\delta x^2 + \&c.,$$

and substituting for $\frac{dy}{dx}$ the value already obtained, the result is

$$2\beta x + (4\gamma - 2\beta)x^2 + (6\delta - 4\gamma)x^3 + \&c. = 4 - 3\alpha + (6\alpha - 3\beta)x + (6\beta - 3\gamma)x^2 + (6\gamma - 3\delta)x^3 + \&c.$$

Therefore we have

$$\begin{aligned} 4 - 3\alpha &= 0, & 6\alpha - 3\beta &= 2\beta, \\ 6\beta - 3\gamma &= 4\gamma - 2\beta, & 6\gamma - 3\delta &= 6\delta - 4\gamma, \end{aligned}$$

from which we get

$$\alpha = \frac{4}{3}, \quad \beta = \frac{4 \cdot 6}{3 \cdot 5}, \quad \gamma = \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7}, \quad \delta = \frac{4 \cdot 6 \cdot 8 \cdot 10}{3 \cdot 5 \cdot 7 \cdot 9}, \&c.$$

Hence we obtain

$$\varepsilon - \sin \varepsilon = \frac{4}{3} \sin^3 \frac{1}{2} \varepsilon \left(1 + \frac{6}{5} \sin^2 \frac{1}{4} \varepsilon + \frac{6 \cdot 8}{5 \cdot 7} \sin^4 \frac{1}{4} \varepsilon + \frac{6 \cdot 8 \cdot 10}{5 \cdot 7 \cdot 9} \sin^6 \frac{1}{4} \varepsilon + \&c. \right), \quad (62)$$

and, in like manner,

$$\delta - \sin \delta = \frac{4}{3} \sin^3 \frac{1}{2} \delta \left(1 + \frac{6}{5} \sin^2 \frac{1}{4} \delta + \frac{6 \cdot 8}{5 \cdot 7} \sin^4 \frac{1}{4} \delta + \frac{6 \cdot 8 \cdot 10}{5 \cdot 7 \cdot 9} \sin^6 \frac{1}{4} \delta + \&c. \right), \quad (63)$$

which, for brevity, may be written

$$\begin{aligned} \varepsilon - \sin \varepsilon &= \frac{4}{3} Q \sin^3 \frac{1}{2} \varepsilon, \\ \delta - \sin \delta &= \frac{4}{3} Q' \sin^3 \frac{1}{2} \delta, \end{aligned} \quad (64)$$

Combining these expressions with (61), and substituting for $\sin \frac{1}{2}\varepsilon$ and $\sin \frac{1}{2}\delta$ their values given by the equations (60), there results

$$6\tau' = Q(r + r'' + x)^{\frac{3}{2}} \mp Q'(r + r'' - x)^{\frac{3}{2}}, \quad (65)$$

the upper sign being used when the heliocentric motion of the body is less than 180° , and the lower sign when it is greater than 180° . The coefficients Q and Q' represent, respectively, the series of terms enclosed in the parentheses in the second members of the equations (62) and (63), and it is evident that their values may be tabulated with the argument ε or δ , as the case may be. It will be observed, however, that the first two terms of the value of Q are identical with the first two terms of the expansion of $(\cos \frac{1}{4}\varepsilon)^{-\frac{1}{2}}$ into a series of ascending powers of $\sin \frac{1}{4}\varepsilon$, while the difference is very small between the coefficients of the third terms. Thus, we have

$$\begin{aligned} (\cos \tfrac{1}{4}\varepsilon)^{-\frac{1}{2}} &= (1 - \sin^2 \tfrac{1}{4}\varepsilon)^{-\frac{1}{2}} = 1 + \tfrac{1}{2} \sin^2 \tfrac{1}{4}\varepsilon + \tfrac{6 \cdot 11}{5 \cdot 10} \sin^4 \tfrac{1}{4}\varepsilon \\ &+ \tfrac{6 \cdot 11 \cdot 16}{5 \cdot 10 \cdot 15} \sin^6 \tfrac{1}{4}\varepsilon + \&c., \end{aligned}$$

and if we put

$$Q = \frac{B_0}{(\cos \tfrac{1}{4}\varepsilon)^{\frac{1}{2}}}, \quad (66)$$

we shall have

$$B_0 = 1 + \tfrac{9}{1 \cdot 7 \cdot 5} \sin^4 \tfrac{1}{4}\varepsilon + \tfrac{1 \cdot 4 \cdot 2}{2 \cdot 6 \cdot 2 \cdot 5} \sin^6 \tfrac{1}{4}\varepsilon + \&c. \quad (67)$$

In a similar manner, if we put

$$Q' = \frac{B'_0}{(\cos \tfrac{1}{4}\delta)^{\frac{1}{2}}}, \quad (68)$$

we find

$$B'_0 = 1 + \tfrac{9}{1 \cdot 7 \cdot 5} \sin^4 \tfrac{1}{4}\delta + \tfrac{1 \cdot 4 \cdot 2}{2 \cdot 6 \cdot 2 \cdot 5} \sin^6 \tfrac{1}{4}\delta + \&c. \quad (69)$$

Table XV. gives the values of B_0 or B'_0 corresponding to ε or δ from 0° to 60° .

For the case of parabolic motion we have

$$Q = 1, \quad Q' = 1,$$

and the equation (65) becomes identical with (56)₃.

In the application of these formulæ, we first compute ε and δ by means of the equations (60), and then, having found B_0 and B'_0 by means of Table XV., we compute the values of Q and Q' from (66) and (68). Finally, the time $\tau' = k(t'' - t)$ will be obtained from (65), and the difference between this result and the observed interval will

indicate whether the assumed value of \varkappa must be increased or diminished. A few trials will give the correct result.

117. Since the interval of time $t'' - t$ cannot be determined with sufficient accuracy from (65) when the chord \varkappa is very small, it becomes necessary to effect a further transformation of this equation. Thus, let us put

$$Q - Q' = 6P, \quad x = \sin^2 \frac{1}{4} \varepsilon, \quad x' = \sin^2 \frac{1}{4} \delta,$$

and we shall have

$$P = \frac{1}{5}(x - x') \left(1 + \frac{8}{7}(x + x') + \frac{8 \cdot 10}{7 \cdot 9}(x^2 + xx' + x'^2) + \&c. \right).$$

Now, when \varkappa is very small, we may put

$$\cos \frac{1}{4} \varepsilon = \cos \frac{1}{4} \delta,$$

and hence

$$x - x' = \sin^2 \frac{1}{4} \varepsilon - \sin^2 \frac{1}{4} \delta = \frac{\sin^2 \frac{1}{2} \varepsilon - \sin^2 \frac{1}{2} \delta}{4 \cos^2 \frac{1}{4} \varepsilon},$$

which, by means of equations (60), becomes

$$x - x' = \frac{\varkappa}{8a \cos^2 \frac{1}{4} \varepsilon}.$$

Therefore we have, when \varkappa is very small,

$$P = \frac{\varkappa}{40a \cos^2 \frac{1}{4} \varepsilon} \left(1 + \frac{1}{7} \sin^2 \frac{1}{4} \varepsilon + \frac{8}{21} \sin^4 \frac{1}{4} \varepsilon + \&c. \right) \quad (70)$$

If we put

$$\tau_0' = \frac{\tau' - P(r + r'' - \varkappa)^{\frac{3}{2}}}{Q}, \quad (71)$$

the equation (65) becomes, using only the upper sign,

$$(r + r'' + \varkappa)^{\frac{3}{2}} - (r + r'' - \varkappa)^{\frac{3}{2}} = 6\tau_0', \quad (72)$$

which is of the same form as (56)₃. Hence, according to the equations (63)₃ and (66)₃, we shall have

$$\varkappa = \frac{2\tau_0'}{\sqrt{r + r''}} \mu, \quad (73)$$

the value of μ being found from Table XI. with the argument

$$\eta = \frac{2\tau_0'}{(r + r'')^{\frac{3}{2}}} \quad (74)$$

It remains, therefore, simply to find a convenient expression for τ_0' , and the determination of κ is effected by a process precisely the same as in the special case of parabolic motion.

Let us now put

$$\frac{P}{Q} = \frac{\kappa}{40a} \cdot \frac{N}{\cos^4 \frac{1}{4}\epsilon},$$

and we shall have

$$N = \frac{\cos^2 \frac{1}{4}\epsilon}{Q} \left(1 + \frac{2 \cdot 8}{7} \sin^2 \frac{1}{4}\epsilon + \frac{3 \cdot 8 \cdot 10}{7 \cdot 9} \sin^4 \frac{1}{4}\epsilon + \frac{4 \cdot 8 \cdot 10 \cdot 12}{7 \cdot 9 \cdot 11} \sin^6 \frac{1}{4}\epsilon + \&c. \right),$$

or, substituting for Q its value in terms of $\sin \frac{1}{4}\epsilon$,

$$N = 1 + \frac{3}{35} \sin^2 \frac{1}{4}\epsilon + \frac{26}{825} \sin^4 \frac{1}{4}\epsilon + \frac{2076}{67375} \sin^6 \frac{1}{4}\epsilon + \&c. \quad (75)$$

Therefore, if we put

$$\Delta\tau_0' = \frac{\kappa}{40a} \cdot \frac{N}{\cos^4 \frac{1}{4}\epsilon} (r + r'' - \kappa)^{\frac{3}{2}}, \quad (76)$$

the expression for τ_0' becomes

$$\tau_0' = \frac{\tau'}{Q} - \Delta\tau_0'. \quad (77)$$

Table XV. gives the value of $\log N$ corresponding to values of ϵ from $\epsilon = 0$ to $\epsilon = 60^\circ$.

If the chord κ is given, and the interval of time $t'' - t$ is required, we compute $\Delta\tau_0'$ by means of (76), and, having found τ_0' from

$$\tau_0' = \frac{\kappa \sqrt{r + r''}}{2\mu},$$

as in the case of parabolic motion, we have

$$t'' - t = \frac{Q(\tau_0' + \Delta\tau_0')}{k}.$$

It should be observed that although equation (76) is derived for the case of a small value of κ , yet it is applicable whenever the difference $\epsilon - \delta$ is very small, whatever may be the value of κ . For orbits which differ but little from the parabolic form, it will in all cases be sufficient to use this expression for $\Delta\tau_0'$; and for cases in which the difference between ϵ and δ is such that the assumption of $\cos \frac{1}{4}\epsilon = \cos \frac{1}{4}\delta$, $x + x' = 2x$, &c., made in deriving equation (70), does

not afford the required accuracy, we may compute both Q and Q' directly, and then we have

$$\Delta\tau'_0 = \frac{1}{6} \left(1 - \frac{Q'}{Q} \right) (r + r'' - \kappa)^{\frac{2}{3}}. \quad (78)$$

The values of the factor $\frac{1}{6} \left(1 - \frac{Q'}{Q} \right)$ may be tabulated directly with $\frac{r+r''}{4a}$ as the vertical argument and $\frac{\kappa}{4a}$ as the horizontal argument; but for the few cases in which the value of N given by the equation (75) is not sufficiently accurate, it will be easy to compute Q and Q' by means of the formulæ (66) and (68), and then find $\Delta\tau'_0$ from (78). Further, when there is any doubt as to the accuracy of the result given by (76), for the final trial in finding κ from $r + r''$ and τ_0 by means of the equations (73) and (74), it will be advisable to compute $\Delta\tau'_0$ from (78).

It appears, therefore, that for nearly all the cases which actually occur the determination of the value of κ , corresponding to given values of a and $M = \frac{\rho''}{\rho}$, is reduced by means of the equation (72) to the method which is adopted in the case of parabolic orbits.

The calculation of the numerical values of $r + r'' + \kappa$ and $r + r'' - \kappa$ will be most conveniently effected by the aid of addition and subtraction logarithms. If the tables of common logarithms are used, we may first compute

$$\sin \gamma' = \frac{\kappa}{r + r''},$$

and then we have

$$\begin{aligned} r + r'' + \kappa &= 2(r + r'') \sin^2(45^\circ + \tfrac{1}{2}\gamma'), \\ r + r'' - \kappa &= 2(r + r'') \cos^2(45^\circ + \tfrac{1}{2}\gamma'). \end{aligned}$$

118. In the case of hyperbolic motion, the semi-transverse axis is negative, and the values of $\sin \frac{1}{2}\epsilon$ and $\sin \frac{1}{2}\delta$ given by the equations (60) become imaginary, so that it is no longer possible to compute the interval of time from $r + r''$ and κ by means of the auxiliary angles ϵ and δ . Let us, therefore, put

$$\sin^2 \tfrac{1}{2}\epsilon = -m^2, \quad \sin^2 \tfrac{1}{2}\delta = -n^2;$$

then, when a is negative, m and n will be real. Now we have

$$\tfrac{1}{2}\epsilon = \sin^{-1} \sqrt{-m^2}, \quad \tfrac{1}{2}\delta = \sin^{-1} \sqrt{-n^2},$$

and

$$\tfrac{1}{2}\epsilon \sqrt{-1} = \log_0 (\cos \tfrac{1}{2}\epsilon + \sqrt{-1} \sin \tfrac{1}{2}\epsilon).$$

Hence we derive

$$\begin{aligned}\varepsilon &= 2 \sin^{-1} \sqrt{-m^2} = \frac{2}{\sqrt{-1}} \log_e (\sqrt{1+m^2} + m), \\ \delta &= 2 \sin^{-1} \sqrt{-n^2} = \frac{2}{\sqrt{-1}} \log_e (\sqrt{1+n^2} + n).\end{aligned}$$

Substituting these values in the equation (61), and writing $-a$ instead of a , since

$$\sin \varepsilon = 2m \sqrt{-1} \cdot \sqrt{1+m^2},$$

we shall have

$$\begin{aligned}\frac{\tau'}{a^{\frac{3}{2}}} &= 2m \sqrt{1+m^2} - 2 \log_e (\sqrt{1+m^2} + m) \\ &\quad \mp (2n \sqrt{1+n^2} - 2 \log_e (\sqrt{1+n^2} + n)),\end{aligned}\tag{79}$$

the upper sign being used when the heliocentric motion is less than 180° , and the lower sign when it is greater than 180° . Therefore, if we compute m and n from

$$m = \sqrt{\frac{r+r''+x}{4a}}, \quad n = \sqrt{\frac{r+r''-x}{4a}},\tag{80}$$

regarding the hyperbolic semi-transverse axis a as positive, the formula (79) will determine the interval of time $\tau' = k(t'' - t)$.

The first two terms of the second member of equation (79) may be expressed in a series of ascending powers of m , and the last two terms in a series of ascending powers of n . Thus, if we put

$$\log_e (\sqrt{1+m^2} + m) = \alpha m + \beta m^2 + \gamma m^3 + \delta m^4 + \&c.,$$

we get, by differentiation,

$$\frac{1}{\sqrt{1+m^2}} = \alpha + 2\beta m + 3\gamma m^2 + 4\delta m^3 + 5\varepsilon m^4 + \&c.;$$

and since

$$\frac{1}{\sqrt{1+m^2}} = 1 - \frac{1}{2}m^2 + \frac{1 \cdot 3}{2 \cdot 4}m^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}m^6 + \&c.,$$

we have

$$\alpha = 1, \quad \beta = 0, \quad \gamma = -\frac{1}{2} \cdot \frac{1}{2}, \quad \delta = 0, \quad \varepsilon = \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4}, \&c.$$

Hence we obtain

$$2 \log_e (\sqrt{1+m^2} + m) = 2m - \frac{1}{2}m^3 + \frac{1}{2} \cdot \frac{3}{4}m^5 - \frac{1}{4} \cdot \frac{3 \cdot 5}{4 \cdot 6}m^7 + \&c.$$

We have, also,

$$2m\sqrt{1+m^2} = 2m + m^3 - \frac{1}{4}m^5 + \frac{1 \cdot 3}{4 \cdot 6}m^7 - \&c.$$

Therefore,

$$2m\sqrt{1+m^2} - 2\log_e(\sqrt{1+m^2} + m) = \frac{4}{3}m^3 \left(1 - \frac{3}{5} \cdot \frac{1}{2}m^2 + \frac{3}{7} \cdot \frac{1 \cdot 3}{2 \cdot 4}m^4 - \&c. \right), \quad (81)$$

and similarly

$$2n\sqrt{1+n^2} - 2\log_e(\sqrt{1+n^2} + n) = \frac{4}{3}n^3 \left(1 - \frac{3}{5} \cdot \frac{1}{2}n^2 + \frac{3}{7} \cdot \frac{1 \cdot 3}{2 \cdot 4}n^4 - \&c. \right). \quad (82)$$

Substituting these values in the equation (79), and denoting the series of terms enclosed in the parentheses by Q and Q' , respectively, we get

$$6\tau' = Q(r + r'' + \kappa)^{\frac{3}{2}} \mp Q'(r + r'' - \kappa)^{\frac{3}{2}} \quad (83)$$

which is identical with equation (65). If we replace m^2 by $-\sin^2 \frac{1}{2}\epsilon$ and n^2 by $-\sin^2 \frac{1}{2}\delta$ in the expressions for Q and Q' , as given by (81) and (82), we shall have the expressions for these quantities in terms of $\sin \frac{1}{2}\epsilon$ and $\sin \frac{1}{2}\delta$, respectively, instead of $\sin \frac{1}{4}\epsilon$ and $\sin \frac{1}{4}\delta$ as given by the equations (62) and (63), namely,

$$Q = 1 + \frac{3}{5} \cdot \frac{1}{2} \sin^2 \frac{1}{2}\epsilon + \frac{3}{7} \cdot \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \frac{1}{2}\epsilon + \frac{3}{9} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin^6 \frac{1}{2}\epsilon + \&c., \quad (84)$$

$$Q' = 1 + \frac{3}{5} \cdot \frac{1}{2} \sin^2 \frac{1}{2}\delta + \frac{3}{7} \cdot \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \frac{1}{2}\delta + \frac{3}{9} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin^6 \frac{1}{2}\delta + \&c.$$

For the case of an elliptic orbit it is most convenient to use the equations (66) and (68) in finding Q and Q' ; but, since the cases of hyperbolic motion are rare, while for those which do occur the eccentricity is very little greater than that of the parabola, it will be sufficient to tabulate Q directly with the argument m . The same table, using n as the argument, will give the value of Q' . Table XVI. gives the values of Q corresponding to values of m from $m=0$ to $m=0.2$.

When the values of $r + r''$, τ' , and a are given, and the chord κ is required, we may compute $\Delta\tau'_0$ from (78), τ'_0 from (77), and finally κ from (73).

It may be remarked, also, that the formulæ for the relation between τ' , $r + r''$, κ , and a suffice to find by trial the value of a when $r + r''$ and κ are given. Hence, in the computation of an orbit from assumed

values of Δ and Δ'' , the value of κ may be computed from r , r'' , and $u'' - u$, and then a may be found in the manner here indicated.

If we substitute in the equations (84) the values of $\sin \frac{1}{2}\epsilon$ and $\sin \frac{1}{2}\delta$ in terms of $r + r''$, κ , and a , and then substitute the resulting values of Q and Q' in the equation (65), we obtain

$$6k(t'' - t) = (r + r'' + \kappa)^{\frac{3}{2}} \mp (r + r'' - \kappa)^{\frac{3}{2}} + \frac{3}{40} \frac{1}{a} \left\{ (r + r'' + \kappa)^{\frac{5}{2}} \mp (r + r'' - \kappa)^{\frac{5}{2}} \right\} \\ + \frac{9}{880} \frac{1}{a^2} \left\{ (r + r'' + \kappa)^{\frac{7}{2}} \mp (r + r'' - \kappa)^{\frac{7}{2}} \right\} + \&c., \quad (85)$$

the lower sign being used when $u'' - u$ exceeds 180° . When the eccentricity is very nearly equal to unity, this series converges with great rapidity. In the case of hyperbolic motion, the sign of a must be changed.

119. The formulæ thus derived for the determination of the chord κ for the cases of elliptic and hyperbolic orbits, enable us to correct an approximate orbit by varying the semi-transverse axis a and the ratio M of two curtate distances. But since the formulæ will generally be applied for the correction of approximate parabolic elements, or those which are nearly parabolic, it will be expedient to use $\frac{1}{a}$ and M as the quantities to be determined.

In the first place, we compute a system of elements from M and $f = \frac{1}{a}$; and, for the determination of the auxiliary quantities preliminary to the calculation of the values of r , r'' , and κ , the equations (41)₃, (50)₃, and (51)₃ will be employed when the ecliptic is the fundamental plane. But when the equator is taken as the fundamental plane, we must first compute g , K , and G by means of the equations (96)₃. Then, by a process entirely analogous to that by which the equations (47)₃ and (50)₃ were derived, we obtain

$$\begin{aligned} h \cos \zeta \cos (H - \alpha'') &= M - \cos (\alpha'' - \alpha), \\ h \cos \zeta \sin (H - \alpha'') &= \sin (\alpha'' - \alpha), \\ h \sin \zeta &= M \tan \delta'' - \tan \delta, \end{aligned} \quad (86)$$

from which to find H , ζ , and h ; and also

$$\cos \varphi = \cos \zeta \cos K \cos (G - H) + \sin \zeta \sin K, \quad (87)$$

from which to find φ . In this case, ζ and H will be referred to the equator as the fundamental plane. The angles ψ and ψ'' will be obtained from the equations (102)₃, or from equations of the form

of (26), and finally the auxiliary quantities A , B , B'' , &c. will be obtained from (51)₃, writing δ and δ'' in place of β and β'' , respectively.

As soon as these auxiliary quantities have been determined, by means of (52)₃ the value of κ must be found which will exactly satisfy equation (65). To effect this, we first compute ε from

$$\sin \frac{1}{2}\varepsilon = \sqrt{\frac{1}{4}f(r + r'' + \kappa)},$$

and, if it be required, we also find δ from

$$\sin \frac{1}{2}\delta = \sqrt{\frac{1}{4}f(r + r'' - \kappa)},$$

using approximate values of $r + r''$ and κ . Then we find Q from (66), and $\Delta\tau'_0$ from (76) or from (78), the logarithms of the auxiliary quantities B_0 and N being found by means of Table XV. with the argument ε . The value of τ'_0 having been found from (77), the equations (73) and (74), in connection with Table XI., enable us to obtain a closer approximation to the correct value of κ . With this we compute new values of r and r'' , and repeat the determination of κ . A few trials will generally give the correct result, and these trials may be facilitated by the use of the formula (67)₃. It will be observed, also, that Q and $\Delta\tau'_0$ are very slightly changed by a small change in the values of $r + r''$ and κ , so that a repetition of the calculation of these quantities only becomes necessary for the final trial in finding the value of κ which completely satisfies the equations (52)₃ and (65). When the value of a is such that the values of Q and N exceed the limits of Table XV., the equation (61) may be employed, and, in the case of hyperbolic motion, when Q and Q' exceed the limits of Table XVI., we may employ the complete expression for the time τ' in terms of m and n as given by (79).

The values of r , r'' , and κ having thus been found, the equations

$$d = \sqrt{\kappa^2 - A^2}, \quad \rho = \frac{d + g \cos \varphi}{h}, \quad \rho'' = M\rho,$$

will determine the curtate distances ρ and ρ'' . When the equator is the fundamental plane, we have

$$\rho = \Delta \cos \delta, \quad \rho'' = \Delta'' \cos \delta''.$$

From ρ , ρ'' , and the corresponding geocentric spherical co-ordinates, the radii-vectores and the heliocentric spherical co-ordinates l , l'' , b , and b'' will be obtained, and thence Ω , i , u , u'' , and the remaining

elements of the orbit, as already illustrated. In the case of elliptic motion, if we compute the auxiliary quantities ϵ and δ by means of the equations (60), we shall have

$$\begin{aligned} e \sin \frac{1}{2}(E'' + E) &= \frac{r'' - r}{2a \sin \frac{1}{2}(\epsilon - \delta)}, \\ e \cos \frac{1}{2}(E'' + E) &= \cos \frac{1}{2}(\epsilon + \delta), \end{aligned}$$

from which e and $\frac{1}{2}(E'' + E)$ may be found, and hence, since $\frac{1}{2}(E'' - E) = \frac{1}{2}(\epsilon - \delta)$, we derive E and E'' . The values of q and v may then be found directly from these and quantities already obtained. Thus, the last of equations (43)₁ gives

$$\frac{\cos \frac{1}{2}v}{\sqrt{q}} = \frac{\cos \frac{1}{2}E}{\sqrt{r}}, \quad \frac{\cos \frac{1}{2}v''}{\sqrt{q}} = \frac{\cos \frac{1}{2}E''}{\sqrt{r''}}.$$

Multiplying the first of these expressions by $\sin \frac{1}{2}v''$, and the second by $-\sin \frac{1}{2}v$, adding the products, and reducing, we obtain

$$\frac{\sin \frac{1}{2}(v'' - v) \sin \frac{1}{2}v}{\sqrt{q}} = \frac{\cos \frac{1}{2}(v'' - v) \cos \frac{1}{2}E}{\sqrt{r}} - \frac{\cos \frac{1}{2}E''}{\sqrt{r''}}.$$

Therefore, we shall have

$$\begin{aligned} \frac{1}{\sqrt{q}} \sin \frac{1}{2}v &= \frac{\cos \frac{1}{2}E}{\sqrt{r} \tan \frac{1}{2}(u'' - u)} - \frac{\cos \frac{1}{2}E''}{\sqrt{r''} \sin \frac{1}{2}(u'' - u)}, \\ \frac{1}{\sqrt{q}} \cos \frac{1}{2}v &= \frac{\cos \frac{1}{2}E}{\sqrt{r}}, \end{aligned} \quad (88)$$

from which q and v may be found as soon as $\cos \frac{1}{2}E$ and $\cos \frac{1}{2}E''$ are known. In the case of parabolic motion the eccentric anomaly is equal to zero, and these equations become identical with (92)₃. The angular distance of the perihelion from the ascending node will be obtained from

$$\omega = u - v.$$

Since $r = a - ae \cos E$, and $q = a(1 - e)$, we have

$$\cos E = \frac{1 - \frac{r}{a}}{1 - \frac{q}{a}} = 1 - \frac{\frac{r}{a} - 1}{\frac{q}{a} - 1},$$

and hence

$$\begin{aligned} \cos^2 \frac{1}{2}E &= 1 - \frac{\frac{r}{a} - 1}{\frac{q}{a} - 1}, \\ \cos^2 \frac{1}{2}E'' &= 1 - \frac{\frac{r''}{a} - 1}{\frac{q}{a} - 1}. \end{aligned} \quad (89)$$

When the eccentricity is nearly equal to unity, the value of q given by approximate elements will be sufficient to compute $\cos \frac{1}{2}E$ and $\cos \frac{1}{2}E''$ by means of these equations, and the results thus derived will be substituted in the equations (88), from which a new value of q results. If this should differ considerably from that used in computing $\cos \frac{1}{2}E$ and $\cos \frac{1}{2}E''$, a repetition of the calculation will give the correct result.

In the case of hyperbolic motion, although E and E'' are imaginary, we may compute the numerical values of $\cos \frac{1}{2}E$ and $\cos \frac{1}{2}E''$ from the equations (89), regarding a as negative, and the results will be used for the corresponding quantities in (88) in the computation of q and v for the hyperbolic orbit.

Next, we compute a second system of elements from M and $f + \delta f$, and a third system from $M + \delta M$ and f , δf and δM denoting the arbitrary increments assigned to f and M respectively. The comparison of these three systems of elements with additional observed places of the comet, will enable us to form the equations of condition for the determination of the most probable values of the corrections ΔM and Δf to be applied to M and f respectively. The formation of these equations is effected in precisely the same manner as in the case of the variation of the geocentric distances or of Ω and i , and it does not require any further illustration. The final elements will be obtained from $M + \Delta M$, and $f + \Delta f$, either directly or by interpolation. We may remark, further, that it will be convenient to use $\log M$ as the quantity to be corrected, and to express the variations of $\log M$ in units of the last decimal place of the logarithms.

When the orbit differs very little from the parabolic form, it will be most expeditious to make two hypotheses in regard to M , putting in each case $\frac{1}{a} = 0$, and only compute elliptic or hyperbolic elements in the third hypothesis, for which we use M and $f = \delta f$. The first and second systems of elements will thus be parabolic.

120. Instead of M and $\frac{1}{a}$ we may use Δ and $\frac{1}{a}$ as the quantities to be corrected. In this case we assume an approximate value of Δ by means of elements already known, and by means of (96)₃, (98)₃, (102)₃, and (103)₃, we compute the auxiliary quantities C , B , B'' , &c., required in the solution of the equations (104)₃. We assume, also, an approximate value of Δ'' and compute the corresponding value of r'' , the value of r having been already found from the assumed value of Δ . Then, by trial, we find the value of x which, in connection with

the assumed value of $\frac{1}{a}$, will satisfy the equations $(104)_3$ and (65) or (61) . The corresponding value of Δ'' is given by

$$\Delta'' = c \pm \sqrt{x^2 - C^2}.$$

When Δ'' has thus been determined, the heliocentric places will be obtained by means of the equations $(106)_3$ and $(107)_3$, and, finally, the corresponding elements of the orbit will be computed. If the ecliptic is taken as the fundamental plane, we put $D=0$, $A=\odot$, and write λ and β in place of α and δ respectively.

If we now compute a second system of elements from $\Delta + \delta\Delta$ and $f = \frac{1}{a}$, and a third system from Δ and $f + \delta f$, the comparison of the three systems of elements with additional observed places will furnish the equations of condition for the determination of the corrections $\Delta\Delta$ and Δf to be applied to Δ and $\frac{1}{a}$ respectively.

When the eccentricity is very nearly equal to unity, we may assume $f=0$ for the first and second hypotheses, and only compute elliptic or hyperbolic elements for the third hypothesis.

121. The comparison of the several observed places of a heavenly body with one of the three systems of elements obtained by varying the two quantities selected for correction, or, when the required differential coefficients are known, with any other system of elements such that the squares and products of the corrections may be neglected, gives a series of equations of the form

$$\begin{aligned} mx + ny &= p, \\ m'x + n'y &= p', \text{ \&c.,} \end{aligned}$$

in which x and y denote the final corrections to be applied to the two assumed quantities respectively. The combination of these equations which gives the most probable values of the unknown quantities, is effected according to the method of least squares. Thus, we multiply each equation by the coefficient of x in that equation, and the sum of all the equations thus formed gives the first normal equation. Then we multiply each equation of condition by the coefficient of y in that equation, and the sum of all the products gives the second normal equation. Let these equations be expressed thus:—

$$\begin{aligned} [mm]x + [mn]y &= [mp], \\ [mn]x + [nn]y &= [np], \end{aligned}$$

in which $[mm] = m^2 + m'^2 + m''^2 + \&c.$, $[mn] = mn + m'n' + m''n'' + \&c.$, and similarly for the other terms. These two final equations give, by elimination, the most probable values of x and y , namely, those for which the sum of the squares of the residuals will be a minimum. It is, however, often convenient to determine x in terms of y , or y in terms of x , so that we may find the influence of a variation of one of the unknown quantities on the differences between computation and observation when the most probable value of the other unknown quantity is used. Thus, if it be desired to find x in terms of y , the most probable value of x will be

$$x = \frac{[mp]}{[mm]} - \frac{[mn]}{[mm]}y.$$

If we substitute this value of x in the original equations of condition, the remaining differences between computation and observation will be expressed in terms of the unknown quantity y , or in the form

$$\Delta\theta = m_0 + n_0y. \quad (90)$$

Then, by assigning different values to y , we may find the corresponding residuals, and thus determine to what extent the correction y may be varied without causing these residuals to surpass the limits of the probable errors of observation.

In the determination of the orbit of a comet there must be more or less uncertainty in the value of a , and if y denotes the correction to be applied to the assumed value of $\frac{1}{a}$, we may thus determine the probable limits within which the true value of the periodic time must be found. In the case of a comet which is identified, by the similarity of elements, with one which has previously appeared, if we compute the system of elements which will best satisfy the series of observations, the supposition being made that the comet has performed but one revolution around the sun during the intervening interval, it will be easy to determine whether the observations are better satisfied by assuming that two or more revolutions have been completed during this interval. Thus, let T denote the periodic time assumed, and the relation between T and a is expressed by

$$T = \frac{2\pi a^{\frac{3}{2}}}{k},$$

in which π denotes the semi-circumference of a circle whose radius

is unity. Let the periodic time corresponding to $\frac{1}{a} + y$ be denoted by $\frac{T}{z}$: then we shall have

$$y = \frac{1}{a} z^{\frac{2}{3}} - \frac{1}{a},$$

and the equations for the residuals are transformed into the form

$$\Delta\theta = (m_0 - n_0 f) + n_0 f z^{\frac{2}{3}}. \quad (91)$$

If we now assign to z , successively, the values 1, 2, 3, &c., the residuals thus obtained will indicate the value of z which best satisfies the series of observations, and hence how many revolutions of the comet have taken place during the interval denoted by T .

122. In the determination of the orbit of a comet from three observed places, a hypothesis in regard to the semi-transverse axis may with facility be introduced simultaneously with the computation of the parabolic elements. The numerical calculation as far as the formation of the equations (52)₃ will be precisely the same for both the parabolic and the elliptic or hyperbolic elements. Then in the one case we find the values of r , r'' , and α which will satisfy equation (56)₃, and in the other case we find those which will satisfy the equation (65), as already explained. From the results thus obtained, the two systems of elements will be computed. Let $f = \frac{1}{a}$, then in the case of the system of parabolic elements we have $f = 0$, and the comparison of the middle place with these and also with the elliptic or hyperbolic elements will give the value of

$$\frac{d\theta}{df} = \frac{\theta_2 - \theta_1}{f},$$

in which θ_1 denotes the geocentric spherical co-ordinate computed from the parabolic elements, and θ_2 that computed from the other system of elements. Further, let $\Delta\theta$ denote the difference between computation and observation for the middle place, and the correction to be applied to f , in order that the computed and the observed values of θ may agree, will be given by

$$\frac{d\theta}{df} \Delta f + \Delta\theta = 0.$$

Hence, the two observed spherical co-ordinates for the middle place will give two equations of condition from which Δf may be found,

and the corresponding elements will be those which best represent the observations, assuming the adopted value of M to be correct.

123. The first determination of the approximate elements of the orbit of a comet is most readily effected by adopting the ecliptic as the fundamental plane. In the subsequent correction of these elements, by varying $\frac{1}{a}$ and M or Δ , it will often be convenient to use the equator as the fundamental plane, and the first assumption in regard to M will be made by means of the values of the distances given by the approximate elements already known. But if it be desired to compute M directly from three observed places in reference to the equator, without converting the right ascensions and declinations into longitudes and latitudes, the requisite formulæ may be derived by a process entirely analogous to that employed when the curtate distances refer to the ecliptic. The case may occur in which only the right ascension for the middle place is given, so that the corresponding longitude cannot be found. It will then be necessary to adopt the equator as the fundamental plane in determining a system of parabolic elements by means of two complete observations and this incomplete middle place. If we substitute the expressions for the heliocentric co-ordinates in reference to the equator in the equations (4)₃ and (5)₃, we shall have

$$\begin{aligned} 0 &= n(\rho \cos \alpha - R \cos D \cos A) - (\rho' \cos \alpha' - R' \cos D' \cos A') \\ &\quad + n''(\rho'' \sin \alpha'' - R'' \cos D'' \cos A''), \\ 0 &= n(\rho \sin \alpha - R \cos D \sin A) - (\rho' \sin \alpha' - R' \cos D' \sin A') \\ &\quad + n''(\rho'' \sin \alpha'' - R'' \cos D'' \sin A''), \\ 0 &= n(\rho \tan \delta - R \sin D) - (\rho' \tan \delta' - R' \sin D') \\ &\quad + n''(\rho'' \tan \delta'' - R'' \sin D''), \end{aligned} \quad (92)$$

in which ρ, ρ', ρ'' denote the curtate distances with respect to the equator, A, A', A'' the right ascensions of the sun, and D, D', D'' its declinations. These equations correspond to (6)₃, and may be treated in a similar manner.

From the first and second of equations (92) we get

$$0 = n(\rho \sin(\alpha' - \alpha) - R \cos D \sin(\alpha' - A)) + R' \cos D' \sin(\alpha' - A') \\ - n''(\rho'' \sin(\alpha'' - \alpha') + R'' \cos D'' \sin(\alpha' - A'')),$$

and hence

$$M = \frac{\rho''}{\rho} = \frac{n}{n''} \cdot \frac{\sin(\alpha' - \alpha)}{\sin(\alpha'' - \alpha')} \quad (93)$$

$$\frac{n R \cos D \sin(\alpha' - A) - R' \cos D' \sin(\alpha' - A') + n'' R'' \cos D'' \sin(\alpha' - A'')}{\rho n'' \sin(\alpha'' - \alpha')}.$$

This formula, being independent of the declination δ' , may be used to compute M when only the right ascension for the middle place is given. For the first assumption in the case of an unknown orbit, we take

$$M = \frac{t'' - t'}{t' - t} \cdot \frac{\sin(a' - a)}{\sin(a'' - a')},$$

and, by means of the results obtained from this hypothesis, the complete expression (93) may be computed. By a process identical with that employed in deriving the equation (36)₃, we derive, from (93), the expression

$$\begin{aligned} \rho'' = \rho \frac{n}{n''} \cdot \frac{\sin(a' - a)}{\sin(a'' - a')} \\ - \frac{1}{6} \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right) \frac{R' \cos D' \sin(a' - A')}{\sin(a'' - a')}; \end{aligned} \quad (94)$$

and, putting

$$\begin{aligned} M_0 = \frac{n}{n''} \cdot \frac{\sin(a' - a)}{\sin(a'' - a')}, \\ F = 1 - \frac{1}{6} \frac{n''}{n} \cdot \frac{\tau\tau'}{\tau''} (\tau' + \tau'') \frac{\cos D' \sin(a' - A')}{\sin(a' - a)} \cdot \frac{R'}{\rho} \left(\frac{1}{r'^3} - \frac{1}{R'^3} \right), \end{aligned}$$

we have

$$M = \frac{\rho''}{\rho} = M_0 F. \quad (95)$$

The calculation of the auxiliary quantities in the equations (52)₃ will be effected by means of the formulæ (96)₃, (86), (87), (102)₃, and (51)₃. The heliocentric places for the times t and t'' will be given by (106)₃ and (107)₃, and from these the elements of the orbit will be found according to the process already illustrated.

124. The methods already given for the correction of the approximate elements of the orbit of a heavenly body by means of additional observations or normal places, are those which will generally be applied. There are, however, modifications of these which may be advantageous in rare and special cases, and which will readily suggest themselves. Thus, if it be desired to correct approximate elements by varying two radii-vectores r and r'' , we may assume an approximate value of each of these, and the three equations (88)₁ will contain only the three unknown quantities A , b , and l . By elimination, these unknown quantities may be found, and in like manner the

values of d'' , b'' , and l'' . It will be most convenient to compute the angles ψ and ψ'' , and then find z and z'' from

$$\sin z = \frac{R \sin \psi}{r}, \quad \sin z'' = \frac{R'' \sin \psi''}{r''},$$

or, putting $x^2 = r^2 - R^2 \sin^2 \psi$, and $x''^2 = r''^2 - R''^2 \sin^2 \psi''$, from

$$\tan z = \frac{R \sin \psi}{x}, \quad \tan z'' = \frac{R'' \sin \psi''}{x''}.$$

The curtate distances will be given by the equations (3), and the heliocentric spherical co-ordinates by means of (4), writing r in place of a . From these $u'' - u$ may be found, and by means of the values of r , r'' , and $u'' - u$ the determination of the elements of the orbit may be completed. Then, assigning to r an increment δr , we compute a second system of elements, and from r and $r'' + \delta r''$ a third system. The comparison of these three systems of elements with an additional or intermediate observed place will furnish the equations for the determination of the corrections Δr and $\Delta r''$ to be applied to r and r'' , respectively. The comparison of the middle place may be made with the observed geocentric spherical co-ordinates directly, or with the radius-vector and argument of the latitude computed directly from the observed co-ordinates; and in the same manner any number of additional observed places may be employed in forming the equations of condition for the determination of Δr and $\Delta r''$.

Instead of r and r'' , we may take the projections of these radii-vectores on the plane of the ecliptic as the quantities to be corrected. Let these projected distances of the body from the sun be denoted by r_0 and r_0'' , respectively; then, by means of the equations (88)₁, we obtain

$$\sin(l - \lambda) = \frac{R \sin(\lambda - \odot)}{r_0}, \quad (96)$$

from which l may be found; and in a similar manner we may find l'' . If we put

$$x_0^2 = r_0^2 - R^2 \sin^2(\lambda - \odot),$$

we have

$$\tan(l - \lambda) = \frac{R \sin(\lambda - \odot)}{x_0}. \quad (97)$$

Let S denote the angle at the sun between the earth and the place of the planet or comet projected on the plane of the ecliptic; then we shall have

$$S = 180^\circ + \odot - l, \quad (98)$$

$$\rho = \frac{R \sin(l - \odot)}{\sin(l - \lambda)},$$

and

$$\tan b = \frac{\rho \tan \beta}{r_0}, \quad (99)$$

by means of which the heliocentric latitudes b and b'' may be found. The calculation of the elements and the correction of r_0 and r_0'' are then effected as in the case of the variation of r and r'' .

In the case of parabolic motion, the eccentricity being known, we may take q and T as the quantities to be corrected. If we assume approximate values of these elements, r , r' , r'' , and v , v' , v'' will be given immediately. Then from r , r' , r'' and the observed spherical co-ordinates of the body we may compute the values of $u'' - u'$ and $u' - u$. In the same manner, by means of the observed places, we compute the angles $u'' - u'$ and $u' - u$ corresponding to $q + \delta q$ and T , and to q and $T + \delta T$, δq and δT denoting the arbitrary increments assigned to q and T , respectively. The comparison of the heliocentric motion, during the intervals $t'' - t'$ and $t' - t$, thus obtained, in the case of each of the three systems of elements, from the observed geocentric places with the corresponding results given by

$$u'' - u' = v'' - v', \quad u' - u = v' - v,$$

enables us to form the equations by which we may find the corrections Δq and ΔT to be applied to the assumed values of q and T , respectively, in order that the values of $u'' - u'$ and $u' - u$ computed by means of the observed places shall agree with those given by the true anomalies computed directly from q and T .

CHAPTER VII.

METHOD OF LEAST SQUARES, THEORY OF THE COMBINATION OF OBSERVATIONS, AND DETERMINATION OF THE MOST PROBABLE SYSTEM OF ELEMENTS FROM A SERIES OF OBSERVATIONS.

125. WHEN the elements of the orbit of a heavenly body are known to such a degree of approximation that the squares and products of the corrections which should be applied to them may be neglected, by computing the partial differential coefficients of these elements with respect to each of the observed spherical co-ordinates, we may form, by means of the differences between computation and observation, the equations for the determination of these corrections. Three complete observations will furnish the six equations required for the determination of the corrections to be applied to the six elements of the orbit; but, if more than three complete places are given, the number of equations will exceed the number of unknown quantities, and the problem will be more than determinate. If the observed places were absolutely exact, the combination of the equations of condition in any manner whatever would furnish the values of these corrections, such that each of these equations would be completely satisfied. The conditions, however, which present themselves in the actual correction of the elements of the orbit of a heavenly body by means of given observed places, are entirely different. When the observations have been corrected for all known instrumental errors, and when all other known corrections have been duly applied, there still remain those accidental errors which arise from various causes, such as the abnormal condition of the atmosphere, the imperfections of vision, and the imperfections in the performance of the instrument employed. These accidental and irregular errors of observation cannot be eliminated from the observed data, and the equations of condition for the determination of the corrections to be applied to the elements of an approximate orbit cannot be completely satisfied by any system of values assigned to the unknown quantities unless the number of equations is the same as the number of these unknown quantities. It becomes an important problem, therefore, to determine the particular combination of these equations of condition, by means of which

the resulting values of the unknown quantities will be those which, while they do not completely satisfy the several equations, will afford the highest degree of probability in favor of their accuracy. It will be of interest also to determine, as far as it may be possible, the degree of accuracy which may be attributed to the separate results. But, in order to simplify the more general problem, in which the quantities sought are determined indirectly by observation, it will be expedient to consider first the simpler case, in which a single quantity is obtained directly by observation.

126. If the accidental errors of observation could be obviated, the different determinations of a magnitude directly by observation would be identical; but since this is impossible when an extreme limit of precision is sought, we adopt a *mean* or average value to be derived from the separate results obtained. The adopted value may or may not agree with any individual result, since it is only necessary that the residuals obtained by comparing the adopted value with the observed values shall be such as to make this adopted value the *most probable* value. It is evident, from the very nature of the case, that we approach here the confines of the unknown, and, before we proceed further, something additional must be assumed.

However irregular and uncertain the law of the accidental errors of observation may be, we may at least assume that small errors are more probable than large errors, and that errors surpassing a certain limit will not occur. We may also assume that in the case of a large number of observations, errors in excess will occur as frequently as errors in defect, so that, in general, positive and negative residuals of equal absolute value are equally probable. It appears, therefore, that the relative frequency of the occurrence of an accidental error Δ in the observed value will depend on the magnitude of this error, and may be expressed by $\varphi(\Delta)$. This function will also express the probability of an error Δ in an observed value. At the limit beyond which an error of the magnitude Δ can never occur, we must have $\varphi(\Delta) = 0$: when $\Delta = 0$, the value of $\varphi(\Delta)$ must be a maximum, and for equal positive and negative values of Δ the values of $\varphi(\Delta)$ must be the same. Hence, in a given series of observations, the number m of observations being supposed to be large, the number of times in which the error Δ occurs will be expressed by $m\varphi(\Delta)$, and the number of times in which the error Δ' occurs will be expressed by $m\varphi(\Delta')$, so that we shall have

$$m = m\varphi(\Delta) + m\varphi(\Delta') + m\varphi(\Delta'') + \&c.,$$

or

$$\Sigma \varphi(\Delta) = 1.$$

The sum Σ must be taken between the limits for which the accidental errors of observation are considered possible; but since the assignment of these limits is, in a certain sense, arbitrary, we must evidently have

$$\sum_{\Delta = -\infty}^{\Delta = +\infty} \varphi(\Delta) = 1, \quad (1)$$

the value of $\varphi(\Delta)$ being absolutely zero for the limits $+\infty$ and $-\infty$.

Within any given limits there are an infinite number of values, any one of which may possibly be the true value of Δ , and hence the number of the functions expressed by $\varphi(\Delta)$ must be infinite. The probability of an error Δ is expressed by $\varphi(\Delta)$, and will be the same as the probability that the error is contained within the limits Δ and $\Delta + d\Delta$. The latter is expressed by the sum of all the functions $\varphi(\Delta)$ between the limits Δ and $\Delta + d\Delta$, or by

$$\varphi(\Delta) d\Delta.$$

We conclude, therefore, that the probability that an error falls between the limits a and b is expressed by the integral

$$\int_a^b \varphi(\Delta) d\Delta,$$

and this integral, taken so as to include all possible accidental errors of observation, is, according to equation (1),

$$\int_{-\infty}^{+\infty} \varphi(\Delta) d\Delta = 1. \quad (2)$$

According to the theory of probabilities, the probability that the errors Δ , Δ' , &c. occur simultaneously is equal to the continued product of the probabilities of the occurrence of these errors separately. Let P denote the probability that these errors occur at the same time in the given series of observed values, and we have

$$P = \varphi(\Delta) \cdot \varphi(\Delta') \cdot \varphi(\Delta'') \dots \quad (3)$$

The most probable value of the quantity sought, which we will denote by x , must evidently be that which makes P a maximum. If

we take the logarithms of both members of equation (3), and differentiate, the condition of a maximum gives

$$0 = \frac{d \log \varphi(\Delta)}{d\Delta} \cdot \frac{d\Delta}{dx} + \frac{d \log \varphi(\Delta')}{d\Delta'} \cdot \frac{d\Delta'}{dx} + \&c. \quad (4)$$

Let $n, n', n'', \&c.$ be the observed values of x , and m the number of observations; then we have

$$\Delta = n - x, \quad \Delta' = n' - x, \quad \Delta'' = n'' - x, \&c.,$$

and hence

$$\frac{d\Delta}{dx} = \frac{d\Delta'}{dx} = \frac{d\Delta''}{dx} \dots = -1.$$

Therefore the equation (4) becomes

$$0 = \frac{d \log \varphi(n - x)}{d(n - x)} + \frac{d \log \varphi(n' - x)}{d(n' - x)} + \&c. \quad (5)$$

This equation will serve to determine the value of x as soon as the form of the function symbolized by φ is known. It becomes necessary, therefore, to make some further assumption in regard to the errors $\Delta, \Delta', \Delta'', \&c.$, in order that the form of this function may be determined; and, although the hypothesis which presents itself gives directly the most probable value of x , since the function $\varphi(\Delta)$ is supposed to be general, we may thus, by the special case, determine the form of this function; and the result will be applicable when, instead of the value of a single quantity, it is required to find the most probable values of several unknown quantities determined indirectly by observation.

127. The principle may be received as an axiom, that when a series of observed values of a quantity is given, if the circumstances under which the separate observations were made are similar, so that there is no reason for preferring one result to another, the most probable value of the quantity sought is the *arithmetical mean* of the several results. Hence we have

$$x = \frac{n + n' + n'' + \dots}{m},$$

m being the number of observed values. This expression gives

$$0 = (n - x) + (n' - x) + (n'' - x) + \&c., \quad (6)$$

from which it appears that the algebraic sum of the residuals is equal to zero. The equation (5) may be written

$$0 = (n-x) \frac{d \log \varphi (n-x)}{(n-x) d(n-x)} + (n'-x) \frac{d \log \varphi (n'-x)}{(n'-x) d(n'-x)} + \&c.,$$

and the comparison of this with (6) shows that

$$\frac{d \log \varphi (n-x)}{(n-x) d(n-x)} = \frac{d \log \varphi (n'-x)}{(n'-x) d(n'-x)} \dots = k, \quad (7)$$

k being a constant quantity. Hence we derive

$$d \log_e \varphi (\Delta) = k \Delta d\Delta,$$

the integration of which gives

$$\log_e \varphi (\Delta) = \frac{1}{2} k \Delta^2 + \log_e c,$$

$\log_e c$ being the constant of integration. From this equation there results

$$\varphi (\Delta) = c e^{\frac{1}{2} k \Delta^2}, \quad (8)$$

in which e is the base of Napierian logarithms. Since $\varphi (\Delta)$ diminishes as Δ increases, the quantity k must be essentially negative, and if we put $\frac{1}{2} k = -h^2$, we shall have

$$\varphi (\Delta) = c e^{-h^2 \Delta^2}. \quad (9)$$

If we substitute this value of $\varphi (\Delta)$ in the equation (2), we have

$$c \int_{-\infty}^{+\infty} e^{-h^2 \Delta^2} d\Delta = 1,$$

or, putting also $t = h\Delta$,

$$\frac{c}{h} \int_{-\infty}^{+\infty} e^{-t^2} dt = 1. \quad (10)$$

This equation will give the value of the constant c , provided that the value of the integral

$$\int_0^{\infty} e^{-t^2} dt$$

is known. Since the definite integral is independent of the variable, let us multiply it by a similar one, in which y is the variable; so that we have

$$\left(\int_0^{\infty} e^{-t^2} dt \right)^2 = \int_0^{\infty} e^{-t^2} dt \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} \int_0^{\infty} e^{-(t^2 + y^2)} dt dy,$$

in which the order of integration is indifferent. If we put $y = tz$,

we have, since t is regarded as constant in the integration with respect to y ,

$$dy = t dz;$$

and hence

$$\left(\int_0^{\infty} e^{-t^2} dt \right)^2 = \int_0^{\infty} dz \int_0^{\infty} e^{-(1+z^2)t^2} t dt.$$

Then, since we have, in general,

$$\int_0^{\infty} e^{-ax^2} x dx = \frac{1}{2a},$$

the preceding equation gives

$$\left(\int_0^{\infty} e^{-t^2} dt \right)^2 = \int_0^{\infty} \frac{dz}{2(1+z^2)} = \frac{1}{2} [\tan^{-1} z]_{z=0}^{z=\infty} = \frac{1}{2} \pi,$$

in which π denotes the semi-circumference of a circle whose radius is unity. Therefore we have

$$\int_0^{\infty} e^{-t^2} dt = \frac{1}{2} \sqrt{\pi}, \quad (11)$$

and the equation (10) gives

$$c = \frac{h}{\sqrt{\pi}}. \quad (12)$$

Hence, the expression for $\varphi(\Delta)$ becomes

$$\varphi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2 \Delta^2}. \quad (13)$$

The constant h , according to the relation $h^2 = -\frac{1}{2}k$, must depend on the nature of the observations, and will be the same in the case of systems of observations in which the probability of an error Δ is the same. Since $h^2 \Delta^2$ must necessarily be an abstract number, Δ and $\frac{1}{h}$ must be homogeneous.

128. In a given series of observations, the probability that for any observation the error will be within the limits $-\delta$ and $+\delta$ will be expressed by

$$\frac{h}{\sqrt{\pi}} \int_{-\delta}^{+\delta} e^{-h^2 \Delta^2} d\Delta; \quad (14)$$

and in another series of observations, more or less precise, the pro-

bability that the error of an observation is within the limits $-\delta'$ and $+\delta'$ will be

$$\frac{h'}{\sqrt{\pi}} \int_{-\delta'}^{+\delta'} e^{-h'^2 \Delta^2} d\Delta. \quad (15)$$

Since

$$\frac{h}{\sqrt{\pi}} \int_{-\delta}^{+\delta} e^{-h^2 \Delta^2} d\Delta = \frac{1}{\sqrt{\pi}} \int_{-h\delta}^{+h\delta} e^{-h^2 \Delta^2} d(h\Delta),$$

it appears that the integrals (14) and (15) are equal when $h\delta = h'\delta'$. Hence, if we put $h' = 2h$, these integrals will be equal when $\delta = 2\delta'$, and an error of a given magnitude in the first series will have the same probability as an error of half that magnitude in the second series. The second series of observations will therefore be twice as accurate as the first series, and the constant h may be called the *measure of precision* of the observations. The greater the degree of precision of the observations, the greater will be the value of h .

The relative accuracy of two series of observations may also be determined by a comparison of the errors which are committed with equal facility in each series. If we arrange the errors of the several observations in each series in the order of their absolute magnitude without reference to the algebraic sign, the errors which occupy the same position in reference to the extremes in each case will serve to determine the relation sought. We select that, however, which occupies the middle place in the series of errors thus arranged, and since the number of errors which exceed this is the same as the number of errors less than this, if we designate the error which occupies the middle place by r , the probability that an error is within the limits $-r$ and $+r$ will be equal to $\frac{1}{2}$. The probability of an error greater than r being the same as the probability of an error less than r , the error r is called the *probable error*.

The relation between r and h is easily determined. Thus, we have

$$\frac{h}{\sqrt{\pi}} \int_{-r}^{+r} e^{-h^2 \Delta^2} d\Delta = \frac{1}{2},$$

or, putting $h\Delta = t$,

$$\int_0^h e^{-t^2} dt = \frac{\sqrt{\pi}}{4} = 0.44311. \quad (16)$$

If we expand e^{-t^2} into a series of ascending powers of t , multiply by dt , and integrate between the limits 0 and T , we get

$$\int_0^T e^{-t^2} dt = T - \frac{1}{3}T^3 + \frac{1}{5}\frac{T^5}{1 \cdot 2} - \frac{1}{7}\frac{T^7}{1 \cdot 2 \cdot 3} + \frac{1}{9}\frac{T^9}{1 \cdot 2 \cdot 3 \cdot 4} - \&c., \quad (17)$$

which converges rapidly when T is small. To find the value of T which corresponds to the value 0.44311 assigned to the integral, we compute the value of the series (17) for the values 0.45, 0.47, and 0.49 assigned to T , successively, and from the results thus obtained it is easily seen that when the sum of the terms of the series is 0.44311, we have

$$T = hr = 0.47694,$$

or

$$r = \frac{0.47694}{h}, \quad (18)$$

which determines the relation between the probable error and the measure of precision.

The probability that the error of an observation, without regard to sign, does not exceed nr , is expressed by

$$\frac{2}{\sqrt{\pi}} \int_0^{nhr} e^{-t^2} dt, \quad (19)$$

and this integral, therefore, indicates the ratio of the number of observations affected with an error which does not exceed nr to the whole number of observations. Hence, if we assign different values to n , the integral (19) computed for the several assumed values of

$$nhr = 0.47694n$$

will give the relative number of errors of a given magnitude. Thus, if we put $n = \frac{1}{2}$, we obtain

$$\frac{2}{\sqrt{\pi}} \int_0^{0.2385} e^{-t^2} dt = 0.264.$$

from which it appears that in a series of 1000 observations there ought to be 264 observations in which the error does not exceed $\frac{1}{2}r$. It has been found, in this manner, that in the case of an extended series of observations the number of errors of a given magnitude assigned by theory agrees very closely with that actually given by the series of observations; and hence we conclude that the error committed in extending the limits of the summation in the expression (1) to $-\infty$ and $+\infty$, instead of the finite limits which it is presumed that the actual errors cannot exceed, is very slight, so that the form

of the function $\varphi(\Delta)$ which has been derived may be regarded as that which best satisfies all the conditions of the problem.

129. The relative accuracy of different series of observations may also be indicated by means of what are called the *mean error* and the *mean of the errors* for each series, the former being the error whose square is equal to the mean of the squares of all the errors of the series, and the latter the mean of these errors without reference to their algebraic sign.

Let ϵ denote the mean error; then, since the number of observations having the error Δ is $m\varphi(\Delta)$, we shall have, according to the definition,

$$\epsilon^2 = \frac{\Delta^2 m\varphi(\Delta) + \Delta'^2 m\varphi(\Delta') + \&c.}{m} = \Sigma \Delta^2 \varphi(\Delta).$$

But the number of possible errors being infinite, the probability of an error Δ is expressed by $\varphi(\Delta) d\Delta$, and we have

$$\epsilon^2 = \int_{-\infty}^{+\infty} \Delta^2 \varphi(\Delta) d\Delta = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-h^2 \Delta^2} \Delta^2 d\Delta,$$

which gives

$$\epsilon^2 = \frac{1}{2h^2}. \quad (20)$$

Hence, by means of (18), we have

$$\begin{aligned} \epsilon &= \frac{1}{h\sqrt{2}} = 1.4826r, \\ r &= 0.67449\epsilon, \end{aligned} \quad (21)$$

which determine the relation between ϵ and r .

Let η denote the mean of the errors, and we shall have

$$\eta = \int_0^{\infty} \Delta \varphi(\Delta) d\Delta = \frac{2h}{\sqrt{\pi}} \int_0^{\infty} e^{-h^2 \Delta^2} \Delta d\Delta,$$

which gives

$$\eta = \frac{1}{h\sqrt{\pi}}. \quad (22)$$

Therefore, we have

$$\begin{aligned} \eta &= 1.1829r, \\ r &= 0.8453\eta, \end{aligned} \quad (23)$$

for the relation between r and η .

130. Let us denote by $v, v', v'',$ &c. the differences between any assumed value of x and the observed values for a given series of observations, the number of observations being denoted by m ; then if we put

$$[vv] = v^2 + v'^2 + v''^2 + \&c., \quad (24)$$

and similarly in the case of the sum of any other series of similar terms, we shall have for the probability of the value $x,$

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2 [vv]}, \quad (25)$$

and this probability will be a maximum when $[vv]$ is a minimum. Now we have

$$v = n - x, \quad v' = n' - x, \quad v'' = n'' - x, \quad \&c.,$$

$n, n', n'',$ &c. being the observed values of x , and hence

$$\begin{aligned} [vv] &= [nn] - 2[n]x + mx^2 \\ &= [nn] - \frac{[n]^2}{m} + m\left(x - \frac{[n]}{m}\right)^2. \end{aligned}$$

It appears, therefore, that $[vv]$ will be a minimum when

$$x = \frac{[n]}{m}, \quad (26)$$

and this is a necessary consequence of the assumption that the arithmetical mean of the observations gives the most probable value of x , according to which the form of the function $\varphi(\Delta)$ was derived. But although the arithmetical mean is the most probable value, yet we cannot affirm that this is the exact value, so long as the number of observations is finite. It becomes important, therefore, to determine the degree of precision of the arithmetical mean.

Let x_0 denote the most probable value of x , for which the residuals are $v, v', v'',$ &c., and let $x_0 + \delta$ be any other value of x . Then, since we may put

$$[v] = v + v' + v'' + \dots = 0,$$

and

$$[vv] = m\varepsilon^2,$$

the probability of the value $x_0 + \delta$ will be

$$P' = \frac{h^m}{\sqrt{\pi^m}} e^{-mh^2(\varepsilon^2 + \delta^2)}.$$

The probability that the error of the arithmetical mean is zero is indicated by

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-m h^2 \epsilon_0^2},$$

and we have

$$P' = P e^{-m h^2 \delta^2}.$$

In the case of a single observation, if P denotes the probability of the error zero, and P' the probability of the error δ , we have

$$P' = P e^{-h^2 \delta^2}.$$

Hence it appears that if h_0 denotes the measure of precision of the arithmetical mean of m observations, the relation between h_0 and h , the measure of precision of an observation, is given by

$$h_0^2 = m h^2; \quad (27)$$

and if r_0 is the probable error of the arithmetical mean, and ϵ_0 its mean error, we have, according to the equations (18) and (20),

$$\begin{aligned} r_0 &= \frac{r}{\sqrt{m}}, \\ \epsilon_0 &= \frac{\epsilon}{\sqrt{m}}. \end{aligned} \quad (28)$$

These expressions determine the probable and the mean error of the arithmetical mean of a number of observations when these errors in the case of a single observation are known.

131. The expressions for the relation between the mean and probable errors have been derived for the case of a very large number of observations, a number so great that the error of the arithmetical mean becomes equal to zero. In the case of a limited number of observed values of x , the residuals given by comparing the arithmetical mean with the several observations will not, in general, give the true errors of the observations; but the greater the number of observations, the nearer will these residuals approach the absolute errors. If Δ , Δ' , Δ'' , &c. are the actual errors of the observations, and v , v' , v'' , &c. those which result from the most probable value of x , we shall have, denoting the arithmetical mean by x_0 , and the true value by $x_0 + \delta$,

$$\Delta = v - \delta, \quad \Delta' = v' - \delta, \quad \Delta'' = v'' - \delta, \text{ \&c.};$$

and hence

$$m\varepsilon^2 = [\Delta\Delta] = [vv] + m\delta^2. \quad (29)$$

This equation will enable us to determine the mean error of an observation when δ is given; but, since this is necessarily unknown, some assumption in regard to its value must be made. If we assume it to be equal to the mean error of the arithmetical mean, the remaining error will be wholly insensible, and hence the equation (29) becomes

$$m\varepsilon^2 = [vv] + m\varepsilon_0^2 = [vv] + \varepsilon^2.$$

Therefore, we shall have

$$\varepsilon = \sqrt{\frac{[vv]}{m-1}}, \quad (30)$$

and, according to (21),

$$r = 0.6745 \sqrt{\frac{[vv]}{m-1}}. \quad (31)$$

These equations give the values of the mean and probable errors of a single observation in terms of the actual residuals found by comparing the arithmetical mean with the several observed values.

The probable and the mean error of the arithmetical mean will be given by

$$\begin{aligned} \varepsilon_0 &= \sqrt{\frac{[vv]}{m(m-1)}}, \\ r_0 &= 0.6745 \sqrt{\frac{[vv]}{m(m-1)}}. \end{aligned} \quad (32)$$

When the number of observations is very large, the probable error of an observation and also that of the arithmetical mean may be determined by means of the mean of the errors. If we suppose the number of positive errors to be the same as the number of negative errors, the mean of the errors without reference to the algebraic sign gives

$$\eta = \frac{[v]}{m},$$

and hence we have, according to (23),

$$r = 0.8453 \frac{[v]}{m}. \quad (33)$$

For the mean error of an observation we have

$$\varepsilon = \eta \sqrt{\frac{2}{\pi}} = 1.2533 \frac{[v]}{m}. \quad (34)$$

If the number of observations is very great, the results given by these equations will agree with those given by (30) and (31); but for any limited series of observed values, the results obtained by means of the mean error will afford the greatest accuracy.

132. The relative accuracy of two or more observed values of a quantity may be expressed by means of what are called their *weights*. If the observations are made under precisely similar circumstances, so that there is no reason for preferring one to the other, they are said to have the same weight. The weight must therefore depend on the measure of precision of the observations, and hence on their probable errors. The unit of the weight is entirely arbitrary, since only the relative weights are required, and if we denote the weight by p , the value of p indicates the number of observations of equal accuracy which must be combined in order that their arithmetical mean may have the same degree of precision as the observation whose weight is p . Hence, if the weight of a single observation is 1, the arithmetical mean of m such observations will have the weight m . Let the probable error of an observation of the weight unity be denoted by r , and the probable error of that whose weight is p' by r' ; then, according to the first of equations (28), we shall have

$$r' = \frac{r}{\sqrt{p'}},$$

or

$$r^2 = p' r'^2.$$

For the case of an observation whose weight is p'' and whose probable error is r'' , we have

$$r^2 = p'' r''^2 = p' r'^2,$$

from which it appears that *the weights of two observations are to each other inversely as the squares of their probable or mean errors, and, according to (18), directly as the squares of their measures of precision.*

Let us now consider two values of x , which may be designated by x' and x'' , the mean errors of these values being, respectively, ϵ' and ϵ'' ; then, if we put

$$X = x' \pm x''$$

and suppose that both x' and x'' have been derived from a large number m of observations (and the same number in each case), so that the residuals $v, v', v'', \&c.$ in the case of x' and the residuals $v, v', v'', \&c.$ in the case of x'' may be regarded as the actual errors of obser-

vation, the errors of the value of X , as determined from the several observations, will be

$$v \pm v_1, \quad v' \pm v'_1, \quad v'' \pm v''_1, \text{ \&c.}$$

Let the mean error of X be denoted by E ; then we have

$$mE^2 = \Sigma (v \pm v_1)^2 = [vv] \pm 2[vv_1] + [v_1v_1];$$

and since the number of observed values is supposed to be so great that the frequency of negative products vv , is the same as that of the similar positive products, so that $[vv] = 0$, this equation gives

$$mE^2 = m\varepsilon'^2 + m\varepsilon''^2,$$

or

$$E^2 = \varepsilon'^2 + \varepsilon''^2.$$

Combining X with a third value x''' whose mean error is ε''' , the mean error of $x' \pm x'' \pm x'''$ will be found in the same manner to be equal to $\varepsilon'^2 + \varepsilon''^2 + \varepsilon'''^2$; and hence we have, for the algebraic sum of any number of separate values,

$$E = \sqrt{\varepsilon^2 + \varepsilon'^2 + \varepsilon''^2 + \text{\&c.}}, \quad (35)$$

and, according to the last of equations (21),

$$R = \sqrt{r^2 + r'^2 + r''^2 + \text{\&c.}}, \quad (36)$$

R being the probable error of the algebraic sum. If the probable errors of the several values are the same, we have

$$r = r' = r'' = \text{\&c.}$$

and the probable error of the sum of m values will be given by

$$R = r\sqrt{m}.$$

Hence the probable error of the arithmetical mean of m observed values will be

$$r_0 = \frac{R}{m} = \frac{r}{\sqrt{m}},$$

which agrees with the first of equations (28).

Let P denote the weight of the sum X , p' the weight of x' , and p'' that of x'' ; then we shall have

$$\frac{p'}{P} = \frac{r'^2 + r''^2}{r'^2}, \quad \frac{p''}{P} = \frac{r'^2 + r''^2}{r''^2},$$

from which we get

$$P = \frac{p'p''}{p' + p''} \quad (37)$$

Since the unit of weight is arbitrary, we may take

$$p' = \frac{1}{r'^2}, \quad p'' = \frac{1}{r''^2}, \text{ \&c. ;}$$

and hence we have, for the weight of the algebraic sum of any number of values,

$$P = \frac{1}{R^2} = \frac{1}{r'^2 + r''^2 + r'''^2 + \text{\&c.}} \quad (38)$$

or, whatever may be the unit of weight adopted,

$$P = \frac{1}{\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''} + \dots} \quad (39)$$

In the case of a series of observed values of a quantity, if we designate by r' the probable error of a residual found by comparing the arithmetical mean with an observed value, by r the probable error of the observation, by x_0 the arithmetical mean, and by n any observed value, the probable error of

$$n = x_0 + v,$$

according to (36), will be

$$r^2 = r_0^2 + r'^2 = \frac{r_0^2}{m} + r'^2,$$

r_0 being the probable error of the arithmetical mean. Hence we derive

$$r = r' \sqrt{\frac{m}{m-1}};$$

and if we adopt the value

$$r' = 0.8453 \frac{[v]}{m},$$

the expression for the probable error of an observation becomes

$$r = 0.8453 \frac{[v]}{\sqrt{m(m-1)}}, \quad (40)$$

in which $[v]$ denotes the sum of the residuals regarded as positive, and m the number of observations.

133. Let $n, n', n'', \text{\&c.}$ denote the observed values of x , and let $p, p', p'', \text{\&c.}$ be their respective weights; then, according to the defi-

tion of the weight, the value n may be regarded as the arithmetical mean of p observations whose weight is unity, and the same is true in the case of n' , n'' , &c. We thus resolve the given values into $p + p' + p'' + \dots$ observations of the weight unity, and the arithmetical mean of all these gives, for the most probable value of x ,

$$x_0 = \frac{pn + p'n' + p''n'' + \&c.}{p + p' + p'' + \&c.} = \frac{[pn]}{[p]}. \quad (41)$$

The unit of weight being entirely arbitrary, it is evident that the relation given by this equation is correct as well when the quantities p , p' , p'' , &c. are fractional as when they are whole numbers. The weight of x_0 as determined by (41) is expressed by the sum

$$p + p' + p'' + p''' + \&c.,$$

and the probable error of x_0 is given by

$$r_0 = \frac{r}{\sqrt{p + p' + p'' + \dots}} = \frac{r}{\sqrt{[p]}}, \quad (42)$$

when r , denotes the probable error of an observation whose weight is unity. The value of r , must be found by means of the observations themselves. Thus, there will be p residuals expressed by $n - x_0$, p' residuals expressed by $n' - x_0$, and similarly in the case of n'' , n''' , &c. Hence, according to equation (31), we shall have

$$r = 0.6745 \sqrt{\frac{[pvv]}{m-1}}, \quad (43)$$

in which m denotes the number of values to be combined, or the number of quantities n , n' , n'' , &c. For the mean error of x_0 , we have the equations

$$\begin{aligned} \epsilon_1 &= \sqrt{\frac{[pvv]}{m-1}}, \\ \epsilon_0 &= \frac{\epsilon_1}{\sqrt{[p]}} = \sqrt{\frac{[pvv]}{(m-1)[p]}}. \end{aligned} \quad (44)$$

If different determinations of the quantity x are given, for which the probable errors are r , r' , r'' , &c., the reciprocals of the squares of these probable errors may be taken as the weights of the respective values n , n' , n'' , &c., and we shall have

$$x_0 = \frac{\frac{n}{r^2} + \frac{n'}{r'^2} + \frac{n''}{r''^2} + \dots}{\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{r''^2} + \dots}, \quad (45)$$

with the probable error

$$r_0 = \frac{1}{\sqrt{\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{r''^2} + \dots}}. \quad (46)$$

The mean errors may be used in these equations instead of the probable errors.

134. The results thus obtained for the case of the direct observation of the quantity sought, are applicable to the determination of the conditions for finding the most probable values of several unknown quantities when only a certain function of these quantities is directly observed. In the actual application of the formulæ it will always be possible to reduce the problem to the case in which the quantity observed is a linear function of the quantities sought. Thus, let V be the quantity observed, and ξ, η, ζ , &c. the unknown quantities to be determined, so that we have

$$V = f(\xi, \eta, \zeta, \dots).$$

Let ξ_0, η_0, ζ_0 , &c. be approximate values of these quantities supposed to be already known by means of previous calculation, and let x, y, z , &c. denote, respectively, the corrections which must be applied to these approximate values in order to obtain their true values. Then, if we suppose that the previous approximation is so close that the squares and products of the several corrections may be neglected, we have

$$V - V_0 = \frac{dV}{d\xi} x + \frac{dV}{d\eta} y + \frac{dV}{d\zeta} z + \dots,$$

and thus the equation is reduced to a linear form. Hence, in general, if we denote by n the difference between the computed and the observed value of the function, and similarly in the case of each observation employed, the equations to be solved are of the following form:—

$$\begin{aligned} ax + by + cz + du + ew + ft + n &= 0, \\ a'x + b'y + c'z + d'u + e'w + f't + n' &= 0, \\ a''x + b''y + c''z + d''u + e''w + f''t + n'' &= 0, \\ &\text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{aligned} \quad (47)$$

which may be extended so as to include any number of unknown quantities. If the number of equations is the same as the number of unknown quantities, the resulting values of these will exactly satisfy the several equations; but if the number of equations exceeds the number of unknown quantities, there will not be any system of

values for these which will reduce the second members absolutely to zero, and we can only determine the values for which the errors for the several equations, which may be denoted by $v, v', v'', \&c.$, will be those which we may regard as belonging to the most probable values of the unknown quantities.

Let $\Delta, \Delta', \Delta'', \&c.$ be the actual errors of the observed quantities; then the probability that these occur in the case of the observations used in forming the equations of condition, will be expressed by

$$P = \varphi(\Delta) \cdot \varphi(\Delta') \cdot \varphi(\Delta'') \dots,$$

and the most probable values of the unknown quantities will be those which make P a maximum. The form of the function $\varphi(\Delta)$ has been already found to be

$$\varphi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2 \Delta^2},$$

and hence we shall have

$$P = \frac{hh'h'' \dots}{\sqrt{\pi^m}} e^{-(h^2 \Delta^2 + h'^2 \Delta'^2 + h''^2 \Delta''^2 + \&c.)},$$

m being the number of observations or equations of condition. In order that P may be a maximum, the value of

$$h^2 \Delta^2 + h'^2 \Delta'^2 + h''^2 \Delta''^2 + \&c.$$

must be a minimum. If the observations are equally good, the expression for P becomes

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2 (\Delta^2 + \Delta'^2 + \Delta''^2 + \&c.)},$$

and the condition of a maximum probability requires that

$$\Delta^2 + \Delta'^2 + \Delta''^2 + \&c.$$

shall be a minimum. Hence it appears that when the observations are equally precise, the most probable values of the unknown quantities are those which render the sum of the squares of the residuals a minimum, and that, in general, if each error is multiplied by its measure of precision, the sum of the squares of the products thus formed must be a minimum.

If we denote the actual residuals by $v, v', v'', \&c.$, and regard the observations as having the same measure of precision, the condition that the sum of their squares shall be a minimum gives

$$\frac{d[vv]}{dx} = 0, \quad \frac{d[vv]}{dy} = 0, \quad \frac{d[vv]}{dz} = 0, \quad \&c.,$$

or

$$\begin{aligned}
 v \frac{dv}{dx} + v' \frac{dv'}{dx} + v'' \frac{dv''}{dx} + \dots &= 0, \\
 v \frac{dv}{dy} + v' \frac{dv'}{dy} + v'' \frac{dv''}{dy} + \dots &= 0, \\
 v \frac{dv}{dz} + v' \frac{dv'}{dz} + v'' \frac{dv''}{dz} + \dots &= 0, \\
 \&c. &\quad \&c.
 \end{aligned}
 \tag{48}$$

If we differentiate the equations

$$\begin{aligned}
 ax + by + cz + du + ew + ft + n &= v, \\
 a'x + b'y + c'z + d'u + e'w + f't + n' &= v', \\
 a''x + b''y + c''z + d''u + e''w + f''t + n'' &= v'', \\
 \&c. &\quad \&c.
 \end{aligned}
 \tag{49}$$

with respect to $x, y, z, \&c.$, successively, we obtain

$$\begin{aligned}
 \frac{dv}{dx} &= a, & \frac{dv'}{dx} &= a', & \frac{dv''}{dx} &= a'', \&c. \\
 \frac{dv}{dy} &= b, & \frac{dv'}{dy} &= b', & \frac{dv''}{dy} &= b'', \&c. \\
 \&c. & & \&c. & & \&c.
 \end{aligned}
 \tag{50}$$

Introducing these values into the equations (48), and substituting for $v, v', v'', \&c.$ their values given by (49), we get

$$\begin{aligned}
 [aa]x + [ab]y + [ac]z + [ad]u + [ae]w + [af]t + [an] &= 0, \\
 [ab]x + [bb]y + [bc]z + [bd]u + [be]w + [bf]t + [bn] &= 0, \\
 [ac]x + [bc]y + [cc]z + [cd]u + [ce]w + [cf]t + [cn] &= 0, \\
 [ad]x + [bd]y + [cd]z + [dd]u + [de]w + [df]t + [dn] &= 0, \\
 [ae]x + [be]y + [ce]z + [de]u + [ee]w + [ef]t + [en] &= 0, \\
 [af]x + [bf]y + [cf]z + [df]u + [ef]w + [ff]t + [fn] &= 0,
 \end{aligned}
 \tag{51}$$

in which

$$\begin{aligned}
 [aa] &= aa + a'a' + a''a'' + \dots \\
 [ab] &= ab + a'b' + a''b'' + \dots \\
 [ac] &= ac + a'c' + a''c'' + \dots \\
 [bb] &= bb + b'b' + b''b'' + \dots \\
 \&c. &\quad \&c.
 \end{aligned}
 \tag{52}$$

The equations of condition are thus reduced to the same number as the number of the unknown quantities, and the solution of these will give the values for which the sum of the squares of the residuals will be a minimum. These final equations are called *normal equations*.

When the observations are not equally precise, in accordance with the condition that $h^2v^2 + h'^2v'^2 + h''^2v''^2 + \&c.$ shall be a minimum,

each equation of condition must be multiplied by the measure of precision of the observation; or, since the weight is proportional to the square of the measure of precision, each equation of condition must be multiplied by the square root of the weight of the observation, and the several equations of condition, being thus reduced to the same unit of weight, must be combined as indicated by the equations (51).

135. It will be observed that the formation of the first normal equation is effected by multiplying each equation of condition by the coefficient of x in that equation and then taking the sum of all the equations thus formed. The second normal equation is obtained in the same manner by multiplying by the coefficient of y ; and thus by multiplying by the coefficient of each of the unknown quantities the several normal equations are formed. These equations will generally give, by elimination, a system of determinate values of the unknown quantities x, y, z , &c. But if one of the normal equations may be derived from one of the others by multiplying it by a constant, or if one of the equations may be derived by a combination of two or more of the remaining equations, the number of distinct relations will be less than the number of unknown quantities, and the problem will thus become indeterminate. In this case an unknown quantity may be expressed in the form of a linear function of one or more of the other unknown quantities. Thus, if the number of independent equations is one less than the number of unknown quantities, the final expressions for all of these quantities except one, will be of the form

$$x = \alpha + \beta t, \quad y = \alpha' + \beta' t, \quad z = \alpha'' + \beta'' t, \text{ \&c.} \quad (53)$$

The coefficients $\alpha, \beta, \alpha', \beta'$, &c. depend on the known terms and coefficients in the normal equations, and if by any means t can be determined independently, the values of x, y, z , &c. become determinate. It is evident, further, that when two of the normal equations may be rendered nearly identical by the introduction of a constant factor, the problem becomes so nearly indeterminate that in the numerical application the resulting values of the unknown quantities will be very uncertain, so that it will be necessary to express them as in the equations (53).

The indetermination in the case of the normal equations results necessarily from a similarity in the original equations of condition, and when the problem becomes nearly indeterminate, the identity of

the equations will be closer in the normal equations than in the equations of condition from which they are derived. It should be observed, also, that when we express x, y, z , &c. in terms of t , as in (53), the normal equation in t , which is the one formed by multiplying by the coefficient of t in each of the equations of condition, is not required.

136. The elimination in the solution of the equations (51) is most conveniently effected by the method of substitution. Thus, the first of these equations gives

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[ad]}{[aa]}u - \frac{[ae]}{[aa]}w - \frac{[af]}{[aa]}t - \frac{[an]}{[aa]};$$

and if we substitute this for x in each of the remaining normal equations, and put

$$\begin{aligned} [bb] - \frac{[ab]}{[aa]}[ab] &= [bb.1], & [bc] - \frac{[ab]}{[aa]}[ac] &= [bc.1], \\ [bd] - \frac{[ab]}{[aa]}[ad] &= [bd.1], & [be] - \frac{[ab]}{[aa]}[ae] &= [be.1], \\ [bf] - \frac{[ab]}{[aa]}[af] &= [bf.1]; \end{aligned} \quad (54)$$

$$\begin{aligned} [cc] - \frac{[ac]}{[aa]}[ac] &= [cc.1], & [cd] - \frac{[ac]}{[aa]}[ad] &= [cd.1], \\ [ce] - \frac{[ac]}{[aa]}[ae] &= [ce.1], & [cf] - \frac{[ac]}{[aa]}[af] &= [cf.1]; \\ [dd] - \frac{[ad]}{[aa]}[ad] &= [dd.1], & [de] - \frac{[ad]}{[aa]}[ae] &= [de.1], \\ [df] - \frac{[ad]}{[aa]}[af] &= [df.1]; \end{aligned} \quad (55)$$

$$\begin{aligned} [ee] - \frac{[ae]}{[aa]}[ae] &= [ee.1], & [ef] - \frac{[ae]}{[aa]}[af] &= [ef.1], \\ [ff] - \frac{[af]}{[aa]}[af] &= [ff.1]; \end{aligned} \quad (56)$$

$$\begin{aligned} [bn] - \frac{[ab]}{[aa]}[an] &= [bn.1], & [cn] - \frac{[ac]}{[aa]}[an] &= [cn.1], \\ [dn] - \frac{[ad]}{[aa]}[an] &= [dn.1], & [en] - \frac{[ae]}{[aa]}[an] &= [en.1], \\ [fn] - \frac{[af]}{[aa]}[an] &= [fn.1], \end{aligned} \quad (57)$$

we obtain

$$\begin{aligned}
[bb.1]y + [be.1]z + [bd.1]u + [be.1]w + [bf.1]t + [bn.1] &= 0, \\
[bc.1]y + [ce.1]z + [cd.1]u + [ce.1]w + [cf.1]t + [cn.1] &= 0, \\
[bd.1]y + [cd.1]z + [dd.1]u + [de.1]w + [df.1]t + [dn.1] &= 0, \\
[be.1]y + [ce.1]z + [de.1]u + [ee.1]w + [ef.1]t + [en.1] &= 0, \\
[bf.1]y + [cf.1]z + [df.1]u + [ef.1]w + [ff.1]t + [fn.1] &= 0.
\end{aligned} \tag{59}$$

These equations are symmetrical, and of the same form as the normal equations, the coefficients being distinguished by writing the numeral 1 within the brackets.

The unknown quantity x is thus eliminated, and by a similar process y may be eliminated from the equations (59), the resulting equations being rendered symmetrical in form by the introduction of the numeral 2 within the brackets. Thus, we put

$$\begin{aligned}
[ce.1] - \frac{[be.1]}{[bb.1]}[be.1] &= [ce.2], & [cd.1] - \frac{[be.1]}{[bb.1]}[bd.1] &= [cd.2], \\
[ce.1] - \frac{[be.1]}{[bb.1]}[be.1] &= [ce.2], & [cf.1] - \frac{[be.1]}{[bb.1]}[bf.1] &= [cf.2];
\end{aligned} \tag{60}$$

$$\begin{aligned}
[dd.1] - \frac{[bd.1]}{[bb.1]}[bd.1] &= [dd.2], & [de.1] - \frac{[bd.1]}{[bb.1]}[be.1] &= [de.2], \\
[df.1] - \frac{[bd.1]}{[bb.1]}[bf.1] &= [df.2];
\end{aligned} \tag{61}$$

$$\begin{aligned}
[ee.1] - \frac{[be.1]}{[bb.1]}[be.1] &= [ee.2], & [ef.1] - \frac{[be.1]}{[bb.1]}[bf.1] &= [ef.2] \\
[ff.1] - \frac{[bf.1]}{[bb.1]}[bf.1] &= [ff.2];
\end{aligned} \tag{62}$$

$$\begin{aligned}
[en.1] - \frac{[be.1]}{[bb.1]}[bn.1] &= [en.2], & [dn.1] - \frac{[bd.1]}{[bb.1]}[bn.1] &= [dn.2], \\
[en.1] - \frac{[be.1]}{[bb.1]}[bn.1] &= [en.2], & [fn.1] - \frac{[bf.1]}{[bb.1]}[bn.1] &= [fn.2],
\end{aligned} \tag{63}$$

and the equations become

$$\begin{aligned}
[ce.2]z + [cd.2]u + [ce.2]w + [cf.2]t + [cn.2] &= 0, \\
[cd.2]z + [dd.2]u + [de.2]w + [df.2]t + [dn.2] &= 0, \\
[ce.2]z + [de.2]u + [ee.2]w + [ef.2]t + [en.2] &= 0, \\
[cf.2]z + [df.2]u + [ef.2]w + [ff.2]t + [fn.2] &= 0.
\end{aligned} \tag{64}$$

To eliminate z from these equations, we put

$$\begin{aligned}
[dd.2] - \frac{[cd.2]}{[ce.2]}[cd.2] &= [dd.3], & [de.2] - \frac{[cd.2]}{[ce.2]}[ce.2] &= [de.3], \\
[df.2] - \frac{[cd.2]}{[ce.2]}[cf.2] &= [df.3];
\end{aligned} \tag{65}$$

$$\begin{aligned}
[ee.2] - \frac{[ce.2]}{[ce.2]} [ce.2] &= [ee.3], & [ef.2] - \frac{[ce.2]}{[ce.2]} [cf.2] &= [ef.3], \\
[ff.2] - \frac{[ef.2]}{[ce.2]} [cf.2] &= [ff.3];
\end{aligned} \tag{66}$$

$$\begin{aligned}
[dn.2] - \frac{[cd.2]}{[ce.2]} [cn.2] &= [dn.3], & [en.2] - \frac{[ce.2]}{[ce.2]} [cn.2] &= [en.3], \\
[fn.2] - \frac{[ef.2]}{[ce.2]} [cn.2] &= [fn.3],
\end{aligned} \tag{67}$$

and we have

$$\begin{aligned}
[dd.3] u + [de.3] w + [df.3] t + [dn.3] &= 0, \\
[de.3] u + [ee.3] w + [ef.3] t + [en.3] &= 0, \\
[df.3] u + [ef.3] w + [ff.3] t + [fn.3] &= 0,
\end{aligned} \tag{68}$$

Again we put, in a similar manner,

$$\begin{aligned}
[ee.3] - \frac{[de.3]}{[dd.3]} [de.3] &= [ee.4], & [ef.3] - \frac{[de.3]}{[dd.3]} [df.3] &= [ef.4], \\
[ff.3] - \frac{[df.3]}{[dd.3]} [df.3] &= [ff.4], & [en.3] - \frac{[de.3]}{[dd.3]} [dn.3] &= [en.4], \\
[fn.3] - \frac{[df.3]}{[dd.3]} [dn.3] &= [fn.4];
\end{aligned} \tag{69}$$

and the equations are

$$\begin{aligned}
[ee.4] w + [ef.4] t + [en.4] &= 0, \\
[ef.4] w + [ff.4] t + [fn.4] &= 0.
\end{aligned} \tag{70}$$

Finally, to eliminate w , we put

$$[ff.4] - \frac{[ef.4]}{[ee.4]} [ef.4] = [ff.5], \quad [fn.4] - \frac{[ef.4]}{[ee.4]} [en.4] = [fn.5], \tag{71}$$

and the resulting equation is

$$[ff.5] t + [fn.5] = 0, \tag{72}$$

which gives

$$t = - \frac{[fn.5]}{[ff.5]}. \tag{73}$$

The value of t thus found enables us to derive that of w by means of the first of equations (70). The value of w being found, that of u will be obtained from the first of equations (68). In like manner, the remaining unknown quantities will be determined by means of the equations (64), (59), and (51). The determination of the unknown quantities is thus reduced to the solution of the following system of equations:

$$\begin{aligned}
x + \frac{[ab]}{[aa]}y + \frac{[ac]}{[aa]}z + \frac{[ad]}{[aa]}u + \frac{[ae]}{[aa]}w + \frac{[af]}{[aa]}t + \frac{[an]}{[aa]} &= 0, \\
y + \frac{[bc.1]}{[bb.1]}z + \frac{[bd.1]}{[bb.1]}u + \frac{[be.1]}{[bb.1]}w + \frac{[bf.1]}{[bb.1]}t + \frac{[bn.1]}{[bb.1]} &= 0, \\
z + \frac{[cd.2]}{[cc.2]}u + \frac{[ce.2]}{[cc.2]}w + \frac{[cf.2]}{[cc.2]}t + \frac{[cn.2]}{[cc.2]} &= 0, \\
u + \frac{[de.3]}{[dd.3]}w + \frac{[df.3]}{[dd.3]}t + \frac{[dn.3]}{[dd.3]} &= 0, \\
w + \frac{[ef.4]}{[ee.4]}t + \frac{[en.4]}{[ee.4]} &= 0, \\
t + \frac{[fn.5]}{[ff.5]} &= 0,
\end{aligned} \tag{74}$$

the coefficients of which will have been found in the process of determining the several auxiliary quantities. It will be observed, further, that both in the normal equations and in those which result after each successive elimination, the coefficients which appear in a horizontal line, with the exception of the coefficient involving the absolute terms of the equations of condition, are found also in the corresponding vertical line. The form of the notation $[bb.1]$, $[bc.1]$, &c. may be symbolized thus :

$$[\beta\gamma.\mu] - \frac{[\alpha\beta.\mu]}{[\alpha\alpha.\mu]} [\alpha\gamma.\mu] = [\beta\gamma.(\mu + 1)], \tag{75}$$

in which α , β , γ , denote any three letters, and μ any numeral.

The equations (74) are derived for the case of six unknown quantities, which is the number usually to be determined in the correction of the elements of the orbit of a heavenly body; but there will be no difficulty in extending the process indicated to the case of a greater number of unknown quantities, except that the number of auxiliaries symbolized generally by (75) increases very rapidly when the number of unknown quantities is increased.

137. In the numerical application of the formulæ, when so many quantities are to be computed, it becomes important to be able to check the accuracy of the calculation in its successive stages. First, then, to prove the calculation of the coefficients in the normal equations, we put

$$\begin{aligned}
a + b + c + d + e + f &= s, \\
a' + b' + c' + d' + e' + f' &= s', \text{ \&c.}
\end{aligned}$$

If we multiply each of the sums thus formed by the corresponding absolute term n , and take the sum of all the products, we have

$$[an] + [bn] + [cn] + [dn] + [en] + [fn] = [sn]. \quad (76)$$

In a similar manner, multiplying by each of the coefficients in the original equations of condition, we find

$$\begin{aligned} [aa] + [ab] + [ac] + [ad] + [ae] + [af] &= [as], \\ [ab] + [bb] + [bc] + [bd] + [be] + [bf] &= [bs], \\ [ac] + [bc] + [cc] + [cd] + [ce] + [cf] &= [cs], \\ [ad] + [bd] + [cd] + [dd] + [de] + [df] &= [ds], \\ [ae] + [be] + [ce] + [de] + [ee] + [ef] &= [es], \\ [af] + [bf] + [cf] + [df] + [ef] + [ff] &= [fs]. \end{aligned} \quad (77)$$

Hence it appears that if we compute the sums $s, s', s'', s''', \&c.$, and form $[as], [bs], [cs], \&c.$ simultaneously with the calculation of the coefficients in the normal equations, the equation (76) must be satisfied when the absolute terms of the normal equations are correct; and the equations (77) must be satisfied when the coefficients of the unknown quantities in the normal equations are correct.

The accuracy of the calculation of the auxiliary quantities symbolized by the equation (75) may be proved in a similar manner. Thus, we have

$$[bs.1] = [bs] - \frac{[ab]}{[aa]} [as],$$

which, by means of the first and second of equations (77), becomes

$$\begin{aligned} [bs.1] &= [bb] - \frac{[ab]}{[aa]} [ab] + [bc] - \frac{[ab]}{[aa]} [ac] + [bd] - \frac{[ab]}{[aa]} [ad] \\ &\quad + [be] - \frac{[ab]}{[aa]} [ae] + [bf] - \frac{[ab]}{[aa]} [af], \end{aligned}$$

or

$$[bs.1] = [bb.1] + [bc.1] + [bd.1] + [be.1] + [bf.1]; \quad (78)$$

and similarly we derive the expressions for $[cs.1], [ds.1], \&c.$ It is obvious, therefore, that the calculation of the coefficients in the equations (59), (64), (68), and (70) will be checked as in the case of the coefficients in the normal equations, the auxiliaries depending on s being determined as if $s, s', s'', \&c.$ were the coefficients of an additional unknown quantity in the several equations of condition. Hence we must have, finally,

$$[fs.5] = [ff.5], \quad [sn.5] = [fn.5]. \quad (79)$$

If we multiply each of the equations (49) by its v , and take the sum of the several products, we get

$$[av]x + [bv]y + [cv]z + [dv]u + [ev]w + [fv]t + [vn] = [vv].$$

But, according to the equations (48) and (50), we have, for the most probable values of the unknown quantities,

$$[av] = 0, \quad [bv] = 0, \quad [cv] = 0, \text{ \&c.};$$

and hence

$$[vn] = [vv]. \quad (80)$$

If we multiply each of the equations (49) by its n , and take the sum of all the products thus formed, substituting $[vv]$ for $[vn]$, there results

$$[an]x + [bn]y + [cn]z + [dn]u + [en]w + [fn]t + [nn] = [vv].$$

Substituting in this the value of x given by the first normal equation, it becomes

$$[bn.1]y + [cn.1]z + [dn.1]u + [en.1]w + [fn.1]t + [nn.1] = [vv],$$

in which

$$[nn.1] = [nn] - \frac{[an]}{[aa]}[an]. \quad (81)$$

Substituting, further, for y its value given by the first of equations (59), and continuing the process as in the elimination of the unknown quantities by successive substitution, we obtain the following equations:

$$\begin{aligned} [cn.2]z + [dn.2]u + [en.2]w + [fn.2]t + [nn.2] &= [vv], \\ [dn.3]u + [en.3]w + [fn.3]t + [nn.3] &= [vv], \\ [en.4]w + [fn.4]t + [nn.4] &= [vv], \\ [fn.5]t + [nn.5] &= [vv], \\ [nn.6] &= [vv]. \end{aligned} \quad (82)$$

The expressions for the auxiliaries $[nn.2]$, $[nn.3]$, &c. are

$$\begin{aligned} [nn.2] &= [nn.1] - \frac{[bn.1]}{[bb.1]}[bn.1], & [nn.3] &= [nn.2] - \frac{[cn.2]}{[cc.2]}[cn.2], \\ [nn.4] &= [nn.3] - \frac{[dn.3]}{[dd.3]}[dn.3], & [nn.5] &= [nn.4] - \frac{[en.4]}{[ee.4]}[en.4], \\ [nn.6] &= [nn.5] - \frac{[fn.5]}{[ff.5]}[fn.5]. \end{aligned} \quad (83)$$

The process here indicated may be readily extended to the case of a greater number of unknown quantities, and we have, in general, when u denotes the number of unknown quantities,

$$[vv] = [nn.u]. \quad (84)$$

This equation affords a complete verification of the entire numerical calculation involved in the determination of the unknown quantities from the original equations of condition. Thus, after the elimination has been completed, we substitute the resulting values of x, y, z , &c. in the equations of condition, and derive the corresponding values of the residuals $v, v', v'',$ &c. Then, taking the sum of the squares of these, the equation (84) must be satisfied within the limits of the unavoidable errors of calculation with the logarithmic tables employed. If this condition is satisfied, it may be inferred that the entire calculation of the values of the unknown quantities from the given equations of condition is correct.

138. If the values of x, y, z , &c. thus found were the absolutely exact values, the residuals $v, v', v'',$ &c. would be the actual errors of observation. But since the results obtained only furnish the most probable values of the unknown quantities, the final residuals may differ slightly from the accidental errors of observation. Further, it is evident that the degree of precision with which the several unknown quantities may be determined by means of the data of the problem may be very different, so that it is desirable to be able to determine the relative weights of the different results.

It will be observed that the expressions for either of the unknown quantities resulting from the elimination of the others is a linear function of $n, n', n'',$ &c., so that we have

$$x + \alpha n + \alpha' n' + \alpha'' n'' + \alpha''' n''' + \dots = 0, \quad (85)$$

in which the coefficients $\alpha, \alpha', \alpha'',$ &c. are functions of the several coefficients of the unknown quantities in the equations of condition. If we now suppose the equations of condition to be reduced to the same unit of weight, the mean error of the several absolute terms of the equations will be the same, and will be the mean error of an observation whose weight is unity. Thus, if ε denotes the mean error of an observation of the weight unity, the mean error of αn will be $\alpha \varepsilon$, that of $\alpha' n'$ will be $\alpha' \varepsilon'$, and similarly for the other terms of (85); and, according to the equation (35), the mean error of x will be

$$\varepsilon_x = \varepsilon \sqrt{\alpha^2 + \alpha'^2 + \alpha''^2 + \dots} = \varepsilon \sqrt{[\alpha \alpha]}. \quad (86)$$

Hence the weight of x will be expressed by

$$p_x = \frac{1}{[\alpha \alpha]}. \quad (87)$$

Let x , denote the true value of x , namely, that which would be obtained if the true values of $v, v', v'',$ &c. were retained in the second members of the equations of condition instead of putting them equal to zero; then it is evident that the expression for x , must be that which would result by substituting $n - v$ in place of n in the formulæ for the most probable value as determined from the actual data. Hence we have

$$x, + \alpha(n - v) + \alpha'(n' - v') + \dots = 0,$$

and comparing this with the expression (85), we obtain

$$x, = x + [\alpha v].$$

Substituting in this the values of $v, v', v'',$ &c. given by the equations (49), there results

$$x, = x + [\alpha a]x + [\alpha b]y + [\alpha c]z + [\alpha d]u + [\alpha e]w + [\alpha f]t + [\alpha n],$$

and since, according to (85), $x + [\alpha n] = 0$, in order to satisfy this expression for $x,$ we must evidently have

$$[\alpha a] = 1, \quad [\alpha b] = 0, \quad [\alpha c] = 0, \quad [\alpha d] = 0, \quad [\alpha e] = 0, \quad [\alpha f] = 0. \quad (88)$$

Since the values of the unknown quantities as determined by the normal equations must be the same by whatever mode the elimination may have been performed, let us suppose the method of indeterminate multipliers to be applied for the determination of x , and let these multipliers be designated by $q, q', q'',$ &c.; then, the values of these factors are determined by the condition that the coefficient of x in the final equation shall be unity, and that the coefficients of the other unknown quantities shall be zero. Hence we shall have

$$\begin{aligned} [aa]q + [ab]q' + [ac]q'' + [ad]q''' + \dots &= 1, \\ [ab]q + [bb]q' + [bc]q'' + [bd]q''' + \dots &= 0, \\ [ac]q + [bc]q' + [cc]q'' + [cd]q''' + \dots &= 0, \\ &\text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{aligned} \quad (89)$$

and also, retaining the residuals $v, v', v'',$ &c. in the formation of the normal equations,

$$x, + [\alpha n]q + [\alpha n']q' + [\alpha n'']q'' + \dots = [\alpha v]q + [\alpha v']q' + [\alpha v'']q'' + \dots \quad (90)$$

Therefore, since

$$x, + [\alpha n] = [\alpha v],$$

and since the first member of this equation must be identical with the first member of (90), we have

$$[\alpha v]q + [\alpha v']q' + [\alpha v'']q'' + \dots = \alpha v + \alpha'v' + \alpha''v'' + \dots,$$

which gives, by expanding the several sums,

$$\begin{aligned} aq + bq' + cq'' + dq''' + \dots &= \alpha, \\ a'q + b'q' + c'q'' + d'q''' + \dots &= \alpha', \\ a''q + b''q' + c''q'' + d''q''' + \dots &= \alpha'', \\ \&c. & \qquad \qquad \qquad \&c. \end{aligned} \quad (91)$$

Multiplying each of these equations by its α , and adding the products, the result is

$$[\alpha\alpha]q + [\alpha b]q' + [\alpha c]q'' + [\alpha d]q''' + \dots = [\alpha\alpha],$$

which, by means of the equations (88), reduces to

$$q = \frac{1}{p_x}. \quad (92)$$

Hence it appears that the eliminating factor q is the reciprocal of the weight of x , and, since the coefficients of $q, q', q'', \&c.$ in the equations (89) are the same as those of $x, y, z, \&c.$ in the normal equations, that if we put $[an] = -1, [bn] = 0, [cn] = 0, \&c.$, in the normal equations, the resulting value of x will be the reciprocal of the weight of the most probable value of this quantity.

The equation (90) shows that if, in the general elimination, by whatever method it may have been effected, we write $[av], [bv], \&c.$ instead of zero in the second members of the normal equations respectively, the coefficient of $[av]$ is the reciprocal of the weight of x . It is obvious that it will not be necessary to know the numerical values of $[av], [bv], \&c.$, since only the coefficient q is required. The most probable value of x is found from (90) by the condition of a minimum of the squares of the residuals, namely, that

$$[av] = 0, \quad [bv] = 0, \quad [cv] = 0, \quad \&c.$$

The process here indicated for the determination of the weight of the final value of x is general, and applies to the case of any other unknown quantity provided that the necessary changes are made in the notation. Thus, the reciprocal of the weight of y is determined by writing, in the normal equations, -1 in place of $[bn]$, and putting $[an], [cn], \&c.$ equal to zero, and completing the elimination. It is also the coefficient of $[bv]$ in the value of y when the elimination is effected with the symbols $[av], [bv], \&c.$ retained in the second members of the normal equations.

139. It may be easily shown that when the elimination is effected by the method of successive substitution, as already explained, the

coefficient of the unknown quantity which is made the last in the elimination, in the final equation for its determination, is equal to the weight of the resulting value of that quantity. Thus, in the case of the equations for six unknown quantities, since the reciprocal of the weight of the most probable value of t is the value of t obtained from the normal equations by putting $[fn] = -1$, and $[an]$, $[bn]$, $[cn]$, &c. equal to zero, the equations (63), (67), (69), and (71) show that we have

$$[fn] = [fn.1] = [fn.2] = [fn.3] = [fn.4] = [fn.5] = -1,$$

and hence, according to (72), for the reciprocal of the weight of t ,

$$[ff.5] \frac{1}{p_t} - 1 = 0,$$

which gives

$$p_t = [ff.5]. \quad (93)$$

The weight of t is therefore equal to its coefficient in the final equation which results from the elimination of the other unknown quantities by successive substitution. Hence, by repeating the elimination, successively changing the order of the quantities, so that each of the unknown quantities may have the last place, the weights will be determined independently, and the agreement of the several sets of values for the unknown quantities will be a proof of the accuracy of the calculation. It is not necessary, however, to make so many repetitions of the elimination, since, in each case, the weights of two of the unknown quantities will be given by means of the auxiliaries used in the elimination. Thus, the reciprocal of the weight of w is obtained by putting $[en] = -1$, and the other absolute terms of the normal equations equal to zero, and finding the corresponding value of w . This operation gives

$$[en.4] = -1, \quad [fn.4] = 0, \quad [fn.5] = \frac{[ef.4]}{[ee.4]}.$$

Hence the equation (73) becomes

$$t = - \frac{[ef.4]}{[ee.4] [ff.5]};$$

and substituting this value of t in the last of equations (70), we get

$$[ef.4] \frac{1}{p_w} - \frac{[ff.4] [ef.4]}{[ff.5] [ee.4]} = 0,$$

or

$$p_w = \frac{[ff.5]}{[ff.4]} [ee.4], \quad (94)$$

which gives the weight of w in terms of the auxiliary quantities required in the determination of its most probable value.

If the order of elimination is now completely reversed, so that x is made the last in the elimination, the weights of x and y will be determined by the equations

$$\begin{aligned} p_x &= [aa.5], \\ p_y &= \frac{[aa.5]}{[aa.4]} [bb.4]. \end{aligned} \quad (95)$$

A third elimination, in which z and u are the unknown quantities first determined, will give the weights of these determinations. It appears, therefore, that when only four unknown quantities are to be found, a single repetition of the elimination, the order of the quantities being completely reversed, will furnish at once the weights of the several results, and check the accuracy of the calculation. When there are only two unknown quantities, the elimination gives directly the values of these quantities and also of their weights.

140. In the case of three or more unknown quantities, the weights of all the results may be determined without repeating the elimination when certain additional auxiliary quantities have been found. The weights of the two which are first determined are given in terms of the auxiliaries required in the elimination, that of the quantity which is next found will require the value of an additional auxiliary quantity, the succeeding one will require two additional auxiliaries, and so on. The equations (74) show that when the substitution is effected analytically the final value of x will have the denominator

$$D = [aa] [bb.1] [cc.2] [dd.3] [ee.4] [ff.5],$$

and this denominator, being the determinant formed from all the coefficients in the normal equations, must evidently have the same value whatever may be the order in which the unknown quantities are eliminated. Let us now suppose that each of the unknown quantities is, in succession, made the last in the elimination, and let the auxiliaries in each elimination be distinguished from those when t is last eliminated by annexing the letter which is the coefficient of the quantity first determined; then we shall have

$$\begin{aligned} D &= [aa] [bb.1] [cc.2] [dd.3] [ee.4] [ff.5] \\ &= [aa]_e [bb.1]_e [cc.2]_e [dd.3]_e [ff.4]_e [ee.5] \\ &= [aa]_d [bb.1]_d [cc.2]_d [ee.3]_d [ff.4]_d [dd.5] \\ &= [aa]_c [bb.1]_c [dd.2]_c [ee.3]_c [ff.4]_c [cc.5] \\ &= [aa]_b [cc.1]_b [dd.2]_b [ee.3]_b [ff.4]_b [bb.5] \\ &= [bb]_a [cc.1]_a [dd.2]_a [ee.3]_a [ff.4]_a [aa.5]. \end{aligned}$$

It will be observed, however, that when the order of elimination is changed, only those auxiliaries which involve the coefficient of the quantity which is made the last in the changed order will be changed. Hence, if we add the distinguishing letter only to those auxiliaries which have a different value in the new order, we have

$$\begin{aligned} D &= [aa] [bb.1] [cc.2] [dd.3] [ee.4] [ff.5] \\ &= [aa] [bb.1] [cc.2] [dd.3] [ff.4] [ee.5] \\ &= [aa] [bb.1] [cc.2] [ee.3] [ff.4]_a [dd.5] \\ &= [aa] [bb.1] [dd.2] [ee.3]_e [ff.4]_a [cc.5] \\ &= [aa] [cc.1] [dd.2]_b [ee.3]_b [ff.4]_b [bb.5] \\ &= [bb] [cc.1]_a [dd.2]_a [ee.3]_a [ff.4]_a [aa.5], \end{aligned}$$

and from these equations we obtain

$$\begin{aligned} p_1 &= [ff.5], \\ p_u &= [ee.5] = \frac{[ff.5]}{[ff.4]} [ee.4], \\ p_u &= [dd.5] = \frac{[ff.5]}{[ff.4]_a} \cdot \frac{[ee.4]}{[ee.3]} [dd.3], \\ p_s &= [cc.5] = \frac{[ff.5]}{[ff.4]_e} \cdot \frac{[ee.4]}{[ee.3]_e} \cdot \frac{[dd.3]}{[dd.2]} [cc.2], \\ p_v &= [bb.5] = \frac{[ff.5]}{[ff.4]_b} \cdot \frac{[ee.4]}{[ee.3]_b} \cdot \frac{[dd.3]}{[dd.2]_b} \cdot \frac{[cc.2]}{[cc.1]} [bb.1], \\ p_x &= [aa.5] = \frac{[ff.5]}{[ff.4]_a} \cdot \frac{[ee.4]}{[ee.3]_a} \cdot \frac{[dd.3]}{[dd.2]_a} \cdot \frac{[cc.2]}{[cc.1]_a} \cdot \frac{[bb.1]}{[bb]} [aa], \end{aligned} \tag{96}$$

by means of which the weights of the six unknown quantities may be determined. The process here indicated may be readily extended to the case of a greater number of unknown quantities. The equation for p_v is identical with (94), the expression for p_u introduces the new auxiliary quantity $[ff.4]_a$, and that for p_s introduces two new auxiliaries.

The expressions for the new auxiliaries $[ff.4]_a$, $[ff.4]_e$, $[ee.3]_e$, &c. are easily formed by observing that all the auxiliaries as far as those which are designated by the numeral 4 are not affected by putting e or f last, that, as far as those which contain the numeral 3, it makes no difference whether d , e , or f is placed last, that those distinguished by the numerals 1 and 2 are not affected by making e , d , e , or f the last, and that those designated by the numeral 1 are unchanged unless a is made the last. Thus, we obtain

$$[ff.4]_a = [ff.3] - \frac{[ef.3]}{[ee.3]} [ef.3], \tag{97}$$

and, also,

$$\begin{aligned} [ef.3]_e &= [ef.2] - \frac{[de.2]}{[dd.2]} [df.2], & [ff.3]_e &= [ff.2] - \frac{[df.2]}{[dd.2]} [ef.2], \\ [ee.3]_e &= [ee.2] - \frac{[de.2]}{[dd.2]} [de.2], & [ff.4]_e &= [ff.3]_e - \frac{[ef.3]_e}{[ee.3]_e} [ef.3]_e. \end{aligned} \quad (98)$$

In like manner we may derive the expressions for the new auxiliaries introduced into the equations for p_y and p_x . It will be expedient, however, in the actual application of the formulæ, to eliminate first in the order x, y, z, u, w, t , and the weights of the results for u, w , and t will be obtained by means of the first three of equations (96), the single additional auxiliary required being found by means of (97). Then the elimination should be performed in the order t, w, u, z, y, x , and we shall have

$$\begin{aligned} p_x &= [aa.5], & p_x &= \frac{[aa.5]}{[aa.4]_e} \cdot \frac{[bb.4]}{[bb.3]} [cc.3], \\ p_y &= \frac{[aa.5]}{[aa.4]} [bb.4], & [aa.4]_e &= [aa.3] - \frac{[ab.3]}{[bb.3]} [ab.3], \end{aligned} \quad (99)$$

by means of which the weights of x, y , and z will be determined. The agreement of the two sets of values of the unknown quantities will prove the accuracy of the numerical calculation in the process of elimination.

141. The weights of the most probable values of the unknown quantities may also be computed separately when certain auxiliary factors have been found, and these factors are those which are introduced when the equations (74) are solved by the method of indeterminate multipliers instead of by successive substitution. Thus, in order to find x , let the first of these equations be multiplied by 1, the second by A' , the third by A'' , the fourth by A''' , and so on, and let the sum of all these products be taken; then the equations of condition for the determination of the several eliminating factors will be

$$\begin{aligned} 0 &= \frac{[ab]}{[aa]} + A', \\ 0 &= \frac{[ac]}{[aa]} + \frac{[bc.1]}{[bb.1]} A' + A'', \\ 0 &= \frac{[ad]}{[aa]} + \frac{[bd.1]}{[bb.1]} A' + \frac{[cd.2]}{[cc.2]} A'' + A''', \\ 0 &= \frac{[ae]}{[aa]} + \frac{[be.1]}{[bb.1]} A' + \frac{[ce.2]}{[cc.2]} A'' + \frac{[de.3]}{[dd.3]} A''' + A^{iv}, \\ 0 &= \frac{[af]}{[aa]} + \frac{[bf.1]}{[bb.1]} A' + \frac{[cf.2]}{[cc.2]} A'' + \frac{[df.3]}{[dd.3]} A''' + \frac{[ef.4]}{[ee.4]} A^{iv} + A^v. \end{aligned} \quad (100)$$

To determine y from the last five of equations (74), let the eliminating factors be denoted by B' , B''' , B^{iv} , and B^v , and we shall have

$$\begin{aligned} 0 &= \frac{[bc.1]}{[bb.1]} + B'', \\ 0 &= \frac{[bd.1]}{[bb.1]} + \frac{[cd.2]}{[cc.2]} B'' + B''', \\ 0 &= \frac{[be.1]}{[bb.1]} + \frac{[ce.2]}{[cc.2]} B'' + \frac{[de.3]}{[dd.3]} B''' + B^{iv}, \\ 0 &= \frac{[bf.1]}{[bb.1]} + \frac{[cf.2]}{[cc.2]} B'' + \frac{[df.3]}{[dd.3]} B''' + \frac{[ef.4]}{[ee.4]} B^{iv} + B^v. \end{aligned} \quad (101)$$

In a similar manner, we obtain the following equations for the determination of the eliminating factors necessary for finding the values of the remaining unknown quantities:

$$\begin{aligned} 0 &= \frac{[cd.2]}{[cc.2]} + C''', \\ 0 &= \frac{[ce.2]}{[cc.2]} + \frac{[de.3]}{[dd.3]} C''' + C^{iv}, \\ 0 &= \frac{[cf.2]}{[cc.2]} + \frac{[df.3]}{[dd.3]} C''' + \frac{[ef.4]}{[ee.4]} C^{iv} + C^v; \\ 0 &= \frac{[de.3]}{[dd.3]} + D^{iv}, \\ 0 &= \frac{[df.3]}{[dd.3]} + \frac{[ef.4]}{[ee.4]} D^{iv} + D^v, \\ 0 &= \frac{[ef.4]}{[ee.4]} + E^v. \end{aligned} \quad (102)$$

The expressions for the values of the unknown quantities will therefore become

$$\begin{aligned} -x &= \frac{[an]}{[aa]} + \frac{[bn.1]}{[bb.1]} A' + \frac{[cn.2]}{[cc.2]} A'' + \frac{[dn.3]}{[dd.3]} A''' + \frac{[en.4]}{[ee.4]} A^{iv} + \frac{[fn.5]}{[ff.5]} A^v, \\ -y &= \frac{[bn.1]}{[bb.1]} + \frac{[cn.2]}{[cc.2]} B'' + \frac{[dn.3]}{[dd.3]} B''' + \frac{[en.4]}{[ee.4]} B^{iv} + \frac{[fn.5]}{[ff.5]} B^v, \\ -z &= \frac{[cn.2]}{[cc.2]} + \frac{[dn.3]}{[dd.3]} C''' + \frac{[en.4]}{[ee.4]} C^{iv} + \frac{[fn.5]}{[ff.5]} C^v, \\ -u &= \frac{[dn.3]}{[dd.3]} + \frac{[en.4]}{[ee.4]} D^{iv} + \frac{[fn.5]}{[ff.5]} D^v, \\ -w &= \frac{[en.4]}{[ee.4]} + \frac{[fn.5]}{[ff.5]} E^v, \\ -t &= \frac{[fn.5]}{[ff.5]}. \end{aligned} \quad (103)$$

The first of these equations will give the reciprocal of the weight of x , when we put $[an] = -1$, and the other absolute terms of the normal equations equal to zero; the second will give the reciprocal of the weight of y by putting $[bn] = -1$, and the other absolute terms of the normal equations equal to zero; and, continuing the process, finally the last equation will give the reciprocal of the weight of t when we put $fn = -1$, and $[an]$, $[bn]$, $[cn]$, &c. equal to zero. It remains, therefore, to determine the particular values of $[bn.1]$, $[cn.2]$, &c., and the expressions for the weights will be complete.

If we multiply the first of equations (100) by $[an]$, it becomes

$$[bn.1] = [an] A' + [bn]. \quad (104)$$

Multiplying the second of equations (100) by $[an]$, and the first of (101) by $[bn]$, adding the products, and introducing the value of $[bn.1]$ just found, we get

$$[cn] - [cn.1] + \frac{[bc.1]}{[bb.1]} [bn.1] + [an] A'' + [bn] B'' = 0,$$

which reduces to

$$[an] A'' + [bn] B'' + [cn] = [cn.2]. \quad (105)$$

Multiplying the third of equations (100) by $[an]$, the second of (101) by $[bn]$, and the first of (102) by $[cn]$, adding the products, and reducing by means of (104) and (105), we obtain

$$0 = [dn] - [dn.1] + \frac{[bd.1]}{[bb.1]} [bn.1] + \frac{[cd.2]}{[cc.2]} [cn.2] + [an] A''' + [bn] B''' + [cn] C''',$$

which, by means of the expressions for the auxiliaries, is further reduced to

$$[an] A''' + [bn] B''' + [cn] C''' + [dn] = [dn.3]. \quad (106)$$

In a similar manner we find, from the remaining equations of (100), (101), and (102), the following expressions:

$$\begin{aligned} [an] A^{iv} + [bn] B^{iv} + [cn] C^{iv} + [dn] D^{iv} + [en] &= [en.4], \\ [an] A^v + [bn] B^v + [cn] C^v + [dn] D^v + [en] E^v + [fn] &= [fn.5]. \end{aligned} \quad (107)$$

The equations (104), (105), (106), and (107), enable us to find the particular values of $[bn.1]$, $[cn.2]$, &c. required in the expressions for the reciprocals of the weights. Thus, for the weight of x , we have

$$[an] = -1, \quad [bn] = [cn] = [dn] = [en] = [fn] = 0;$$

and these equations give

$$\begin{aligned} [bn.1] &= -A', & [en.2] &= -A'', & [dn.3] &= -A''', \\ [en.4] &= -A^{iv}, & [fn.5] &= -A^v. \end{aligned}$$

For the case of the weight of y , we have

$$[bn] = -1, \quad [an] = [en] = [dn] = [en] = [fn] = 0,$$

and the same equations give

$$\begin{aligned} [bn.1] &= -1, & [en.2] &= -B'', & [dn.3] &= -B''', \\ [en.4] &= -B^{iv}, & [fn.5] &= -B^v. \end{aligned}$$

We have, also, for the weight of z ,

$$[en.2] = -1, \quad [dn.3] = -C''', \quad [en.4] = -C^{iv}, \quad [fn.5] = -C^v,$$

for the weight of u ,

$$[dn.3] = -1, \quad [en.4] = -D^{iv}, \quad [fn.5] = -D^v;$$

for the weight of w ,

$$[en.4] = -1, \quad [fn.5] = -E^v;$$

and finally, for the weight of t ,

$$[fn.5] = -1.$$

Introducing these particular values into the equations (103), the corresponding values of the unknown quantities are the reciprocals of the weights of their most probable values, respectively; and hence we derive

$$\begin{aligned} \frac{1}{p_x} &= \frac{1}{[aa]} + \frac{A'A'}{[bb.1]} + \frac{A''A''}{[cc.2]} + \frac{A'''A'''}{[dd.3]} + \frac{A^{iv}A^{iv}}{[ee.4]} + \frac{A^vA^v}{[ff.5]}, \\ \frac{1}{p_y} &= \frac{1}{[bb.1]} + \frac{B'B''}{[cc.2]} + \frac{B'''B'''}{[dd.3]} + \frac{B^{iv}B^{iv}}{[ee.4]} + \frac{B^vB^v}{[ff.5]}, \\ \frac{1}{p_z} &= \frac{1}{[cc.2]} + \frac{C'''C'''}{[dd.3]} + \frac{C^{iv}C^{iv}}{[ee.4]} + \frac{C^vC^v}{[ff.5]}, \\ \frac{1}{p_u} &= \frac{1}{[dd.3]} + \frac{D^{iv}D^{iv}}{[ee.4]} + \frac{D^vD^v}{[ff.5]}, \\ \frac{1}{p_w} &= \frac{1}{[ee.4]} + \frac{E^vE^v}{[ff.5]}, \\ \frac{1}{p_t} &= \frac{1}{[ff.5]}. \end{aligned} \tag{108}$$

The equations (103) and (108) will serve to determine separately the value of each unknown quantity and also that of its weight, the

auxiliary factors A' , A'' , B'' , &c. having been found from the equations (100), (101), and (102). If we reverse the operation and re-compose the equations (74) by means of the expressions for the unknown quantities given by (103), the conditions which immediately follow furnish another series of equations for the determination of the auxiliary factors. The equations thus derived will give first the values of A' , B'' , C''' , D^{iv} , and E^v ; then, those of A'' , B''' , C^{iv} , D^v ; and so on. They are equally as convenient as those already given, provided that the values of all the unknown quantities are required as well as their respective weights.

142. The formulæ already given for the relations between the data of the problem and the weights of the most probable values of the unknown quantities, are those which are of the greatest practical value. It will be apparent from what has been derived that there must be a variety of methods which may be applied, but that all of these methods involve essentially the same numerical operations. The peculiar symmetry of the normal equations affords also a variety of expressions applicable to the different phases under which the problem presents itself.

According to the general theory of elimination, the expression for any unknown quantity, as determined from the normal equations, may be put in the form

$$x = -\frac{A}{D}[an] - \frac{A'}{D}[bn] - \frac{A''}{D}[cn] - \&c., \quad (109)$$

in which D is the determinant formed from all the coefficients of the unknown quantities in the normal equations, and in which A , A' , A'' , &c. are the partial determinants required in the elimination. Thus, A is the determinant formed from the coefficients of all the unknown quantities except x , in all the equations except the first; A' is the determinant formed from the coefficients of y , z , &c. in all the equations except the second; and the values of A'' , A''' , &c. are formed in a similar manner. Now, since the value of x which results when we put $[an] = -1$, and the other absolute terms of the normal equations equal to zero, is the reciprocal of the weight of the most probable value of this unknown quantity as given by (109), we have

$$p_x = \frac{D}{A}. \quad (110)$$

In like manner, the expression for the most probable value of y will be

$$y = -\frac{B}{D}[an] - \frac{B'}{D}[bn] - \frac{B''}{D}[cn] - \&c., \quad (111)$$

$B, B', B'', \&c.$ being the partial determinants formed when the coefficients of y are omitted; and for its weight we have

$$p_y = \frac{D}{B'}. \quad (112)$$

The formulæ for the most probable value of z and for its weight are entirely analogous to those for x and y , so that the process here indicated may be extended to the case of any number of unknown quantities. It appears, therefore, that the weight of the most probable value of any unknown quantity is found by dividing the complete determinant of all the coefficients by the partial determinant formed when we omit the normal equation corresponding particularly to this unknown quantity, and when we omit also the coefficients of this quantity in the remaining normal equations.

The peculiar arrangement of the coefficients in the normal equations abbreviates somewhat the expressions for the several determinants. Thus, in the case of three unknown quantities, we have

$$A = [bb][cc] - [bc]^2, \quad B' = [aa][cc] - [ac]^2, \quad C'' = [aa][bb] - [ab]^2, \\ D = [aa][bb][cc] + 2[ab][bc][ac] - [aa][bc]^2 - [bb][ac]^2 - [cc][ab]^2,$$

which are all the quantities required for finding simply the weights of the most probable values of x, y , and z . The expression for the weight of z is

$$p_z = \frac{D}{C''}.$$

When there are but two unknown quantities, we have

$$A = [bb], \quad B' = [aa], \quad D = [aa][bb] - [ab]^2,$$

and hence

$$p_x = \frac{[aa][bb] - [ab]^2}{[bb]}, \quad p_y = \frac{[aa][bb] - [ab]^2}{[aa]}.$$

When the number of unknown quantities is increased, the expressions for the determinants necessarily become much more complicated, and hence the convenience of other auxiliary quantities is manifest.

143. The case has been already alluded to in which the determination of the values of the unknown quantities is rendered uncertain by the similarity of the signs and coefficients in the normal equations,

and in which the problem becomes nearly indeterminate. Sometimes it will be possible to overcome the difficulty thus encountered by a suitable change of the elements to be determined; but, generally, for a complete and satisfactory solution, additional data will be required. It often happens, however, that several of the unknown quantities may be accurately determined from the given equations when the values of the others are known, but that the certainty of the determination of the same quantities is very greatly impaired when all the unknown quantities are derived simultaneously from the same equations. Let us suppose that one of the unknown quantities is, from the very nature of the problem, not susceptible of an accurate determination from the data employed. The equations will then present themselves in a form approaching that in which the number of independent relations is one less than the number of unknown quantities, so that it will be necessary to determine the other unknown quantities in terms of that whose value is necessarily uncertain. In this case the elimination should be so arranged that the quantity which is regarded as uncertain is that whose value would be first determined. Then, if its coefficient in the final equation, corresponding to (72), is very small, a circumstance which indicates at once the existence of the uncertainty when it is not otherwise suspected, the process of elimination should not be completed, and the auxiliary quantities should be determined only as far as those required in the formation of the equation which corresponds to the first of (70). Thus, let t be the uncertain quantity, and we have

$$w = -\frac{[ef.4]}{[ee.4]}t - \frac{[en.4]}{[ee.4]},$$

which must be substituted for w in the first of equations (68). We thus obtain w , u , z , y , and x as functions of t . If the solution is effected by means of the equations (103), let x_0 , y_0 , z_0 , &c. denote the values of these unknown quantities when we put $t=0$; and then we shall have

$$\begin{aligned} x_0 &= -\frac{[an]}{[aa]} - \frac{[bn.1]}{[bb.1]}A' - \frac{[cn.2]}{[cc.2]}A'' - \frac{[dn.3]}{[dd.3]}A''' - \frac{[en.4]}{[ee.4]}A^{iv}, \\ y_0 &= -\frac{[bn.1]}{[bb.1]} - \frac{[cn.2]}{[cc.2]}B'' - \frac{[dn.3]}{[dd.3]}B''' - \frac{[en.4]}{[ee.4]}B^{iv}, \\ z_0 &= -\frac{[cn.2]}{[cc.2]} - \frac{[dn.3]}{[dd.3]}C''' - \frac{[en.4]}{[ee.5]}C^{iv}, \end{aligned} \quad (114)$$

$$\begin{aligned} u_0 &= -\frac{[dn.3]}{[dd.3]} - \frac{[en.4]}{[ee.4]} D^v, \\ w_0 &= -\frac{[en.4]}{[ee.4]}, \end{aligned} \quad (113)$$

and hence

$$\begin{aligned} x &= x_0 + A^v t, & y &= y_0 + B^v t, & z &= z_0 + C^v t, \\ u &= u_0 + D^v t, & w &= w_0 + E^v t. \end{aligned} \quad (114)$$

As soon as t is determined by some independent condition or relation, these equations will give the corresponding values of x, y, z , &c. The mean errors of x_0, y_0, z_0 , &c. having been determined by neglecting t entirely, if we denote the mean error of the final adopted value of t by ε_t , the mean errors of the corresponding values of the other variables will be given by

$$\begin{aligned} \varepsilon_x^2 &= (\varepsilon_x)^2 + A^v A^v \varepsilon_t^2, & \varepsilon_y^2 &= (\varepsilon_y)^2 + B^v B^v \varepsilon_t^2, & \varepsilon_z^2 &= (\varepsilon_z)^2 + C^v C^v \varepsilon_t^2, \\ \varepsilon_u^2 &= (\varepsilon_u)^2 + D^v D^v \varepsilon_t^2, & \varepsilon_w^2 &= (\varepsilon_w)^2 + E^v E^v \varepsilon_t^2, \end{aligned} \quad (115)$$

in which $(\varepsilon_x), (\varepsilon_y)$, &c. denote the mean errors of x_0, y_0 , &c. These formulæ show, also, that when one of the variables is neglected, the equations assign too great a degree of precision to the results thus obtained.

When there are two or more unknown quantities which cannot be determined from the data with sufficient certainty, the problem must be treated in a manner entirely analogous to that here indicated; but, since cases of this kind will rarely, if ever, occur, it is not necessary to pursue the subject further.

144. The weights which are obtained for the most probable values of the unknown quantities enable us to find the mean and probable errors of these values. Let ε denote the mean error of an observation whose weight is unity; then the mean error of x will be

$$\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}}, \quad (116)$$

and, in like manner, the expressions for the mean errors of y, z, u , &c. will be

$$\varepsilon_y = \frac{\varepsilon}{\sqrt{p_y}}, \quad \varepsilon_z = \frac{\varepsilon}{\sqrt{p_z}}, \quad \varepsilon_u = \frac{\varepsilon}{\sqrt{p_u}}, \quad \&c. \quad (117)$$

It remains, therefore, to determine the value of ε by means of the final residuals obtained by comparing the observed values of the function with those given by the most probable values of the va-

riables. If these residuals were the actual fortuitous errors of observation, the mean error of an observation would be

$$\varepsilon = \sqrt{\frac{[vv]}{m}},$$

m being the number of equations of condition. This value is evidently an approximation to the correct result; but since by supposing the residuals $v, v', v'',$ &c. to be the actual errors of the several observed values of the function, we assign too high a degree of precision to the several results, the true value of ε must necessarily be greater than that given by this equation. Let the true values of the unknown quantities be $x + \Delta x, y + \Delta y, z + \Delta z,$ &c., the substitution of which in the several equations of condition would give the residuals $\Delta, \Delta', \Delta'',$ &c.; then we shall have

$$\begin{aligned} a\Delta x + b\Delta y + c\Delta z + d\Delta u \dots + v &= \Delta, \\ a'\Delta x + b'\Delta y + c'\Delta z + d'\Delta u \dots + v' &= \Delta', \\ &\text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{aligned} \tag{118}$$

If we multiply each of these equations by its Δ , and take the sum of all the products, we get

$$[a\Delta]\Delta x + [b\Delta]\Delta y + [c\Delta]\Delta z + [d\Delta]\Delta u + \dots + [v\Delta] = [\Delta\Delta].$$

But if we multiply each of the same equations by its v , take the sum of the products, and reduce by means of (48) and (50), we obtain

$$[vv] = [v\Delta];$$

and hence we derive

$$[\Delta\Delta] = [vv] + [a\Delta]\Delta x + [b\Delta]\Delta y + [c\Delta]\Delta z + [d\Delta]\Delta u + \dots \tag{119}$$

If we form the normal equations from (118), it will be observed that they are of the same form as the normal equations formed from the original equations of condition, provided that we write $-\Delta$ in place of n ; and hence, according to (85), we have

$$\Delta x = a\Delta + a'\Delta' + a''\Delta'' + \dots$$

We have, also,

$$[a\Delta] = a\Delta + a'\Delta' + a''\Delta'' + \dots,$$

and the product of these equations gives

$$\begin{aligned} [a\Delta]\Delta x &= aa\Delta^2 + a'a'\Delta'^2 + a''a''\Delta''^2 + \dots \\ &\quad + aa'\Delta\Delta' + aa''\Delta\Delta'' + \dots \end{aligned}$$

The mean value of the terms containing $\Delta\Delta', \Delta\Delta'',$ &c. is zero, and

for the mean values of Δ^2 , Δ'^2 , Δ''^2 , &c. we must, in each case, write ϵ^2 . Hence the mean value of the product $[a\Delta] \Delta x$ will be

$$[a\alpha] \epsilon^2,$$

and this, by means of the first of equations (88), is further reduced to

$$[a\Delta] \Delta x = \epsilon^2.$$

In a similar manner, we obtain the value ϵ^2 for the mean value of each of the products $[b\Delta] \Delta y$, $[c\Delta] \Delta z$, &c. Now, the terms added to $[vv]$ in the second member of the equation (119) are necessarily very small, and, although their exact value cannot be determined, we may without sensible error adopt the mean values of the several terms as here determined, so that the equation becomes

$$[\Delta\Delta] = [vv] + \mu\epsilon^2, \quad (120)$$

μ being the number of unknown quantities. Therefore, since $[\Delta\Delta] = m\epsilon^2$, we shall have

$$\epsilon = \sqrt{\frac{[vv]}{m-\mu}} = \sqrt{\frac{[nn.\mu]}{m-\mu}}, \quad (121)$$

by means of which the mean error of an observation whose weight is unity may be determined. When $\mu=1$, this equation becomes identical with (30).

For the determination of the probable errors of the final values of the unknown quantities, if r denotes the probable error of an observation of the weight unity, we have the following equations:—

$$r = 0.67449 \sqrt{\frac{[vv]}{m-\mu}}, \quad (122)$$

$$r_x = \frac{r}{\sqrt{p_x}}, \quad r_y = \frac{r}{\sqrt{p_y}}, \quad \&c.$$

145. The formulæ which result from the theory of errors according to which the method of least squares is derived, enable us to combine the data furnished by observation so as to overcome, in the greatest degree possible, the effect of those accidental errors which no refinement of theory can successfully eliminate. The problem of the correction of the approximate elements of the orbit of a heavenly body by means of a series of observed places, requires the application of nearly all the distinct results which have been derived. The first approximate elements of the orbit of the body will be determined on three or four observed places according to the methods which

have been already explained. In the case of a planet, if the inclination is not very small, the method of three geocentric places may be employed, but it will, in general, afford greater accuracy and require but little additional labor to base the first determination on four observed places, according to the process already illustrated. In the case of a comet, the first assumption made is that the orbit is a parabola, and the elements derived in accordance with this hypothesis may be successively corrected, until it is apparent whether it is necessary to make any further assumption in regard to the value of the eccentricity. In all cases, the approximate elements derived from a few places should be further corrected by means of more extended data before any attempt is made to obtain a more complete determination of the elements. The various methods by which this preliminary correction may be effected have been already sufficiently developed.

The fundamental places adopted as the basis of the correction may be single observed places separated by considerable intervals of time; but it will be preferable to use places which may be regarded as the average of a number of observations made on the same day or during a few days before and after the date of the average or *normal* place. The ephemeris computed from the approximate elements known may be assumed to represent the actual path so closely that, for an interval of a few days, the difference between computation and observation may be regarded as being constant, or at least as varying proportionally to the time. Let $n, n', n'',$ &c. be the differences between computation and observation, in the case of either spherical co-ordinate, for the dates $t, t', t'',$ &c., respectively; then, if the interval between the extreme observations to be combined in the formation of the normal place is not too great, and if we regard the observations as equally precise, the normal difference n_0 between computation and observation will be found by taking the arithmetical mean of the several values of n , and this being applied with the proper sign to the computed spherical co-ordinate for the date t_0 , which is the mean of $t, t', t'',$ &c., will give the corresponding normal place. But when different weights $p, p', p'',$ &c. are assigned to the observations, the value of n_0 must be found from

$$n_0 = \frac{np + n'p' + n''p'' + \dots}{p + p' + p'' + \dots}, \quad (123)$$

and the weight of this value will be equal to the sum

$$p + p' + p'' + \dots$$

The date of the normal place will be determined by

$$t_0 = \frac{pt + p't' + p''t'' + \dots}{p + p' + p'' + \dots}. \quad (124)$$

If the error of the ephemeris can be considered as nearly constant, it is not necessary to determine t_0 with great precision, since any date not differing much from the average of all may be adopted with sufficient accuracy. It should be observed further that, in order to obtain the greatest accuracy practicable, the spherical co-ordinates of the body for the date t_0 should be computed directly from the elements, so that the resulting normal place may be as free as possible from the effect of neglected differences in the interpolation of the ephemeris.

When the differences between the computed and the observed places to be combined for the formation of a normal place cannot be considered as varying proportionally to the time, we may derive the error of the ephemeris from an equation of the form of (53)₆, namely,

$$\Delta\theta = A + B\tau + C\tau^2,$$

the coefficients A , B , and C being found from equations of condition formed by means of the several known values of $\Delta\theta$ in the case of each of the spherical co-ordinates.

146. In this way we obtain normal places at convenient intervals throughout the entire period during which the body was observed. From three or more of these normal places, a new system of elements should be computed by means of some one of the methods which have already been given; and these fundamental places being judiciously selected, the resulting elements will furnish a pretty close approximation to the truth, so that the residuals which are found by comparing them with all the directly observed places may be regarded as indicating very nearly the actual errors of those places. We may then proceed to investigate the character of the observations more fully. But since the observations will have been made at many different places, by different observers, with instruments of different sizes, and under a variety of dissimilar attendant circumstances, it may be easily understood that the investigation will involve much that is vague and uncertain. In the theory of errors which has been developed in this chapter, it has been assumed that all constant errors have been duly eliminated, and that the only errors which remain are those accidental errors which must ever continue in a greater or less degree undetermined. The greater the number and

perfection of the observations employed, the more nearly will these errors be determined, and the more nearly will the law of their distribution conform to that which has been assumed as the basis of the method of least squares.

When all known errors have been eliminated, there may yet remain constant errors, and also other errors whose law of distribution is peculiar, such as may arise from the idiosyncrasies of the different observers, from the systematic errors of the adopted star-places in the case of differential observations, and from a variety of other sources; and since the observations themselves furnish the only means of arriving at a knowledge of these errors, it becomes important to discuss them in such a manner that all errors which may be regarded, in a sense more or less extended, as *regular* may be eliminated. When this has been accomplished, the residuals which still remain will enable us to form an estimate of the degree of accuracy which may be attributed to the different series of observations, in order that they may not only be combined in the most advantageous manner, but that also no refinements of calculation may be introduced which are not warranted by the quality of the material to be employed.

The necessity of a preliminary calculation in which a high degree of accuracy is already obtained, is indicated by the fact that, however conscientious the observer may be, his judgment is unconsciously warped by an inherent desire to produce results harmonizing well among themselves, so that a limited series of places may agree to such an extent that the probable error of an observation as derived from the relative discordances would assign a weight vastly in excess of its true value. The combination, however, of a large number of independent data, by exhibiting at least an approximation to the absolute errors of the observations, will indicate nearly what the measure of precision should be. As soon, therefore, as provisional elements which nearly represent the entire series of observations have been found, an attempt should be made to eliminate all errors which may be accurately or approximately determined. The places of the comparison-stars used in the observations should be determined with care from the data available, and should be reduced, by means of the proper systematic corrections, to some standard system. The reduction of the mean places of the stars to apparent places should also be made by means of uniform constants of reduction. The observations will thus be uniformly reduced. Then the perturbations arising from the action of the planets should be computed by means of formulæ which will be investigated in the next chapter, and the observed

places should be freed from these perturbations so as to give the places for a system of osculating elements for a given date.

147. The next step in the process will be to compare the provisional elements with the entire series of observed places thus corrected; and in the calculation of the ephemeris it will be advantageous to correct the places of the sun given by the tables whenever observations are available for that purpose. Then, selecting one or more epochs as the origin, if we compute the coefficients A , B , C in the equation

$$\Delta\theta = A + B\tau + C\tau^2, \quad (125)$$

in the case of each of the spherical co-ordinates, by means of equations of condition formed from all the observations, the standard ephemeris may be corrected so that it may be regarded as representing the actual path of the body during the period included by the observations. When the number of observations is considerable, it will be more convenient to divide the observations into groups, and use the differences between computation and observation for provisional normal places in the formation of the equations of condition for the determination of A , B , and C . It thus appears that the corrected ephemeris which is so essential to a determination of the constant errors peculiar to each series of observations, is obtained without first having determined the most probable system of elements. The corrections computed by means of the equation (125) being applied to the several residuals of each series, we obtain what may be regarded as the actual errors of these observations. The arithmetical or probable mean of the corrected residuals for the series of observations made by each observer may be regarded as the average error of observation for that series. The mean of the average errors of the several series may be regarded as the actual constant error pertaining to all the observations, and the comparison of this final mean with the means found for the different series, respectively, furnishes the probable value of the constant errors due to the peculiarities of the observers; and the constant correction thus found for each observer should be applied to the corresponding residuals already obtained.

In this investigation, if the number of comparisons or the number of wires taken is known, relative weights proportional to the number of comparisons may be adopted for the combination of the residuals for each series. In this manner, observations which, on account of the peculiarities of the observers, are in a certain sense heterogeneous, may be rendered homogeneous, being reduced to a standard which

approaches the absolute in proportion as the number and perfection of the distinct series combined are increased. Whatever constant error remains will be very small, and, besides, will affect all places alike.

The residuals which now remain must be regarded as consisting of the actual errors of observation and of the error of the adopted place of the comparison-star. Hence they will not give the probable error of observation, and will not serve directly for assigning the measures of precision of the series of observations by each observer. Let us, therefore, denote by ϵ_s the mean error of the place of the comparison-star, by ϵ , the mean error of a single comparison; then will $\frac{\epsilon_s}{\sqrt{m}}$ be the mean error of m comparisons, and the mean error of the resulting place of the body will, according to equation (35), be given by

$$\epsilon_0^2 = \frac{\epsilon_s^2}{m} + \epsilon^2. \quad (126)$$

The value of ϵ_0 , in the case of each series, will be found by means of the residuals finally corrected for the constant errors, and the value of ϵ_s is supposed to be determined in the formation of the catalogue of star-places adopted. Hence the actual mean error of an observation consisting of a single comparison will be

$$\epsilon = \sqrt{m(\epsilon_0^2 - \epsilon_s^2)}. \quad (127)$$

The value of ϵ , for each observer having been found in accordance with this equation, the mean error of an observation consisting of m comparisons will be

$$\frac{\epsilon_s}{\sqrt{m}}$$

The mean error of an observation whose weight is unity being denoted by ϵ , the weight of an observation based on m comparisons will be

$$p = \frac{m\epsilon^2}{\epsilon_s^2}. \quad (128)$$

The value of ϵ may be arbitrarily assigned, and we may adopt for it $\pm 10''$ or any other number of seconds for which the resulting values of p will be convenient numbers.

When all the observations are differential observations, and the stars of comparison are included in the fundamental list, if we do not take into account the number of comparisons on which each observed

place depends, it will not be necessary to consider ε_s , and we may then derive ε_s directly from the residuals corrected for constant errors. Further, in the case of meridian observations, the error which corresponds to ε_s will be extremely small, and hence it is only when these are combined with equatorial observations, or when equatorial observations based on different numbers of comparisons are combined, that the separation of the errors into the two component parts becomes necessary for a proper determination of the relative weights.

According to the complete method here indicated, after having eliminated as far as possible all constant errors, including the corrections assigned by equation (125) to be applied to the provisional ephemeris, we find the value of ε_s given by the equation

$$n\varepsilon_s^2 = [mvv] - [m]\varepsilon_s^2, \quad (129)$$

in which n denotes the number of observations; $m, m', m'',$ &c. the number of comparisons for the respective observations; and $v, v', v'',$ &c. the corresponding residuals. Then, by means of equation (128), assuming a convenient number for ε_s , we compute the weight of each observation. Thus, for example, let the residuals and corresponding values of m be as follows:—

| $\Delta\theta$ | m | $\Delta\theta$ | m |
|----------------|-----|----------------|-----|
| + 2".0 | 5, | — 1".0 | 7, |
| — 1 .8 | 5, | + 1 .5 | 5, |
| — 0 .4 | 10, | + 4 .1 | 8, |
| — 5 .5 | 5, | 0 .0 | 5. |

Let the mean error of the place of a comparison-star be

$$\varepsilon_s = \pm 2''.0;$$

then we have $n = 8$, and, according to (129),

$$8\varepsilon_s^2 = 341.78 - 200.0,$$

which gives

$$\varepsilon_s = \pm 4''.2.$$

Let us now adopt as the unit of weight that for which the mean error is

$$\varepsilon = \pm 3''.0;$$

then we obtain by means of equation (128), for the weights of the observations,

$$2.5, \quad 2.5, \quad 5.1, \quad 2.5, \quad 3.6, \quad 2.5, \quad 4.1, \quad 2.5,$$

respectively.

In this manner the weights of the observations in the series made by each observer must be determined, using throughout the same value of ϵ . Then the differences between the places computed from the provisional elements to be corrected and the observed places corrected for the constant error of the observer, must be combined according to the equations (123) and (125), the adopted values of p , p' , p'' , &c. being those found from (128). Thus will be obtained the final residuals for the formation of the equations of condition from which to derive the most probable value of the corrections to be applied to the elements. The relative weights of these normals will be indicated by the sums formed by adding together the weights of the observations combined in the formation of each normal, and the unit of weight will depend on the adopted value of ϵ . If it be desired to adopt a different unit of weight in the case of the solution of the equations of condition, such, for example, that the weight of an equation of average precision shall be unity, we may simply divide the weights of the normals by any number p_0 which will satisfy the condition imposed. The mean error of an observation whose weight is unity will then be given by

$$\frac{\epsilon}{\sqrt{p_0}},$$

the value of ϵ being that used in the determination of the weights p , p' , &c.

148. The observations of comets are liable to be affected by other errors in addition to those which are common to these and to planetary observations. Different observers will fix upon different points as the proper point to be observed, and all of these may differ from the actual position of the centre of gravity of the comet; and further, on account of changes in the physical appearance of the comet, the same observer may on different nights select different points. These circumstances concur to vitiate the normal places, inasmuch as the resulting errors, although in a certain sense fortuitous, are yet such that the law of their distribution is evidently different from that which is adopted as the basis of the method of least squares. The impossibility of assigning the actual limits and the law of distribution of many errors of this class, renders it necessary to adopt empirical methods, the success of which will depend on the discrimination of the computer.

If ϵ_0 denotes the mean error of an observation based on m com-

parisons, and ϵ_c the mean error to be feared on account of the peculiarities of the physical appearance of the comet,

$$\sqrt{\epsilon_0^2 + \epsilon_c^2}$$

will express the mean error of the residuals; and if n of these residuals are combined in the formation of a normal place, the mean error of the normal will be given by

$$\epsilon_n^2 = \frac{[\epsilon_0^2]}{n} + \epsilon_c^2. \quad (130)$$

The value of ϵ_c^2 may be determined approximately from the data furnished by the observations. Thus, if the mean error of a single comparison, for the different observers, has been determined by means of the differences between single comparisons and the arithmetical mean of a considerable number of comparisons, and if the mean error of the place of a comparison-star has also been determined, the equation (126) will give the corresponding value of ϵ_0^2 ; then the actual differences between computation and observation obtained by eliminating the error of the ephemeris and such constant errors as may be determined, will furnish an approximate value of ϵ_c by means of the formula

$$\epsilon_c^2 = \frac{[vv]}{n} - \epsilon_0^2,$$

in which n denotes the number of observations combined.

Sometimes, also, in the case of comets, in order to detect the operation of any abnormal force or circumstance producing different effects in different parts of the orbit, it may be expedient to divide the observations into two distinct groups, the first including the observations made before the time of perihelion passage, and the other including those subsequent to that epoch.

149. The circumstances of the problem will often suggest appropriate modifications of the complete process of determining the relative weights of the observations to be combined, or indeed a relaxation from the requirements of the more rigorous method. Thus, if on account of the number or quality of the data it is not considered necessary to compute the relative weights with the greatest precision attainable, it will suffice, when the discussion of the observations has been carried to an extent sufficient to make an approximate estimate of the relative weights, to assume, without considering the number of comparisons, a weight 1 for the observations at one observatory, a

weight $\frac{2}{3}$ for another class of observations, $\frac{1}{3}$ for a third class, and so on. It should be observed, also, that when there are but few observations to be combined, the application of the formulæ for the mean or probable errors may be in a degree fallacious, the resulting values of these errors being little more than rude approximations; still the mean or probable errors as thus determined furnish the most reliable means of estimating the relative weights of the observations made by different observers, since otherwise the scale of weights would depend on the arbitrary discretion of the computer. Further, in a complete investigation, even when the very greatest care has been taken in the theoretical discussion, on account of independent known circumstances connected with some particular observation, it may be expedient to change arbitrarily the weight assigned by theory to certain of the normal places. It may also be advisable to reject entirely those observations whose weight is less than a certain limit which may be regarded as the standard of excellence below which the observations should be rejected; and it will be proper to reject observations which do not afford the data requisite for a homogeneous combination with the others according to the principles already explained. But in all cases the rejection of apparently doubtful observations should not be carried to any considerable extent unless a very large number of good observations are available. The mere apparent discrepancy between any residual and the others of a series, is not in itself sufficient to warrant its rejection unless facts are known which would independently assign to it a low degree of precision.

A doubtful observation will have the greatest influence in vitiating the resulting normal place when but a small number of observed places are combined; and hence, since we cannot assume that the law of the distribution of errors, according to which the method of least squares is derived, will be complied with in the case of only a few observations, it will not in general be safe to reject an observation provided that it surpasses a limit which is fixed by the adopted theory of errors. If the number of observations is so large that the distribution of the errors may be assumed to conform to the theory adopted, it will be possible to assign a limit such that a residual which surpasses it may be rejected. Thus, in a series of m observations, according to the expression (19), the number of errors greater than nr will be

$$m \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{nr} e^{-t^2} dt \right);$$

and when n has a value such that the value of this expression is less than 0.5, the error nr will have a greater probability against it than for it, and hence it may be rejected. The expression for finding the limiting value of n therefore becomes

$$\frac{2}{\sqrt{\pi}} \int_0^{nhr} e^{-t^2} dt = 1 - \frac{1}{2m}. \quad (131)$$

By means of this equation we derive for given values of m the corresponding values of $nhr = 0.47694n$, and hence the values of n . For convenient application, it will be preferable to use ϵ instead of r , and if we put $n' = 0.67449n$, the limiting error will be $n'\epsilon$, and the values of n' corresponding to given values of m will be as exhibited in the following table.

TABLE.

| m | n' | m | n' | m | n' | m | n' |
|-----|-------|-----|-------|-----|-------|-----|-------|
| 6 | 1.732 | 20 | 2.241 | 55 | 2.608 | 90 | 2.773 |
| 8 | 1.863 | 25 | 2.326 | 60 | 2.638 | 95 | 2.791 |
| 10 | 1.960 | 30 | 2.394 | 65 | 2.665 | 100 | 2.807 |
| 12 | 2.037 | 35 | 2.450 | 70 | 2.690 | 200 | 3.020 |
| 14 | 2.100 | 40 | 2.498 | 75 | 2.713 | 300 | 3.143 |
| 16 | 2.154 | 45 | 2.539 | 80 | 2.734 | 400 | 3.224 |
| 18 | 2.200 | 50 | 2.576 | 85 | 2.754 | 500 | 3.289 |

According to this method, we first find the mean error of an observation by means of all the residuals. Then, with the value of m as the argument, we take from the table the corresponding value of n' , and if one of the residuals exceeds the value $n'\epsilon$ it must be rejected. Again, finding a new value of ϵ from the remaining $m - 1$ residuals, and repeating the operation, it will be seen whether another observation should be rejected; and the process may be continued until a limit is reached which does not require the further rejection of observations. Thus, for example, in the case of 50 observations in which the residuals $-11''.5$ and $+7''.8$ occur, let the sum of the squares of the residuals be

$$[vv] = 320.4.$$

Then, according to equation (30), we shall have

$$\epsilon = \pm 2''.56.$$

Corresponding to the value $m = 50$, the table gives $n' = 2.576$, and the limiting value of the error becomes

$$n'\varepsilon = \pm 6''.6;$$

and hence the residuals $-11''.5$ and $+7''.8$ are rejected. Recomputing the mean error of an observation, we have

$$\varepsilon = \sqrt{\frac{320.4 - 193.09}{47}} = \pm 1''.65.$$

In the formation of a normal place, when the mean error of an observation has been inferred from only a small number of observations, according to what has been stated, it will not be safe to rely upon the equation (131) for the necessity of the rejection of a doubtful observation. But if any abnormal influence is suspected, or if any antecedent discussion of observations by the same observer, made under similar circumstances, seems to indicate that an error of a given magnitude is highly improbable, the application of this formula will serve to confirm or remove the doubt already created. Much will therefore depend on the discrimination of the computer, and on his knowledge of the various sources of error which may conspire continuously or discontinuously in the production of large apparent errors. It is the business of the observer to indicate the circumstances peculiar to the phenomenon observed, the instruments employed, and the methods of observation; and the discussion of the data thus furnished by different observers, as far as possible in accordance with the strict requirements of the adopted theory of errors, will furnish results which must be regarded as the best which can be derived from the evidence contributed by all the observations.

150. When the final normal places have been derived, the differences between these and the corresponding places computed from the provisional elements to be corrected, taken in the sense computation minus observation, give the values of n , n' , n'' , &c. which are the absolute terms of the equations of condition. By means of these elements we compute also the values of the differential coefficients of each of the spherical co-ordinates with respect to each of the elements to be corrected. These differential coefficients give the values of the coefficients a , b , c , a' , b' , &c. in the equations of condition. The mode of calculating these coefficients, for different systems of co-ordinates, and the mode of forming the equations of condition, have been fully developed in the second chapter. It is of great import-

ance that the numerical values of these coefficients should be carefully checked by direct calculation, assigning variations to the elements, or by means of differences when this test can be successfully applied. In assigning increments to the elements in order to check the formation of the equations, they should not be so large that the neglected terms of the second order become sensible, nor so small that they do not afford the required certainty by means of the agreement of the corresponding variations of the spherical co-ordinates as obtained by substitution and by direct calculation.

As soon as the equations of condition have been thus formed, we multiply each of them by the square root of its weight as given by the adopted relative weights of the normal places; and these equations will thus be reduced to the same weight. In general, the numerical values of the coefficients will be such that it will be convenient, although not essential, to adopt as the unit of weight that which is the average of the weights of the normals, so that the numbers by which most of the equations will be multiplied will not differ much from unity. The reduction of the equations to a uniform measure of precision having been effected, it remains to combine them according to the method of least squares in order to derive the most probable values of the unknown quantities, together with the relative weights of these values. It should be observed, however, that the numerical calculation in the combination and solution of these equations, and especially the required agreement of some of the checks of the calculation, will be facilitated by having the numerical values of the several coefficients not very unequal. If, therefore, the coefficient a of any unknown quantity x is in each of the equations numerically much greater or much less than in the case of the other unknown quantities, we may adopt as the corresponding unknown quantity to be determined, not x but νx , ν being any entire or fractional number such that the new coefficients $\frac{a}{\nu}$, $\frac{a'}{\nu}$, &c. shall be made to agree in magnitude with the other coefficients. The unknown quantity whose value will then be derived by the solution of the equations will be νx , and the corresponding weight will be that of νx . To find the weight of x from that of νx , we have the equation

$$p_x = \nu^2 p_{\nu x} \quad (132)$$

In the same manner, the coefficient of any other unknown quantity may be changed, and the coefficients of all the unknown quantities may thus be made to agree in magnitude within moderate limits, the

advantage of which, in the numerical solution of the equations, will be apparent by a consideration of the mode of proving the calculation of the coefficients in the normal equations. It will be expedient, also, to take for ν some integral power of 10, or, when a fractional value is required, the corresponding decimal. It may be remarked, further, that the introduction of ν is generally required only when the coefficient of one of the unknown quantities is very large, as frequently happens in the case of the variation of the mean daily motion μ .

When the coefficients of some of the unknown quantities are extremely small in all the equations of condition to be combined, an approximate solution, and often one which is sufficiently accurate for the purposes required, may be obtained by first neglecting these quantities entirely, and afterwards determining them separately. In general, however, this can only be done when it is certainly known that the influence of the neglected terms is not of sensible magnitude, or when at least approximate values of these terms are already given. When we adopt the approximate plane of the orbit as the fundamental plane, the equations for the longitude involve only four elements, and the coefficients of the variations of these elements in the equations for the latitudes are always very small. Hence, for an approximate solution, we may first solve the equations involving four unknown quantities as furnished by the longitudes, and then, substituting the resulting values in the equations for the latitudes, they will contain but two unknown quantities, namely, those which give the corrections to be applied to Ω and i .

151. When the number of equations of condition is large, the computation of the numerical values of the coefficients in the normal equations will entail considerable labor; and hence it is desirable to arrange the calculation in a convenient form, applying also the checks which have been indicated. The most convenient arrangement will be to write the logarithms of the absolute terms $n, n', n'',$ &c. in a horizontal line, directly under these the logarithms of the coefficients $a, a', a'',$ &c., then the logarithms of $b, b', b'',$ &c., and so on. Then writing, in a corresponding form, the values of $\log n, \log n',$ &c. on a slip of paper, by bringing this successively over each line, the sums $[nn], [an], [bn],$ &c. will be readily formed. Again, writing on another slip of paper the logarithms of $a, a', a'',$ &c., and placing this slip successively over the lines containing the coefficients, we derive the values $[aa], [ab], [ac],$ &c. The multiplication by $b, c, d,$

&c. successively is effected in a similar manner; and thus will be derived $[bb]$, $[bc]$, $[bd]$, &c., and finally $[ff]$ in the case of six unknown quantities. In forming these sums, in the cases of sums of positive and negative quantities, it is convenient as well as conducive to accuracy to write the positive values in one vertical column and the negative values in a separate column, and take the difference of the sums of the numbers in the respective columns. The proof of the calculation of the coefficients of the normal equations is effected by introducing s , s' , s'' , &c., the algebraic sums of all the coefficients in the respective equations of condition, and treating these as the coefficients of an additional unknown quantity, thus forming directly the sums $[sn]$, $[as]$, $[bs]$, $[cs]$, &c. Then, according to the equations (76) and (77), the values thus found should agree with those obtained by taking the corresponding sums of the coefficients in the normal equations.

The normal equations being thus derived, the next step in the process is the determination of the values of the auxiliary quantities necessary for the formation of the equations (74). An examination of the equations (54), (55), &c., by means of which these auxiliaries are determined, will indicate at once a convenient and systematic arrangement of the numerical calculation. Thus, we first write in a horizontal line the values of $[aa]$, $[ab]$, $[ac]$, \dots $[as]$, $[an]$, and directly under them the corresponding logarithms. Next, we write under these, commencing with $[ab]$, the values of $[bb]$, $[bc]$, $[bd]$, \dots $[bs]$, $[bn]$; then, adding the logarithm of the factor $\frac{[ab]}{[aa]}$ to the logarithms of $[ab]$, $[ac]$, &c. successively, we write the value of $\frac{[ab]}{[aa]}$ under $[bb]$, that of $\frac{[ab]}{[aa]}$ under $[bc]$, and so on. Subtracting the numbers in this line from those in the line above, the differences give the values of $[bb.1]$, $[bc.1]$, \dots $[bs.1]$, $[bn.1]$, to be written in the next line, and the logarithms of these we write directly under them. Then we write in a horizontal line the values of $[cc]$, $[cd]$, \dots $[cs]$, $[cn]$, placing $[cc]$ under $[bc.1]$, and, having added the logarithm of $\frac{[ac]}{[aa]}$ to the logarithms of $[ac]$, $[ad]$, &c. in succession, we derive, according to the equations (55) and (58), the values of $[cc.1]$, $[cd.1]$, \dots $[cs.1]$, $[cn.1]$, which are to be placed under the corresponding quantities $[cc]$, $[cd]$, &c. Next, we subtract from these, respectively, the products

$$\frac{[bc.1]}{[bb.1]}[bc.1], \quad \frac{[bc.1]}{[bb.1]}[bd.1], \quad \dots \quad \frac{[bc.1]}{[bb.1]}[bs.1], \quad \frac{[bc.1]}{[bb.1]}[bn.1],$$

and thus derive the values of $[cc.2]$, $[cd.2]$, $\dots [cs.2]$, $[cn.2]$, which are to be written in the next horizontal line and under them their logarithms. Then we introduce, in a similar manner, the coefficients $[dd]$, $[de]$, $\dots [dn]$, writing $[dd]$ under $[cd.2]$; and from each of these in succession we subtract the products

$$\frac{[ad]}{[aa]}[ad], \dots \frac{[ad]}{[aa]}[as], \quad \frac{[ad]}{[aa]}[an],$$

thus finding the values of $[dd.1]$, $[de.1]$, $\dots [dn.1]$. From these we subtract the products

$$\frac{[bd.1]}{[bb.1]}[bd.1], \quad \frac{[bd.1]}{[bb.1]}[be.1], \dots \frac{[bd.1]}{[bb.1]}[bn.1],$$

respectively, which operation gives the values of $[dd.2]$, $[de.2]$, $\dots [dn.2]$. From these results we subtract the products

$$\frac{[cd.2]}{[cc.2]}[cd.2], \quad \frac{[cd.2]}{[cc.2]}[ce.2], \dots \frac{[cd.2]}{[cc.2]}[cn.2],$$

and derive $[dd.3]$, $[de.3]$, $\dots [dn.3]$ under which we write the corresponding logarithms. Then we introduce $[ee]$, $[ef]$, $[es]$, and $[en]$, writing $[ee]$ under $[de.3]$. First, subtracting $\frac{[ae]}{[aa]}[ae]$, $\frac{[ae]}{[aa]}[af]$, $\dots \frac{[ae]}{[aa]}[an]$, we get $[ee.1]$, $[ef.1]$, $[es.1]$, and $[en.1]$; then subtracting from these the products

$$\frac{[be.1]}{[bb.1]}[be.1], \quad \frac{[be.1]}{[bb.1]}[bf.1], \dots \frac{[be.1]}{[bb.1]}[bn.1],$$

we obtain the values of $[ee.2]$, $[ef.2]$, $[es.2]$, and $[en.2]$. Again, subtracting

$$\frac{[ce.2]}{[cc.2]}[ce.2], \quad \frac{[ce.2]}{[cc.2]}[cf.2], \dots \frac{[ce.2]}{[cc.2]}[cn.2],$$

we have the values of $[ee.3]$, $[ef.3]$, $[es.3]$, $[en.3]$; and finally, subtracting from these the products

$$\frac{[de.3]}{[dd.3]}[de.3], \quad \frac{[de.3]}{[dd.3]}[df.3], \dots \frac{[de.3]}{[dd.3]}[dn.3],$$

we derive the results for $[ee.4]$, $[ef.4]$, $[es.4]$, and $[en.4]$; under which the corresponding logarithms are to be written.

If there are six unknown quantities to be determined, we must further write in a horizontal line the values of $[ff]$, $[fs]$, and $[fn]$,

placing $[ff]$ under $[ef.4]$, and by means of five successive subtractions entirely analogous to what precedes, and as indicated by the remaining equations for the auxiliaries, we obtain the values of $[ff.5]$, $[fs.5]$, and $[fn.5]$.

The values of $[bs.1]$, $[cs.1]$, $[cs.2]$, &c. serve to check the calculation of the successive auxiliary coefficients. Thus we must have

$$\begin{aligned} [bb.1] + [bc.1] + [bd.1] + [be.1] + [bf.1] &= [bs.1] \\ [bc.1] + [cc.1] + [cd.1] + [ce.1] + [cf.1] &= [cs.1], \text{ \&c.}, \\ [cc.2] + [cd.2] + [ce.2] + [cf.2] &= [cs.2], \\ [cd.2] + [dd.2] + [de.2] + [df.2] &= [ds.2], \text{ \&c.} \end{aligned}$$

Hence it appears that when the numerical calculation is arranged as above suggested, the auxiliary containing s must, in each line, be equal to the sum of all the terms to the left of it in the same line and of those terms containing the same distinguishing numeral found in a vertical column over the last quantity at the left of this line.

There will yet remain only the auxiliaries which are derived from $[sn]$ and $[nn]$ to be determined. These additional auxiliaries will be found by means of the formulæ

$$\begin{aligned} [sn.1] &= [sn] - \frac{[an]}{[aa]}[as], & [sn.2] &= [sn.1] - \frac{[bn.1]}{[bb.1]}[bs.1], \\ [sn.3] &= [sn.2] - \frac{[cn.2]}{[cc.2]}[cs.2], & [sn.4] &= [sn.3] - \frac{[dn.3]}{[dd.3]}[ds.3], \text{ (133)} \\ [sn.5] &= [sn.4] - \frac{[en.4]}{[ee.4]}[es.4], & [sn.6] &= [sn.5] - \frac{[fn.5]}{[ff.5]}[fs.5], \end{aligned}$$

and the equations (81) and (83). The arrangement of the numerical process should be similar to that already explained.

The values of $[sn.1]$, $[sn.2]$, &c. check the accuracy of the results for $[bn.1]$, $[cn.1]$, $[cn.2]$, $[dn.3]$, &c. by means of the equations

$$\begin{aligned} [bn.1] + [cn.1] + [dn.1] + [en.1] + [fn.1] &= [sn.1], \\ [cn.2] + [dn.2] + [en.2] + [fn.2] &= [sn.2], \\ [dn.3] + [en.3] + [fn.3] &= [sn.3], \\ [en.4] + [fn.4] &= [sn.4], \\ [fn.5] &= [sn.5]. \end{aligned} \quad (134)$$

It appears further, that, in the case of six unknown quantities, since $[fs.5] = [ff.5]$, we have $[sn.6] = 0$.

Having thus determined the numerical values of the auxiliaries required, we are prepared to form at once the equations (74), by means of which the values of the unknown quantities will be determined

by successive substitution, first finding t from the last of these equations, then substituting this result in the equation next to the last and thus deriving the value of w , and so on until all the unknown quantities have been determined. It will be observed that the logarithms of the coefficients of the unknown quantities in these equations will have been already found in the computation of the auxiliaries.

If we add together the several equations of (74), first clearing them of fractions, we get

$$\begin{aligned} 0 = [aa]x + ([ab] + [bb.1])y + ([ac] + [bc.1] + [cc.2])z \\ + ([ad] + [bd.1] + [cd.2] + [dd.3])u \\ + ([ae] + [be.1] + [ce.2] + [de.3] + [ee.4])w \\ + ([af] + [bf.1] + [cf.2] + [df.3] + [ef.4] + [ff.5])t \\ + [an] + [bn.1] + [cn.2] + [dn.3] + [en.4] + [fn.5]; \end{aligned} \quad (135)$$

and this equation must be satisfied by the values of x, y, z , &c. found from (74).

152. EXAMPLE.—The arrangement of the calculation in the case of any other number of unknown quantities is precisely similar; and to illustrate the entire process let us take the following equations, each of which is already multiplied by the square root of its weight:—

$$\begin{aligned} 0.707x + 2.052y - 2.372z - 0.221u + 6''.58 &= 0, \\ 0.471x + 1.347y - 1.715z - 0.085u + 1.63 &= 0, \\ 0.260x + 0.770y - 0.356z + 0.483u - 4.40 &= 0, \\ 0.092x + 0.343y + 0.235z + 0.469u - 10.21 &= 0, \\ 0.414x + 1.204y - 1.506z - 0.205u + 3.99 &= 0, \\ 0.040x + 0.150y + 0.104z + 0.206u - 4.34 &= 0. \end{aligned}$$

First, we derive

$$\begin{aligned} [nn] &= 204.313, \\ [an] &= + 4.815, [aa] = + 0.971, \\ [bn] &= + 12.961, [ab] = + 2.821, [bb] = + 8.208, \\ [cn] &= - 25.697, [ac] = - 3.175, [bc] = - 9.168, [cc] = + 11.028, \\ [dn] &= - 10.218, [ad] = - 0.104, [bd] = - 0.251, [cd] = + 0.933, [dd] = + 0.594, \\ [sn] &= - 18.139, [as] = + 0.513, [bs] = + 1.610, [cs] = - 0.377, [ds] = + 1.177. \end{aligned}$$

The values of $[sn]$, $[as]$, $[bs]$, $[cs]$, and $[ds]$, found by taking the sums of the normal coefficients, agree exactly with the values computed directly, thus proving the calculation of these coefficients. The normal equations are, therefore,

$$\begin{aligned} 0.971x + 2.821y - 3.175z - 0.104u + 4.815 &= 0, \\ 2.821x + 8.208y - 9.168z - 0.251u + 12.961 &= 0, \\ -3.175x - 9.168y + 11.028z + 0.938u - 25.697 &= 0, \\ -0.104x - 0.251y + 0.938z + 0.594u - 10.218 &= 0. \end{aligned}$$

It will be observed that the coefficients in these equations are numerically greater than in the equations of condition; and this will generally be the case. Hence, if we use logarithms of five decimals in forming the normal equations, it will be expedient to use tables of six or seven decimals in the solution of these equations.

Arranging the process of elimination in the most convenient form, the successive results are as follows:—

| | | | | |
|---------------------------|---------------------|---------------------|----------------------|---------------------|
| $[b\bar{b}.1] = +0.0123,$ | $[bc.1] = +0.0562,$ | $[bd.1] = +0.0511,$ | $[bs.1] = +0.1196,$ | $[bn.1] = -1.0278,$ |
| $[cc.1] = +0.6463,$ | $[cd.1] = +0.5979,$ | $[cs.1] = +1.3004,$ | $[cn.1] = -9.9528,$ | |
| $[ce.2] = +0.3895,$ | $[cd.2] = +0.3644,$ | $[cs.2] = +0.7539,$ | $[cn.2] = -5.2567,$ | |
| | $[dd.1] = +0.5829,$ | $[ds.1] = +1.2319,$ | $[dn.1] = -9.7023,$ | |
| | $[dd.2] = +0.3706,$ | $[ds.2] = +0.7350,$ | $[dn.2] = -5.4323,$ | |
| | $[dd.3] = +0.0297,$ | $[ds.3] = +0.0297$ | $[dn.3] = -0.5143,$ | |
| | | $[nn.1] = 180.436,$ | $[sn.1] = -20.6828,$ | |
| | | $[nn.2] = 94.552,$ | $[sn.2] = -10.6869,$ | |
| | | $[nn.3] = 23.608,$ | $[sn.3] = -0.5143,$ | |
| | | $[nn.4] = 14.698,$ | $[sn.4] = 0.$ | |

The several checks agree completely, and only the value of $[nn.4]$ remains to be proved. The equations (74) therefore give

$$\begin{aligned} x + 2.9052y - 3.2698z - 0.1071u + 4.9588 &= 0, \\ y + 4.5691z + 4.1545u - 83.5610 &= 0, \\ z + 0.9356u - 13.4960 &= 0, \\ u - 17.3165 &= 0, \end{aligned}$$

and from these we get

$$u = +17''.316, \quad z = -2''.705, \quad y = +23''.977, \quad x = -81''.608.$$

Then the equation (135) becomes

$$0 = +0.9710x + 2.8333y - 2.7293z + 0.3412u - 1.9838,$$

which is satisfied by the preceding values of the unknown quantities.

If we substitute these values of x , y , z , and u in the equations of condition already reduced to the same weight by multiplication by the square roots of their weights, we obtain the residuals

$$+0''.67, \quad -1''.34, \quad +2''.17, \quad -2''.01, \quad -0''.40, \quad -0''.72,$$

The sum of the squares of these gives

$$[vv] = [nn.4] = 11.672,$$

and the difference between this result and the value 14.698 already

found is due to the decimals neglected in the computation of the numerical values of the several auxiliaries. The sum of all the equations of condition gives generally

$$[a]x + [b]y + [c]z + [d]u + \dots + [n] = [v], \quad (136)$$

which may be used to check the substitution of the numerical values in the determination of v , v' , &c. Thus, we have, for the values here given,

$$1.984x + 5.866y - 5.610z + 0.647u - 6.75 = [v] = -1.''63.$$

It remains yet to determine the relative weights of the resulting values of the unknown quantities. For this purpose we may apply any of the various methods already given. The weights of u and z may be found directly from the auxiliaries whose values have been computed. Thus, we have

$$p_u = [dd.3] = 0.0297, \quad p_z = \frac{[dd.3]}{[dd.2]}[cc.2] = 0.0312.$$

If we now completely reverse the order of elimination from the normal equations, and determine x first, we obtain the values

$$\begin{aligned} [bb.2] &= +0.0425, & [aa.2] &= +0.0033, \\ [aa.3] &= +0.00056, & [nn.4] &= 14.665, \end{aligned}$$

and also

$$x = -82.''750, \quad y = +24.''365, \quad z = -2.''699, \quad u = +17.''272.$$

The small differences between these results and those obtained by the first elimination arise from the decimals neglected. This second elimination furnishes at once the weights of x and y , namely,

$$p_x = [aa.3] = 0.00056, \quad p_y = \frac{[aa.3]}{[aa.2]}[bb.2] = 0.0072.$$

We may also compute the weights by means of the equations (96). Thus, to find the weight of y , we have

$$[dd.2]_b = [dd.1] - \frac{[cd.1]}{[ce.1]}[cd.1] = +0.02977,$$

and hence

$$p_y = \frac{[dd.3]}{[dd.2]_b} \cdot \frac{[cc.2]}{[ce.1]}[bb.1] = 0.0074.$$

The equations (103) and (108) are convenient for the determination of the values and weights of the unknown quantities separately.

Thus, by means of the values of the auxiliaries obtained in the first elimination, we find from the equations (100), (101), and (102),

$$\begin{array}{lll} A' = -2.9052, & A'' = +16.5442, & A''' = -3.3012, \\ B' = -4.5691, & B'' = +0.1202, & C''' = -0.9356, \end{array}$$

and then the equations (103) and (108) give

$$\begin{array}{llll} x = -81''.609, & y = +23''.977, & z = -2''.705, & u = +17''.316, \\ p_x = 0.00057, & p_y = 0.0074, & p_z = 0.0312, & p_u = 0.0297, \end{array}$$

agreeing with the results obtained by means of the other methods. The weights are so small that it may be inferred at once that the values of x, y, z , and u are very uncertain, although they are those which best satisfy the given equations. It will be observed that if we multiply the first normal equation by 2.9, the resulting equation will differ very little from the second normal equation, and hence we have nearly the case presented in which the number of independent relations is one less than the number of unknown quantities.

The uncertainty of the solution will be further indicated by determining the probable errors of the results, although on account of the small number of equations the probable or mean errors obtained may be little more than rude approximations. Thus, adopting the value of $[vv]$ obtained by direct substitution, we have

$$\epsilon = \sqrt{\frac{[nn.4]}{m-\mu}} = \sqrt{\frac{11.672}{6-4}} = 2.416,$$

and hence

$$r = \pm 1''.629,$$

which is the probable error of the absolute term of an equation of condition whose weight is unity. Then the equations

$$r_x = \frac{r}{\sqrt{p_x}}, \quad r_y = \frac{r}{\sqrt{p_y}}, \quad r_z = \frac{r}{\sqrt{p_z}}, \text{ \&c.,}$$

give

$$r_x = \pm 68''.25, \quad r_y = \pm 18''.94, \quad r_z = \pm 9''.22, \quad r_u = \pm 9''.45.$$

It thus appears that the probable error of z exceeds the value obtained for the quantity itself, and that although the sum of the squares of the residuals is reduced from 204.31 to 11.67, the results are still quite uncertain.

153. The certainty of the solution will be greatest when the coefficients in the equations of condition and also in the normal equations

differ very considerably both in magnitude and in sign. In the correction of the elements of the orbit of a planet when the observations extend only over a short interval of time, the coefficients will generally change value so slowly that the equations for the direct determination of the corrections to be applied to the elements will not afford a satisfactory solution. In such cases it will be expedient to form the equations for the determination of a less number of quantities from which the corrected elements may be subsequently derived. Thus we may determine the corrections to be applied to two assumed geocentric distances or to any other quantities which afford the required convenience in the solution of the problem, various formulæ for which have been given in the preceding chapter. The quantities selected for correction should be known functions of the elements, and such that the equations to be solved, in order to combine all the observed places, shall not be subject to any uncertainty in the solution. But when the observations extend over a long period, the most complete determination of the corrections to be applied to the provisional elements will be obtained by forming the equations for these variations directly, and combining them as already explained. A complete proof of the accuracy of the entire calculation will be obtained by computing the normal places directly from the elements as finally corrected, and comparing the residuals thus derived with those given by the substitution of the adopted values of the unknown quantities in the original equations of condition.

If the elements to be corrected differ so much from the true values that the squares and products of the corrections are of sensible magnitude, so that the assumption of a linear form for the equations does not afford the required accuracy, it will be necessary to solve the equations first provisionally, and, having applied the resulting corrections to the elements, we compute the places of the body directly from the corrected elements, and the differences between these and the observed places furnish new values of n , n' , n'' , &c., to be used in a repetition of the solution. The corrections which result from the second solution will be small, and, being applied to the elements as corrected by the first solution, will furnish satisfactory results. In this new solution it will not in general be necessary to recompute the coefficients of the unknown quantities in the equations of condition, since the variations of the elements will not be large enough to affect sensibly the values of their differential coefficients with respect to the observed spherical co-ordinates. Cases may occur, however, in which it may become necessary to recompute the coefficients of one

or more of the unknown quantities, but only when these coefficients are very considerably changed by a small variation in the adopted values of the elements employed in the calculation. In such cases the residuals obtained by substitution in the equations of condition will not agree with those obtained by direct calculation unless the corrections applied to the corresponding elements are very small. It may also be remarked that often, and especially in a repetition of the solution so as to include terms of the second order, it will be sufficiently accurate to relax a little the rigorous requirements of a complete solution, and use, instead of the actual coefficients, equivalent numbers which are more convenient in the numerical operations required. Although the greatest confidence should be placed in the accuracy of the results obtained as far as possible in strict accordance with the requirements of the theory, yet the uncertainty of the determination of the relative weights in the combination of a series of observations, as well as the effect of uneliminated constant errors, may at least warrant a little latitude in the numerical application, provided that the weights of the results are not thereby much affected. A constant error may in fact be regarded as an unknown quantity to be determined, and since the effect of the omission of one of the unknown quantities is to diminish the probable errors of the resulting values of the others, it is evident that, on account of the existence of constant errors not determined, the values of the variables obtained by the method of least squares from different corresponding series of observations may differ beyond the limits which the probable errors of the different determinations have assigned. Further, it should be observed that, on account of the unavoidable uncertainty in the estimation of the weights of the observations in the preliminary combination, the probable error of an observed place whose weight is unity as determined by the final residuals given by the equations of condition, may not agree exactly with that indicated by the prior discussion of the observations.

154. In the case of very eccentric orbits in which the corrections to be applied to certain elements are not indicated with certainty by the observations, it will often become necessary to make that whose weight is very small the last in the elimination, and determine the other corrections as functions of this one; and whenever the coefficients of two of the unknown quantities are nearly equal or have nearly the same ratio to each other in all the different equations of condition, this method is indispensable unless the difficulty is reme

died by other means, such as the introduction of different elements or different combinations of the same elements. The equations (113) furnish the values of the unknown quantities when we neglect that which is to be determined independently; and then the equations (114) give the required expressions for the complete values of these quantities. Thus, when a comet has been observed only during a brief period, the ellipticity of the orbit, however, being plainly indicated by the observations, the determination of the correction to be applied to the mean daily motion as given by the provisional elements, in connection with the corrections of the other elements, will necessarily be quite uncertain, and this uncertainty may very greatly affect all the results. Hence the elimination will be so arranged that $\Delta\mu$ shall be the last, and the other corrections will be determined as functions of this quantity. The substitution of the results thus derived in the equations of condition will give for each residual an expression of the following form:—

$$\Delta\theta = v_0 + \gamma\Delta\mu.$$

Therefore we shall have

$$[vv] = [v_0v_0] + 2[v_0\gamma]\Delta\mu + [\gamma\gamma]\Delta\mu^2, \quad (137)$$

which may be applied more conveniently in the equivalent form

$$[vv] = [v_0v_0] - \frac{[v_0\gamma]}{[\gamma\gamma]}[v_0\gamma] + [\gamma\gamma]\left(\Delta\mu + \frac{[v_0\gamma]}{[\gamma\gamma]}\right)^2. \quad (138)$$

The most probable value of $\Delta\mu$ will be that which renders $[vv]$ a minimum, or

$$\Delta\mu = -\frac{[v_0\gamma]}{[\gamma\gamma]}, \quad (139)$$

and the corresponding value of the sum of the squares of the residuals is

$$[vv] = [v_0v_0] - \frac{[v_0\gamma]}{[\gamma\gamma]}[v_0\gamma]. \quad (140)$$

The correction given by equation (139) having been applied to μ , the result may be regarded as the most probable value of that element, and the corresponding values of the corrections of the other elements as determined by the equations (114) having been also duly applied, we obtain the most probable system of elements. These, however, may still be expressed in the form

$$\Omega + A_0\Delta\mu, \quad i + B_0\Delta\mu, \quad \pi + C_0\Delta\mu, \text{ \&c.}$$

the coefficients A_0 , B_0 , C_0 , &c. being those given by the equations (114), and thus the elements may be derived which correspond to any assumed value of μ differing from its most probable value. The unknown quantity $\Delta\mu$ will also be retained in the values of the residuals. Hence, if we assign small increments to μ , it may easily be seen how much this element may differ from its most probable value without giving results for the residuals which are incompatible with the evidence furnished by the observations.

If the dimensions of the orbit are expressed by means of the elements q and e , it may occur that the latter will not be determined with certainty by the observations, and hence it should be treated as suggested in the case of μ ; and we proceed in a similar manner when the correction to be applied to a given value of the semi-transverse axis a is one of the unknown quantities to be determined.

CHAPTER VIII.

INVESTIGATION OF VARIOUS FORMULÆ FOR THE DETERMINATION OF THE SPECIAL
PERTURBATIONS OF A HEAVENLY BODY.

155. WE have thus far considered the circumstances of the undisturbed motion of the heavenly bodies in their orbits; but a complete determination of the elements of the orbit of any body revolving around the sun, requires that we should determine the alterations in its motion due to the action of the other bodies of the system. For this purpose, we shall resume the general equations (18)₁, namely,

$$\begin{aligned}\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3} &= k^2(1+m)\frac{d\Omega}{dx}, \\ \frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3} &= k^2(1+m)\frac{d\Omega}{dy}, \\ \frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} &= k^2(1+m)\frac{d\Omega}{dz},\end{aligned}\quad (1)$$

which determine the motion of a heavenly body relative to the sun when subject to the action of the other bodies of the system. We have, further,

$$\Omega = \frac{m'}{1+m} \left(\frac{1}{\rho} - \frac{xx' + yy' + zz'}{r'^3} \right) + \frac{m''}{1+m} \left(\frac{1}{\rho'} - \frac{xx'' + yy'' + zz''}{r''^3} \right) + \&c.,$$

which is called the *perturbing function*, of which the partial differential coefficients, with respect to the co-ordinates, are

$$\begin{aligned}\frac{d\Omega}{dx} &= \frac{m'}{1+m} \left(\frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right) + \frac{m''}{1+m} \left(\frac{x'' - x}{\rho'^3} - \frac{x''}{r''^3} \right) + \&c., \\ \frac{d\Omega}{dy} &= \frac{m'}{1+m} \left(\frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right) + \frac{m''}{1+m} \left(\frac{y'' - y}{\rho'^3} - \frac{y''}{r''^3} \right) + \&c., \\ \frac{d\Omega}{dz} &= \frac{m'}{1+m} \left(\frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right) + \frac{m''}{1+m} \left(\frac{z'' - z}{\rho'^3} - \frac{z''}{r''^3} \right) + \&c.,\end{aligned}\quad (2)$$

and in which m' , m'' , &c. denote the ratios of the masses of the several disturbing planets to the mass of the sun, and m the ratio of the mass of the disturbed planet to that of the sun. These partial differential coefficients, when multiplied by $k^2(1+m)$, express the

sum of the components of the disturbing force resolved in directions parallel to the three rectangular axes respectively.

When we neglect the consideration of the perturbations, the general equations of motion become

$$\begin{aligned}\frac{d^2x_0}{dt^2} + k^2(1+m) \frac{x_0}{r_0^3} &= 0, \\ \frac{d^2y_0}{dt^2} + k^2(1+m) \frac{y_0}{r_0^3} &= 0, \\ \frac{d^2z_0}{dt^2} + k^2(1+m) \frac{z_0}{r_0^3} &= 0,\end{aligned}\tag{3}$$

the complete integration of which furnishes as arbitrary constants of integration the six elements which determine the orbital motion of a heavenly body. But if we regard these elements as representing the actual orbit of the body for a given instant of time t , and conceive of the effect of the disturbing forces due to the action of the other bodies of the system, it is evident that, on account of the change arising from the force thus introduced, the body at another instant different from the first will be moving in an orbit for which the elements are in some degree different from those which satisfy the original equations. Although the action of the disturbing force is continuous, we may yet regard the elements as unchanged during the element of time dt , and as varying only after each interval dt . Let us now designate by t_0 the epoch to which the elements of the orbit belong, and let these elements be designated by $M_0, \pi_0, \Omega_0, i_0, e_0$, and a_0 ; then will the equations (3) be exactly satisfied by means of the expressions for the co-ordinates in terms of these rigorously-constant elements. These elements will express the motion of the body subject to the action of the disturbing forces only during the infinitesimal interval dt , and at the time $t_0 + dt$ it will commence to describe a new orbit of which the elements will differ from these constant elements by increments which are called the *perturbations*.

According to the principle of the variation of parameters, or of the constants of integration, the differential equations (1) will be satisfied by integrals of the same form as those obtained when the second members are put equal to zero, provided only that the arbitrary constants of the latter integration are no longer regarded as pure constants but as subject to variation. Consequently, if we denote the variable elements by M, π, Ω, i, e , and a , they will be connected with the constant elements, or those which determine the orbit at the instant t_0 , by the equations

$$\begin{aligned}
 M &= M_0 + \int \frac{dM}{dt} dt, & \pi &= \pi_0 + \int \frac{d\pi}{dt} dt, & \Omega &= \Omega_0 + \int \frac{d\Omega}{dt} dt, \\
 i &= i_0 + \int \frac{di}{dt} dt, & e &= e_0 + \int \frac{de}{dt} dt, & a &= a_0 + \int \frac{da}{dt} dt,
 \end{aligned} \tag{4}$$

in which $\frac{dM}{dt}$, $\frac{d\pi}{dt}$, &c. denote the differential coefficients of the elements depending on the disturbing forces. When these differential coefficients are known, we may determine, by simple quadrature, the perturbations δM , $\delta \pi$, &c. to be added to the constant elements in order to obtain those corresponding to any instant for which the place of the body is required. These differential coefficients, however, are functions of the partial differential coefficients of \mathcal{Q} with respect to the elements, and before the integration can be performed it becomes necessary to find the expressions for these partial differential coefficients. For this purpose we expand the function \mathcal{Q} into a converging series and then differentiate each term of this series relatively to the elements. This function is usually developed into a converging series arranged in reference to the ascending powers of the eccentricities and inclinations, and so as to include an indefinite number of revolutions; and the final integration will then give what are called the *absolute* or *general perturbations*. When the eccentricities and inclinations are very great, as in the case of the comets, this development and analytical integration, or quadrature, becomes no longer possible, and even when it is possible it may, on account of the magnitude of the eccentricity or inclination, become so difficult that we are obliged to determine, instead of the absolute perturbations, what are called the *special perturbations*, by methods of approximation known as *mechanical quadratures*, according to which we determine the variations of the elements from one epoch t_0 to another epoch t . This method is applicable to any case, and may be advantageously employed even when the determination of the absolute perturbations is possible, and especially when a series of observations extending through a period of many years is available and it is desired to determine, for any instant t_0 , a system of elements, usually called *osculating elements*, on which the complete theory of the motion may be based.

Instead of computing the variations of the elements of the orbit directly, we may find the perturbations of any known functions of these elements; and the most direct and simple method is to determine the variations, due to the action of the disturbing forces, of any system of three co-ordinates by means of which the position of

the body or the elements themselves may be found. We shall, therefore, derive various formulæ for this purpose before investigating the formulæ for the direct variation of the elements.

156. Let x_0, y_0, z_0 be the rectangular co-ordinates of the body at the time t computed by means of the osculating elements M_0, π_0, Ω_0 , &c., corresponding to the epoch t_0 . Let x, y, z be the actual co-ordinates of the disturbed body at the time t ; and we shall have

$$x = x_0 + \delta x, \quad y = y_0 + \delta y, \quad z = z_0 + \delta z,$$

$\delta x, \delta y$, and δz being the perturbations of the rectangular co-ordinates from the epoch t_0 to the time t . If we substitute these values of x, y , and z in the equations (1), and then subtract from each the corresponding one of equations (3), we get

$$\begin{aligned} \frac{d^2 \delta x}{dt^2} + k^2(1+m) \left(\frac{x_0 + \delta x}{r^3} - \frac{x_0}{r_0^3} \right) &= k^2(1+m) \frac{d\Omega}{dx}, \\ \frac{d^2 \delta y}{dt^2} + k^2(1+m) \left(\frac{y_0 + \delta y}{r^3} - \frac{y_0}{r_0^3} \right) &= k^2(1+m) \frac{d\Omega}{dy}, \\ \frac{d^2 \delta z}{dt^2} + k^2(1+m) \left(\frac{z_0 + \delta z}{r^3} - \frac{z_0}{r_0^3} \right) &= k^2(1+m) \frac{d\Omega}{dz}. \end{aligned} \quad (5)$$

Let us now put $r = r_0 + \delta r$; then to terms of the order δr^2 , which is equivalent to considering only the first power of the disturbing force, we have

$$\begin{aligned} \frac{x_0 + \delta x}{r^3} - \frac{x_0}{r_0^3} &= \frac{1}{r_0^3} \left(\delta x - 3 \frac{x_0}{r_0} \delta r \right), \\ \frac{y_0 + \delta y}{r^3} - \frac{y_0}{r_0^3} &= \frac{1}{r_0^3} \left(\delta y - 3 \frac{y_0}{r_0} \delta r \right), \\ \frac{z_0 + \delta z}{r^3} - \frac{z_0}{r_0^3} &= \frac{1}{r_0^3} \left(\delta z - 3 \frac{z_0}{r_0} \delta r \right), \end{aligned}$$

and hence

$$\begin{aligned} \frac{d^2 \delta x}{dt^2} &= k^2(1+m) \frac{d\Omega}{dx} + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{x_0}{r_0} \delta r - \delta x \right), \\ \frac{d^2 \delta y}{dt^2} &= k^2(1+m) \frac{d\Omega}{dy} + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{y_0}{r_0} \delta r - \delta y \right), \\ \frac{d^2 \delta z}{dt^2} &= k^2(1+m) \frac{d\Omega}{dz} + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{z_0}{r_0} \delta r - \delta z \right). \end{aligned} \quad (6)$$

We have also from

$$r^2 = x^2 + y^2 + z^2,$$

neglecting terms of the second order,

$$\delta r = \frac{x_0}{r_0} \delta x + \frac{y_0}{r_0} \delta y + \frac{z_0}{r_0} \delta z. \quad (7)$$

The integration of the equations (6) will give the perturbations δx , δy , and δz to be applied to the rectangular co-ordinates x_0 , y_0 , z_0 computed by means of the osculating elements, in order to find the actual co-ordinates of the body for the date to which the integration belongs. But since the second members contain the quantities δx , δy , δz which are sought, the integration must be effected indirectly by successive approximations; and from the manner in which these are involved in the second members of the equations, it will appear that this integration is possible.

If we consider only a single disturbing planet, according to the equations (2), we shall have

$$\begin{aligned} k^2(1+m) \frac{d\Omega}{dx} &= m'k^2 \left(\frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right), \\ k^2(1+m) \frac{d\Omega}{dy} &= m'k^2 \left(\frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right), \\ k^2(1+m) \frac{d\Omega}{dz} &= m'k^2 \left(\frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right), \end{aligned} \quad (8)$$

and these forces we will designate by X , Y , and Z respectively; then, if in these expressions we neglect the terms of the order of the square of the disturbing force, writing x_0 , y_0 , z_0 in place of x , y , z , the equations (6) become

$$\begin{aligned} \frac{d^2\delta x}{dt^2} &= X_0 + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{x_0}{r_0} \delta r - \delta x \right), \\ \frac{d^2\delta y}{dt^2} &= Y_0 + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{y_0}{r_0} \delta r - \delta y \right), \\ \frac{d^2\delta z}{dt^2} &= Z_0 + \frac{k^2(1+m)}{r_0^3} \left(3 \frac{z_0}{r_0} \delta r - \delta z \right), \end{aligned} \quad (9)$$

which are the equations for computing the perturbations of the rectangular co-ordinates with reference only to the first power of the masses or disturbing forces. We have, further,

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2, \quad (10)$$

in which, when terms of the second order are neglected, we use the values x_0 , y_0 , z_0 for x , y , and z respectively.

157. From the values of δx , δy , and δz computed with regard to the first power of the masses we may, by a repetition of part of the calculation, take into account the squares and products and even the higher powers of the disturbing forces. The equations (5) may be written thus:—

$$\begin{aligned}\frac{d^2\delta x}{dt^2} &= X + \frac{k^2(1+m)}{r_0^3} \left(\left(1 - \frac{r_0^3}{r^3} \right) x - \delta x \right), \\ \frac{d^2\delta y}{dt^2} &= Y + \frac{k^2(1+m)}{r_0^3} \left(\left(1 - \frac{r_0^3}{r^3} \right) y - \delta y \right), \\ \frac{d^2\delta z}{dt^2} &= Z + \frac{k^2(1+m)}{r_0^3} \left(\left(1 - \frac{r_0^3}{r^3} \right) z - \delta z \right),\end{aligned}\quad (11)$$

in which nothing is neglected. In the application of these formulæ, as soon as δx , δy , and δz have been found for a few successive intervals, we may readily derive approximate values of these quantities for the date next following, and with these find

$$x = x_0 + \delta x, \quad y = y_0 + \delta y, \quad z = z_0 + \delta z,$$

and hence the complete values of the forces X , Y , and Z , by means of the equations (8). To find an expression for the factor

$$1 - \frac{r_0^3}{r^3}$$

which will be convenient in the numerical calculation, we have

$$\begin{aligned}r^2 &= (x_0 + \delta x)^2 + (y_0 + \delta y)^2 + (z_0 + \delta z)^2 \\ &= r_0^2 + 2x_0\delta x + 2y_0\delta y + 2z_0\delta z + \delta x^2 + \delta y^2 + \delta z^2,\end{aligned}$$

and therefore

$$\frac{r^2}{r_0^2} = 1 + 2 \frac{(x_0 + \frac{1}{2}\delta x)\delta x + (y_0 + \frac{1}{2}\delta y)\delta y + (z_0 + \frac{1}{2}\delta z)\delta z}{r_0^2}.$$

Let us now put

$$q = \frac{x_0 + \frac{1}{2}\delta x}{r_0^2} \delta x + \frac{y_0 + \frac{1}{2}\delta y}{r_0^2} \delta y + \frac{z_0 + \frac{1}{2}\delta z}{r_0^2} \delta z, \quad (12)$$

and

$$fq = 1 - \frac{r_0^3}{r^3} = 1 - (1 + 2q)^{-\frac{3}{2}};$$

then we shall have

$$f = 3 \left(1 - \frac{5}{2}q + \frac{5 \cdot 7}{2 \cdot 3}q^2 - \frac{5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 4}q^3 + \&c. \right), \quad (13)$$

and the values of f may be tabulated with the argument q . The equations (11) therefore become

$$\begin{aligned}\frac{d^2\delta x}{dt^2} &= X + \frac{k^2(1+m)}{r_0^3} (fqx - \delta x), \\ \frac{d^2\delta y}{dt^2} &= Y + \frac{k^2(1+m)}{r_0^3} (fgy - \delta y), \\ \frac{d^2\delta z}{dt^2} &= Z + \frac{k^2(1+m)}{r_0^3} (fqz - \delta z).\end{aligned}\quad (14)$$

The coefficients of δx , δy , and δz in equation (12) may be found at once, with sufficient accuracy, by means of the approximate values of these quantities; and having found the value of f corresponding to the resulting value of q , the numerical values of $\frac{d^2\delta x}{dt^2}$, $\frac{d^2\delta y}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$, which include the squares and products of the masses, will be obtained. The integration of these will give more exact values of δx , δy , and δz , and then, recomputing q and the other quantities which require correction, a still closer approximation to the exact values of the perturbations will result.

Table XVII. gives the values of $\log f$ for positive or negative values of q at intervals of 0.0001 from $q = 0$ to $q = 0.03$. Unless the perturbations are very large, q will be found within the limits of this table; and in those cases in which it exceeds the limits of the table, the value of

$$fq = 1 - \frac{r_0^3}{r^3}$$

may be computed directly, using the value of r in terms of r_0 and δx , δy , δz .

In the application of the preceding formulæ, the positions of the disturbed and disturbing bodies may be referred to any system of rectangular co-ordinates. It will be advisable, however, to adopt either the plane of the equator or that of the ecliptic as the fundamental plane, the positive axis of x being directed to the vernal equinox. By choosing the plane of the elliptic orbit at the time t_0 as the plane of xy , the co-ordinate z will be of the order of the perturbations, and the calculation of this part of the action of the disturbing force will be very much abbreviated; but unless the inclination is very large there will be no actual advantage in this selection, since the computation of the values of the components of the disturbing forces will require more labor than when either the equator or the ecliptic is taken as the fundamental plane. The perturbations computed for one fundamental plane may be converted into those referred to another plane or to a different position of the axes in the same plane by means of the formulæ which give the transformation of the co-ordinates directly.

158. We shall now investigate the formulæ for the integration of the linear differential equations of the second order which express the variation of the co-ordinates, and generally the formulæ for finding the integrals of expressions of the form $\int f(x) dx$ and $\iint f(x) dx^2$

when the values of $f(x)$ are computed for successive values of x increasing in arithmetical progression. First, therefore, we shall find the integral of $f(x) dx$ within given limits.

Within the limits for which x is continuous, we have

$$f(x) = a + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4 + \dots; \quad (15)$$

and if we consider only three terms of this series, the resulting equation

$$f(x) = a + \beta x + \gamma x^2$$

is that of the common parabola of which the abscissa is x and the ordinate $f(x)$, and the integral of $f(x) dx$ is the area included by the abscissa, two ordinates, and the included arc of this curve. Generally, therefore, we may consider the more complete expression for $f(x)$ as the equation of a parabolic curve whose degree is one less than the number of terms taken. Hence, if we take n terms of the series as the value of $f(x)$, we shall derive the equation for a parabola whose degree is $n - 1$, and which has n points in common with the curve represented by the exact value of $f(x)$.

If we multiply equation (15) by dx and integrate between the limits 0 and x' , we get

$$\int_0^{x'} f(x) dx = ax' + \frac{1}{2}\beta x'^2 + \frac{1}{3}\gamma x'^3 + \frac{1}{4}\delta x'^4 + \dots \quad (16)$$

If now the values of $f(x)$ for different values of x from 0 to x' are known, each of these, by means of equation (15), will furnish an equation for the determination of α , β , γ , &c.; and the number of terms which may be taken will be equal to the number of different known values of $f(x)$. As soon as α , β , γ , &c. have thus been found, the equation (16) will give the integral required.

If the values of $f(x)$ are computed for values of x at equal intervals and we integrate between the limits $x = 0$, and $x = n\Delta x$, Δx being the constant interval between the successive values of x , and n the number of intervals from the beginning of the integration, we obtain

$$\int_0^{n\Delta x} f(x) dx = an\Delta x + \frac{1}{2}\beta n^2\Delta x^2 + \frac{1}{3}\gamma n^3\Delta x^3 + \&c.$$

Let us now suppose a quadratic parabola to pass through the points of the curve represented by $f(x)$, corresponding to $x = 0$, $x = \Delta x$,

and $x = 2\Delta x$; then will the area included by the arc of this parabola, the extreme ordinates, and the axis of abscissas be

$$\int_0^{2\Delta x} f(x) dx = \Delta x (2\alpha + 2\beta\Delta x + \frac{8}{3}\gamma\Delta x^2).$$

The equation of the curve gives, if we designate the ordinates of the three successive points by y_0 , y_1 , and y_2 ,

$$\alpha = y_0, \quad \beta = -\frac{1}{2\Delta x} (y_2 - 4y_1 + 3y_0), \quad \gamma = \frac{1}{2\Delta x^2} (y_2 - 2y_1 + y_0),$$

and hence we derive

$$\int_0^{2\Delta x} f(x) dx = \frac{1}{3}\Delta x (y_0 + 4y_1 + y_2).$$

In a similar manner, the area included by the ordinates y_2 and y_4 ,—corresponding to $x = 2\Delta x$ and $x = 4\Delta x$,—the axis of abscissas, and the parabola passing through the three points corresponding to y_2 , y_3 , and y_4 , is found to be

$$\int_{2\Delta x}^{4\Delta x} f(x) dx = \frac{1}{3}\Delta x (y_2 + 4y_3 + y_4);$$

and hence we have, finally,

$$\int_{(n-2)\Delta x}^{n\Delta x} f(x) dx = \frac{1}{3}\Delta x (y_{n-2} + 4y_{n-1} + y_n).$$

The sum of all these gives

$$\begin{aligned} \int_0^{n\Delta x} f(x) dx \\ = \frac{1}{3}\Delta x ((y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots y_{n-1}) + 2(y_2 + y_4 + \dots y_{n-2})), \end{aligned} \quad (17)$$

by means of which the approximate value of the integral within the given limits may be found.

If we consider the curve which passes through four points corresponding to y_0 , y_1 , y_2 , and y_3 , we have

$$y = f(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$$

for the equation of the curve, and hence, giving to x the values 0, Δx , $2\Delta x$, and $3\Delta x$, successively, we easily find

$$\begin{aligned} \alpha &= y_0, \\ \beta &= \frac{1}{6\Delta x} (2y_3 - 9y_2 + 18y_1 - 11y_0), \\ \gamma &= \frac{1}{2\Delta x^2} (-y_3 + 4y_2 - 5y_1 + 2y_0), \\ \delta &= \frac{1}{6\Delta x^3} (y_3 - 3y_2 + 3y_1 - y_0). \end{aligned}$$

Therefore we shall have

$$\int_0^{3\Delta x} f(x) dx = \frac{3}{8} \Delta x (y_0 + 3y_1 + 3y_2 + y_3). \quad (18)$$

In like manner, by taking successively an additional term of the series, we may derive

$$\begin{aligned} \int_0^{4\Delta x} f(x) dx &= \frac{2\Delta x}{45} (7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4), \\ \int_0^{5\Delta x} f(x) dx &= \frac{5\Delta x}{288} (19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5). \end{aligned} \quad (19)$$

This process may be continued so as to include the extreme values of x for which $f(x)$ is known; but in the calculation of perturbations it will be more convenient to use the finite differences of the function instead of the function itself directly. We may remark, further, that the intervals of quadrature when the function itself is used, may be so determined that the degree of approximation will be much greater than when these intervals are uniform.

159. Let us put $\Delta x = \omega$, and let the value of x for which $n = 0$ be designated by a ; then will the general value be

$$f(x) = f(a + n\omega),$$

ω being the constant interval at which the values of $f(x)$ are given. Hence we shall have

$$\begin{aligned} dx &= \omega dn, \\ \int f(x) dx &= \omega \int f(a + n\omega) dn. \end{aligned}$$

If we expand the function $f(a + n\omega)$, we have

$$f(a + n\omega) = f(a) + n\omega \frac{df(a)}{da} + \frac{n^2\omega^2}{1 \cdot 2} \cdot \frac{d^2f(a)}{da^2} + \frac{n^3\omega^3}{1 \cdot 2 \cdot 3} \cdot \frac{d^3f(a)}{da^3} + \&c., \quad (20)$$

and hence

$$\int f(a + n\omega) dn = C + nf(a) + \frac{1}{2}n^2\omega \frac{df(a)}{da} + \frac{1}{6}n^3\omega^2 \frac{d^2f(a)}{da^2} + \frac{1}{24}n^4\omega^3 \frac{d^3f(a)}{da^3} + \&c., \quad (21)$$

C being the constant of integration. The equations (54)₆ give

$$\begin{aligned} \omega \frac{df(a)}{da} &= f'(a) - \frac{1}{6}f'''(a) + \frac{1}{360}f^{(v)}(a) - \frac{1}{1440}f^{(vii)}(a) + \dots, \\ \omega^2 \frac{d^2f(a)}{da^2} &= f''(a) - \frac{1}{12}f^{(iv)}(a) + \frac{1}{960}f^{(vi)}(a) - \frac{1}{5600}f^{(viii)}(a) + \dots, \\ \omega^3 \frac{d^3f(a)}{da^3} &= f'''(a) - \frac{1}{4}f^{(v)}(a) + \frac{7}{120}f^{(vii)}(a) - \dots, \\ \omega^4 \frac{d^4f(a)}{da^4} &= f^{(iv)}(a) - \frac{1}{6}f^{(vi)}(a) + \frac{7}{240}f^{(viii)}(a) - \dots, \\ \omega^5 \frac{d^5f(a)}{da^5} &= f^{(v)}(a) - \frac{1}{3}f^{(vii)}(a) + \dots, \\ \omega^6 \frac{d^6f(a)}{da^6} &= f^{(vi)}(a) - \frac{1}{4}f^{(viii)}(a) + \dots, \end{aligned} \quad (22)$$

in which the functional symbols in the second members denote the different orders of finite differences of the function. Hence we obtain

$$\begin{aligned} \int f(a + n\omega) dn &= C + nf(a) \\ &+ \frac{1}{2}n^2(f'(a) - \frac{1}{6}f'''(a) + \frac{1}{360}f^{(v)}(a) - \frac{1}{1440}f^{(vii)}(a) + \dots) \\ &+ \frac{1}{6}n^3(f''(a) - \frac{1}{12}f^{(iv)}(a) + \frac{1}{960}f^{(vi)}(a) - \frac{1}{5600}f^{(viii)}(a) + \dots) \\ &+ \frac{1}{24}n^4(f'''(a) - \frac{1}{4}f^{(v)}(a) + \frac{7}{120}f^{(vii)}(a) - \dots) \\ &+ \frac{1}{120}n^5(f^{(iv)}(a) - \frac{1}{6}f^{(vi)}(a) + \frac{7}{240}f^{(viii)}(a) - \dots) \\ &+ \frac{1}{720}n^6(f^{(v)}(a) - \frac{1}{3}f^{(vii)}(a) + \dots) \\ &+ \frac{1}{5040}n^7(f^{(vi)}(a) - \frac{1}{4}f^{(viii)}(a) + \dots) \\ &+ \frac{1}{40320}n^8f^{(vii)}(a) - \dots + \frac{1}{362880}n^9f^{(viii)}(a) - \&c. \end{aligned} \quad (23)$$

If we take the integral between the limits $-n'$ and $+n'$, the terms containing the even powers of n disappear. Further, since the values of the function are supposed to be known for a series of values of n at intervals of a unit, it will evidently be convenient to determine the integral between the required limits by means of the sum of a series of integrals whose limits are successively increased by a unit, such that the difference between the superior and the inferior limit of each integral shall be a unit. Hence we take the first integral between the limits $-\frac{1}{2}$ and $+\frac{1}{2}$, and the equation (23) gives, after reduction,

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} f(a + n\omega) dn = f(a) + \frac{1}{24}f''(a) - \frac{1}{5760}f^{iv}(a) + \frac{1}{967680}f^{vi}(a) - \frac{1}{4644486400}f^{viii}(a) + \&c. \quad (24)$$

It is evident that by writing, in succession, $a + \omega$, $a + 2\omega$, $a + i\omega$ in place of a , we simply add 1 to each limit successively, so that we have

$$\begin{aligned} \int_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\omega) dn &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} f((a + i\omega) + (n - i)\omega) d(n - i) \\ &= f(a + i\omega) + \frac{1}{24}f''(a + i\omega) - \frac{1}{5760}f^{iv}(a + i\omega) + \frac{1}{967680}f^{vi}(a + i\omega) - \&c. \end{aligned}$$

But since

$$\int_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\omega) dn = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(a + n\omega) dn + \int_{\frac{1}{2}}^{\frac{3}{2}} f(a + n\omega) dn + \dots + \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\omega) dn,$$

if we give to i successively the values 0, 1, 2, 3, &c. in the preceding equation, and add the results, we get

$$\begin{aligned} \int_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\omega) dn &= \sum_{n=0}^{n=i} f(a + n\omega) + \frac{1}{24} \sum_{n=0}^{n=i} f''(a + n\omega) \\ &\quad - \frac{1}{5760} \sum_{n=0}^{n=i} f^{iv}(a + n\omega) + \frac{1}{967680} \sum_{n=0}^{n=i} f^{vi}(a + n\omega) - \&c. \end{aligned} \quad (25)$$

Let us now consider the functions $f(a)$, $f(a + n\omega)$, &c. as being themselves the finite differences of other functions symbolized by $'f$, the first of which is entirely arbitrary, so that we may put, in accordance with the adopted notation,

$$\begin{aligned} f(a) &= 'f(a + \tfrac{1}{2}\omega) - 'f(a - \tfrac{1}{2}\omega), \\ f(a + \omega) &= 'f(a + \tfrac{3}{2}\omega) - 'f(a + \tfrac{1}{2}\omega), \\ &\dots \dots \dots \\ f(a + n\omega) &= 'f(a + (n + \tfrac{1}{2})\omega) - 'f(a + (n - \tfrac{1}{2})\omega). \end{aligned}$$

Therefore we shall have

$$\sum_{n=0}^{n=i} f(a + n\omega) = 'f(a + (i + \tfrac{1}{2})\omega) - 'f(a - \tfrac{1}{2}\omega),$$

and also

$$\begin{aligned} \sum_{n=0}^{n=i} f''(a + n\omega) &= 'f'(a + (i + \tfrac{1}{2})\omega) - 'f'(a - \tfrac{1}{2}\omega), \\ \sum_{n=0}^{n=i} f^{iv}(a + n\omega) &= f^{iv}(a + (i + \tfrac{1}{2})\omega) - f^{iv}(a - \tfrac{1}{2}\omega), \&c \end{aligned}$$

Further, since the quantity $f(a - \frac{1}{2}\omega)$ is entirely arbitrary, we may assign to it a value such that the sum of all the terms of the equation which have the argument $a - \frac{1}{2}\omega$ shall be zero, namely,

$$f(a - \frac{1}{2}\omega) = -\frac{1}{24}f'(a - \frac{1}{2}\omega) + \frac{1}{5760}f'''(a - \frac{1}{2}\omega) - \frac{3}{987680}f^v(a - \frac{1}{2}\omega) + \&c. \quad (26)$$

Substituting these values in (25), it reduces to

$$\begin{aligned} \int_{a - \frac{1}{2}\omega}^{a + (i + \frac{1}{2})\omega} f(x) dx &= \omega \int_{-\frac{1}{2}}^{i + \frac{1}{2}} f(a + n\omega) dn \\ &= \omega \{ f(a + (i + \frac{1}{2})\omega) + \frac{1}{24}f'(a + (i + \frac{1}{2})\omega) \\ &\quad - \frac{1}{5760}f'''(a + (i + \frac{1}{2})\omega) + \frac{3}{987680}f^v(a + (i + \frac{1}{2})\omega) - \&c. \} \end{aligned} \quad (27)$$

In the calculation of the perturbations of a heavenly body, the dates for which the values of the function are computed may be so arranged that for $n = -\frac{1}{2}$, corresponding to the inferior limit, the integral shall be equal to zero, the epoch of $f(a - \frac{1}{2}\omega)$ being that of the osculating elements. It will be observed that the equation (26) expresses this condition, the constant of integration being included in $f(a - \frac{1}{2}\omega)$. If, instead of being equal to zero, the integral has a given value when $n = -\frac{1}{2}$, it is evidently only necessary to add this value to $f(a - \frac{1}{2}\omega)$ as given by (26).

160. The interval ω and the arguments of the function may always be so taken that the equation (27) will furnish the required integral, either directly or by interpolation; but it will often be convenient to integrate for other limits directly, thus avoiding a subsequent interpolation. The derivation of the required formulæ of integration may be effected in a manner entirely analogous to that already indicated. Thus, let it be required to find the expression for the integral taken between the limits $-\frac{1}{2}$ and i .

The general formula (23) gives

$$\begin{aligned} \int_0^i f(a + n\omega) dn &= \frac{1}{2}f(a) + \frac{1}{8}f'(a) + \frac{1}{48}f''(a) - \frac{7}{384}f'''(a) - \frac{1}{11520}f^{iv}(a) \\ &\quad + \frac{1}{46080}f^v(a) + \frac{3}{1183360}f^v(a) - \&c.; \end{aligned}$$

and since, according to the notation adopted,

$$\begin{aligned} f'(a) &= \frac{1}{2}(f'(a - \frac{1}{2}\omega) + f'(a + \frac{1}{2}\omega)) \\ &= f'(a + \frac{1}{2}\omega) - \frac{1}{2}f''(a), \\ f'''(a) &= f'''(a + \frac{1}{2}\omega) - \frac{1}{2}f^{iv}(a), \\ f^v(a) &= f^v(a + \frac{1}{2}\omega) - \frac{1}{2}f^{vi}(a), \&c., \end{aligned} \quad (28)$$

this becomes

$$\int_0^{\frac{1}{2}} f(a+n\omega) dn = \frac{1}{2}f(a) + \frac{1}{8}f'(a+\frac{1}{2}\omega) - \frac{1}{24}f''(a) - \frac{7}{384}f'''(a+\frac{1}{2}\omega) \\ + \frac{1}{1440}f^{iv}(a) + \frac{1}{46080}f^v(a+\frac{1}{2}\omega) - \frac{1}{120960}f^{vi}(a) - \&c. \quad (29)$$

Therefore we obtain

$$\int_i^{i+\frac{1}{2}} f(a+n\omega) dn = \frac{1}{2}f(a+i\omega) + \frac{1}{8}f'(a+(i+\frac{1}{2})\omega) - \frac{1}{24}f''(a+i\omega) \\ - \frac{7}{384}f'''(a+(i+\frac{1}{2})\omega) + \frac{1}{1440}f^{iv}(a+i\omega) + \frac{1}{46080}f^v(a+(i+\frac{1}{2})\omega) \\ - \frac{1}{120960}f^{vi}(a+i\omega) - \&c. \quad (30)$$

Now we have

$$\int_{-\frac{1}{2}}^i f(a+n\omega) dn = \int_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a+n\omega) dn - \int_i^{i+\frac{1}{2}} f(a+n\omega) dn;$$

and if we substitute the values already found for the terms in the second member, and also

$$\begin{aligned} f(a+i\omega) &= f(a+(i+\frac{1}{2})\omega) - f(a+(i-\frac{1}{2})\omega), \\ f''(a+i\omega) &= f''(a+(i+\frac{1}{2})\omega) - f''(a+(i-\frac{1}{2})\omega), \\ f^{iv}(a+i\omega) &= f^{iv}(a+(i+\frac{1}{2})\omega) - f^{iv}(a+(i-\frac{1}{2})\omega), \\ f^{vi}(a+i\omega) &= f^{vi}(a+(i+\frac{1}{2})\omega) - f^{vi}(a+(i-\frac{1}{2})\omega), \&c. \end{aligned} \quad (31)$$

we get

$$\int_{a-\frac{1}{2}\omega}^{a+i\omega} f(x) dx = \omega \int_{-\frac{1}{2}}^i f(a+n\omega) dn \\ = \omega \{ \frac{1}{2}f(a+(i+\frac{1}{2})\omega) + \frac{1}{2}f'(a+(i-\frac{1}{2})\omega) - \frac{1}{24}f''(a+(i+\frac{1}{2})\omega) \\ - \frac{1}{24}f''(a+(i-\frac{1}{2})\omega) + \frac{1}{1440}f'''(a+(i+\frac{1}{2})\omega) + \frac{1}{1440}f'''(a+(i-\frac{1}{2})\omega) \\ - \frac{1}{120960}f^v(a+(i+\frac{1}{2})\omega) - \frac{1}{120960}f^v(a+(i-\frac{1}{2})\omega) + \&c. \}, \quad (32)$$

which is the required integral between the limits $-\frac{1}{2}$ and i .

161. The methods of integration thus far considered apply to the cases in which but a single integration is required, and when applied to the integration of the differential equations for the variations of the co-ordinates on account of the action of disturbing bodies, they will only give the values of $\frac{d\delta x}{dt}$, $\frac{d\delta y}{dt}$, and $\frac{d\delta z}{dt}$, and another integration becomes necessary in order to obtain the values of δx , δy , and δz . We will therefore proceed to derive formulæ for the determination of the double integral directly.

For the double integral $\iint f(x) dx^2$ we have, since $dx^2 = \omega^2 dn^2$,

$$\iint f(x) dx^2 = \omega^2 \iint f(a + n\omega) dn^2.$$

The value of the function designated by $f(a)$ being so taken that when $n = -\frac{1}{2}$,

$$\int f(a + n\omega) dn = 0,$$

the equation (23) gives

$$C = \int_{-\frac{1}{2}}^0 f(a + n\omega) dn.$$

Therefore, the general equation is

$$\begin{aligned} \int f(a + n\omega) dn &= \int_{-\frac{1}{2}}^0 f(a + n\omega) dn + nf(a) \\ &\quad + \frac{1}{2}\alpha n^2 + \frac{1}{6}\beta n^3 + \frac{1}{24}\gamma n^4 + \frac{1}{120}\delta n^5 + \&c. \end{aligned}$$

the values of $\alpha, \beta, \gamma, \dots$ being given by the equations (22). Multiplying this by dn , and integrating, we get

$$\begin{aligned} \iint f(a + n\omega) dn^2 &= C' + n \int_{-\frac{1}{2}}^0 f(a + n\omega) dn + \frac{1}{2}n^2 f(a) \\ &\quad + \frac{1}{6}\alpha n^3 + \frac{1}{24}\beta n^4 + \frac{1}{120}\gamma n^5 + \&c., \end{aligned}$$

C' being the new constant of integration. If we take the integral between the limits $-\frac{1}{2}$ and $+\frac{1}{2}$, we find

$$\iint_{-\frac{1}{2}}^{+\frac{1}{2}} f(a + n\omega) dn^2 = \int_{-\frac{1}{2}}^0 f(a + n\omega) dn + \frac{1}{24}\alpha + \frac{1}{1920}\gamma + \frac{1}{322560}\delta + \&c$$

From the equation (32) we get, for $i = 0$,

$$\int_{-\frac{1}{2}}^0 f(a + n\omega) dn = f(a) - \frac{1}{12}f'(a) + \frac{1}{720}f'''(a) - \frac{1}{86400}f^{(5)}(a) + \&c. \quad (33)$$

Substituting this value, and also the values of $\alpha, \gamma, \delta, \&c.$,—which are given by the second members of the equations (22),—in the preceding equation, and reducing, we get

$$\iint_{-\frac{1}{2}}^{+\frac{1}{2}} f(a + n\omega) dn^2 = f(a) - \frac{1}{24}f'(a) + \frac{5}{5760}f'''(a) - \frac{1}{967680}f^{(5)}(a) + \&c. \quad (34)$$

Hence

$$\begin{aligned} \iint_{i-\frac{1}{2}}^{i+\frac{1}{2}} f(a+n\omega) dn^2 &= f(a+i\omega) - \frac{1}{24} f'(a+i\omega) \\ &\quad + \frac{1}{1920} f'''(a+i\omega) - \frac{3}{1920} f^{iv}(a+i\omega) + \&c. \end{aligned}$$

and

$$\begin{aligned} \iint_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a+n\omega) dn^2 &= \sum_{n=0}^{n=i} f(a+n\omega) - \frac{1}{24} \sum_{n=0}^{n=i} f'(a+n\omega) \\ &\quad + \frac{1}{1920} \sum_{n=0}^{n=i} f'''(a+n\omega) - \frac{3}{1920} \sum_{n=0}^{n=i} f^{iv}(a+n\omega) + \&c. \end{aligned} \quad (35)$$

We may evidently consider $f(a-\frac{1}{2}\omega)$, $f(a+\frac{1}{2}\omega)$, &c. as the differences of other functions, the first of which is arbitrary, so that we have

$$\begin{aligned} f(a) &= \frac{1}{2} f(a+\frac{1}{2}\omega) + \frac{1}{2} f(a-\frac{1}{2}\omega) = \frac{1}{2} f(a+\omega) - \frac{1}{2} f''(a-\omega), \\ f(a+\omega) &= \frac{1}{2} f(a+\frac{3}{2}\omega) + \frac{1}{2} f(a+\frac{1}{2}\omega) = \frac{1}{2} f(a+2\omega) - \frac{1}{2} f''(a), \\ &\vdots \\ f(a+n\omega) &= \frac{1}{2} f(a+(n+\frac{1}{2})\omega) + \frac{1}{2} f(a+(n-\frac{1}{2})\omega) = \frac{1}{2} f(a+(n+1)\omega) \\ &\quad - \frac{1}{2} f''(a+(n-1)\omega). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{n=i} f(a+n\omega) &= \frac{1}{2} f(a+(i+1)\omega) + \frac{1}{2} f(a+i\omega) - \frac{1}{2} f(a) - \frac{1}{2} f(a-\omega), \\ \sum_{n=0}^{n=i} f'(a+n\omega) &= \frac{1}{2} f'(a+(i+1)\omega) + \frac{1}{2} f'(a+i\omega) - \frac{1}{2} f'(a) - \frac{1}{2} f'(a-\omega), \\ \sum_{n=0}^{n=i} f'''(a+n\omega) &= \frac{1}{2} f'''(a+(i+1)\omega) + \frac{1}{2} f'''(a+i\omega) - \frac{1}{2} f'''(a) - \frac{1}{2} f'''(a-\omega), \\ \sum_{n=0}^{n=i} f^{iv}(a+n\omega) &= \frac{1}{2} f^{iv}(a+(i+1)\omega) + \frac{1}{2} f^{iv}(a+i\omega) - \frac{1}{2} f^{iv}(a) - \frac{1}{2} f^{iv}(a-\omega), \&c. \end{aligned}$$

Substituting these values in equation (35), and observing that

$$\begin{aligned} f(a) + f(a-\omega) &= 2f(a) - f'(a-\frac{1}{2}\omega), \\ f'(a) + f'(a-\omega) &= 2f'(a) - f''(a-\frac{1}{2}\omega), \\ f''(a) + f''(a-\omega) &= 2f''(a) - f'''(a-\frac{1}{2}\omega), \&c., \\ f(a-\frac{1}{2}\omega) &= -\frac{1}{24} f''(a-\frac{1}{2}\omega) + \frac{1}{5760} f^{iv}(a-\frac{1}{2}\omega) - \frac{3}{5760} f^{iv}(a-\frac{1}{2}\omega) + \dots, \end{aligned}$$

and that, since $f(a-\omega)$ is arbitrary, we may put

$$\begin{aligned} f(a-\omega) &= \frac{1}{24} f(a) - \frac{1}{5760} (2f''(a) + f''(a-\omega)) \\ &\quad + \frac{3}{5760} (3f^{iv}(a) + 2f^{iv}(a-\omega)) - \&c., \end{aligned} \quad (36)$$

the integral becomes

$$\begin{aligned} \iint_{a-\frac{1}{2}\omega}^{a+(i+\frac{1}{2})\omega} f(x) dx^2 &= \omega^2 \iint_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a+n\omega) dn^2 \\ &= \omega^2 \left\{ \frac{1}{2} f(a+(i+1)\omega) + \frac{1}{2} f(a+i\omega) - \frac{1}{48} f(a+(i+1)\omega) \right. \\ &\quad - \frac{1}{48} f(a+i\omega) + \frac{3}{840} f''(a+(i+1)\omega) + \frac{3}{840} f''(a+i\omega) \\ &\quad \left. - \frac{3}{87072} f^{iv}(a+(i+1)\omega) - \frac{3}{87072} f^{iv}(a+i\omega) + \&c. \right\}, \end{aligned} \quad (37)$$

which is the expression for the double integral between the limits $-\frac{1}{2}$ and $i+\frac{1}{2}$.

The value of $f'(a-\omega)$ given by equation (36) is in accordance with the supposition that for $n=-\frac{1}{2}$ the double integral is equal to zero, and this condition is fulfilled in the calculation of the perturbations when the argument $a-\frac{1}{2}\omega$ corresponds to the date for which the osculating elements are given. If, for $n=-\frac{1}{2}$, neither the single nor the double integral is to be taken equal to zero, it is only necessary to add the given value of the single integral for this argument to the value of $f'(a-\frac{1}{2}\omega)$ given by equation (26), and to add the given value of the double integral for the same argument to the value of $f'(a-\omega)$ given by (36).

162. In a similar manner we may find the expressions for the double integral between other limits. Thus, let it be required to find the double integral between the limits $-\frac{1}{2}$ and i .

Between the limits 0 and $\frac{1}{2}$ we have

$$\begin{aligned} \iint_0^{\frac{1}{2}} f(a+n\omega) dn^2 &= \frac{1}{2} \int_{-\frac{1}{2}}^0 f(a+n\omega) dn + \frac{1}{8} f(a) + \frac{1}{48} \alpha \\ &\quad + \frac{1}{384} \beta + \frac{1}{3840} \gamma + \frac{1}{48080} \delta + \&c. \end{aligned}$$

which gives

$$\begin{aligned} \iint_0^{\frac{1}{2}} f(a+n\omega) dn^2 &= \frac{1}{2} f(a) + \frac{1}{8} f(a) - \frac{1}{48} f'(a) + \frac{1}{384} f''(a) + \frac{1}{3840} f'''(a) \\ &\quad - \frac{1}{5120} f^{iv}(a) - \frac{3}{87072} f^{iv}(a) + \frac{7}{30965760} f^{vi}(a) + \&c.; \end{aligned} \quad (38)$$

and this again, by means of (28), gives

$$\begin{aligned} \iint_i^{i+\frac{1}{2}} f(a+n\omega) dn^2 &= \frac{1}{2} f(a+(i+\frac{1}{2})\omega) - \frac{1}{8} f(a+i\omega) - \frac{1}{48} f'(a+(i+\frac{1}{2})\omega) \\ &\quad + \frac{5}{84} f''(a+i\omega) + \frac{3}{840} f'''(a+(i+\frac{1}{2})\omega) - \frac{3}{15360} f^{iv}(a+i\omega) \\ &\quad - \frac{3}{87072} f^{iv}(a+(i+\frac{1}{2})\omega) + \frac{1}{30965760} f^{vi}(a+i\omega) + \&c. \end{aligned}$$

Therefore, since

$$\iint_{-\frac{1}{2}}^i f(a + n\omega) dn^2 = \iint_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\omega) dn^2 - \iint_i^{i+\frac{1}{2}} f(a + n\omega) dn^2,$$

and

$$\begin{aligned} f'(a + (i + \tfrac{1}{2})\omega) &= f'(a + (i + 1)\omega) - f'(a + i\omega), \\ f''(a + (i + \tfrac{1}{2})\omega) &= f''(a + (i + 1)\omega) - f''(a + i\omega), \\ f'''(a + (i + \tfrac{1}{2})\omega) &= f'''(a + (i + 1)\omega) - f'''(a + i\omega), \text{ \&c.} \end{aligned}$$

we shall have

$$\begin{aligned} \iint_{a-\frac{1}{2}\omega}^{a+i\omega} f(x) dx^2 &= \omega^2 \iint_{-\frac{1}{2}}^i f(a + n\omega) dn^2 \\ &= \omega^2 \left\{ f'(a + i\omega) + \frac{1}{12} f''(a + i\omega) - \frac{1}{240} f'''(a + i\omega) + \frac{3}{80480} f^{iv}(a + i\omega) - \text{\&c.} \right\}, \end{aligned} \quad (39)$$

which gives the required integral between the limits $-\frac{1}{2}$ and i .

163. It will be observed that the coefficients of the several terms of the formulæ of integration converge rapidly, and hence, by a proper selection of the interval at which the values of the function are computed, it will not be necessary to consider the terms which depend on the fourth and higher orders of differences, and rarely those which depend on the second and third differences. The value assigned to the interval ω must be such that we may interpolate with certainty, by means of the values computed directly, all values of the function intermediate to the extreme limits of the integration; and hence, if the fourth and higher orders of differences are sensible, it will be necessary to extend the direct computation of the values of the function beyond the limits which would otherwise be required, in order to obtain correct values of the differences for the beginning and end of the integration. It will be expedient, therefore, to take ω so small that the fourth and higher differences may be neglected, but not smaller than is necessary to satisfy this condition, since otherwise an unnecessary amount of labor would be expended in the direct computation of the values of the function. It is better, however, to have the interval ω smaller than what would appear to be strictly required, in order that there may be no uncertainty with respect to the accuracy of the integration. On account of the rapidity with which the higher orders of differences decrease as we diminish ω , a limit for the magnitude of the adopted interval will speedily be obtained. The magnitude of the interval will therefore be suggested by the rapidity of the change of value of the function. In the com-

putation of the perturbations of the group of small planets between Mars and Jupiter we may adopt uniformly an interval of forty days; but in the determination of the perturbations of comets it will evidently be necessary to adopt different intervals in different parts of the orbit. When the comet is in the neighborhood of its perihelion, and also when it is near a disturbing planet, the interval must necessarily be much smaller than when it is in more remote parts of its orbit or farther from the disturbing body.

It will be observed, further, that since the double integral contains the factor ω^2 , if we multiply the computed values of the function by ω^2 , this factor will be included in all the differences and sums, and hence it will not appear as a factor in the formulæ of integration. If, however, the values of the function are already multiplied by ω^2 , and only the single integral is sought, the result obtained by the formula of integration, neglecting the factor ω^2 , will be ω times the actual integral required, and it must be divided by ω in order to obtain the final result.

164. In the computation of the perturbations of one of the asteroid planets for a period of two or three years it will rarely be necessary to take into account the effect of the terms of the second order with respect to the disturbing force. In this case the numerical values of the expressions for the forces will be computed by using the values of the co-ordinates computed from the osculating elements for the beginning of the integration, instead of the actual disturbed values of these co-ordinates as required by the formulæ (8). The values of the second differential coefficients of δx , δy , and δz with respect to the time, will be determined by means of the equations (9). If the interval ω is such that the higher orders of differences may be neglected, the values of the forces must be computed for the successive dates separated by the interval ω , and commencing with the date $t_0 - \frac{1}{2}\omega$ corresponding to the argument $a - \omega$, t_0 being the date to which the osculating elements belong. Then, since the last terms of the formulæ for $\frac{d^2\delta x}{dt^2}$, $\frac{d^2\delta y}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$ involve δx , δy , and δz , which are the quantities sought, the subsequent determination of the differential coefficients must be performed by successive trials. Since the integral must in each case be equal to zero for the date t_0 , it will be admissible to assume first, for the dates $t_0 - \frac{1}{2}\omega$ and $t_0 + \frac{1}{2}\omega$ corresponding to the arguments $a - \omega$ and a , that $\delta x = 0$, $\delta y = 0$, and $\delta z = 0$, and hence that the three differential coefficients, for each

date, are respectively equal to X_0 , Y_0 , and Z_0 . We may now by integration derive the actual or the very approximate values of the variations of the co-ordinates for these two dates. Thus, in the case of each co-ordinate, we compute the value of $'f(a - \frac{1}{2}\omega)$ by means of the equation (26), using only the first term, and the value of $''f(a - \omega)$ from (36), using in this case also only the first term. The value of the next function symbolized by $''f$ will be given by

$$''f(a) = ''f(a - \omega) + 'f(a - \frac{1}{2}\omega).$$

Then the formula (39), putting first $i = -1$ and then $i = 0$, and neglecting second differences, will give the values of the variations of the co-ordinates for the dates $a - \omega$ and a . These operations will be performed in the case of each of the three co-ordinates; and, by means of the results, the corrected values of the differential coefficients will be obtained from the equations (9), the value of δr being computed by means of (7). With the corrected values thus derived a new table of integration will be commenced; and the values of $'f(a - \frac{1}{2}\omega)$ and $''f(a - \omega)$ will also be recomputed. Then we obtain, also, by adding $'f(a - \frac{1}{2}\omega)$ to $f(a)$, the value of $'f(a + \frac{1}{2}\omega)$, and, by adding this to $''f(a)$, the value of $''f(a + \omega)$.

An approximate value of $f(a + \omega)$ may now be readily estimated, and two terms of the equation (39), putting $i = 1$, will give an approximate value of the integral. This having been obtained for each of the co-ordinates, the corresponding complete values of the differential coefficients may be computed, and these having been introduced into the table of integration, the process may, in a similar manner, be carried one step farther, so as to determine first approximate values of δx , δy , and δz for the date represented by the argument $a + 2\omega$, and then the corresponding values of the differential coefficients. We may thus by successive partial integrations determine the values of the unknown quantities near enough for the calculation of the series of differential coefficients, even when the integrals are involved directly in the values of the differential coefficients. If it be found that the assumed value of the function is, in any case, much in error, a repetition of the calculation may become necessary; but when a few values have been found, the course of the function will indicate at once an approximation sufficiently close, since whatever error remains affects the approximate integral by only one-twelfth part of the amount of this error. Further, it is evident that, in cases of this kind, when the determination of the values of the differential coefficients requires a preliminary approximate inte-

gration, it is necessary, in order to avoid the effect of the errors in the values of the higher orders of differences, that the interval ω should be smaller than when the successive values of the function to be integrated are already known. In the case of the small planets an interval of 40 days will afford the required facility in the approximations; but in the case of the comets it may often be necessary to adopt an interval of only a few days. The necessity of a change in the adopted value of ω will be indicated, in the numerical application of the formulæ, by the manner in which the successive assumptions in regard to the value of the function are found to agree with the corrected results.

The values of the differential coefficients, and hence those of the integrals, are conveniently expressed by adopting for unity the unit of the seventh decimal place of their values in terms of the unit of space.

165. Whenever it is considered necessary to commence to take into account the perturbations due to the second and higher powers of the disturbing force, the complete equations (14) must be employed. In this case the forces X , Y , and Z should not be computed at once for the entire period during which the perturbations are to be determined. The values computed by means of the osculating elements will be employed only so long as simply the first power of the disturbing force is considered, and by means of the approximate values of δx , δy , and δz which would be employed in computing, for the next place, the last terms of the equations (9), we must compute also the corrected values of X , Y , and Z . These will be given by the second members of (8), using the values of x , y , and z obtained from

$$x = x_0 + \delta x, \quad y = y_0 + \delta y, \quad z = z_0 + \delta z.$$

We compute also q from (12), and then from Table XVII. find the corresponding value of f . The corrected values of $\frac{d^2\delta x}{dt^2}$, $\frac{d^2\delta y}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$ will be given by the equations (14), and these being introduced, in the continuation of the table of integration, we obtain new values of δx , δy , and δz for the date under consideration. If these differ much from those previously assumed, a repetition of the calculation will be necessary in order to secure extreme accuracy. In this repetition, however, it will not be necessary to recompute the coefficients of δx , δy , and δz in the formula for q , their values being given with sufficient accuracy by means of the previous assumption; and gene-

rally a repetition of the calculation of X , Y , and Z will not be required.

Next, the values of δx , δy , and δz may be determined approximately, as already explained, for the following date, and by means of these the corresponding values of the forces X , Y , and Z will be found, and also f and the remaining terms of (14), after which the integration will be completed and a new trial made, if it be considered necessary. In the final integration, all the terms of the formulæ of integration which sensibly affect the result may be taken into account. By thus performing the complete calculation of each successive place separately, the determination of the perturbations in the values of the co-ordinates may be effected in reference to all powers of the masses, provided that we regard the masses and co-ordinates of the disturbing bodies as being accurately known; and it is apparent that this complete solution of the problem requires very little more labor than the determination of the perturbations when only the first power of the disturbing force is considered. But although the places of the disturbing bodies as given by the tables of their motion may be regarded as accurately known, there are yet the errors of the adopted osculating elements of the disturbed body to detract from the absolute accuracy of the computed perturbations; and hence the probable errors of these elements should be constantly kept in view, to the end that no useless extension of the calculation may be undertaken. When the osculating elements have been corrected by means of a very extended series of observations, it will be expedient to determine the perturbations with all possible rigor.

When there are several disturbing planets, the forces for all of these may be computed simultaneously and united in a single sum, so that in the equations (14) we shall have ΣX , ΣY , and ΣZ instead of X , Y , and Z respectively; and the integration of the expressions for $\frac{d^2\delta x}{dt^2}$, $\frac{d^2\delta y}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$ will then give the perturbations due to the action of all the disturbing bodies considered. However, when the interval ω for the different disturbing planets may be taken differently, it may be considered expedient to compute the perturbations separately, and especially if the adopted values of the masses of some of the disturbing bodies are regarded as uncertain, and it is desired to separate their action in order to determine the probable corrections to be applied to the values of m , m' , &c., or to determine the effect of any subsequent change in these values without repeating the calculation of the perturbations.

166. EXAMPLE.—To illustrate the numerical application of the formulæ for the computation of the perturbations of the rectangular co-ordinates, let it be required to compute the perturbations of *Eurynome* [⊙] arising from the action of *Jupiter* from 1864 Jan. 1.0 Berlin mean time to 1865 Jan. 15.0 Berlin mean time, assuming the osculating elements to be the following:—

Epoch = 1864 Jan. 1.0 Berlin mean time.

$$M_0 = 1^\circ 29' 5''.65$$

$$\pi_0 = 44 \ 17 \ 12 \ .17$$

$$\Omega_0 = 206 \ 39 \ 5 \ .69$$

$$i_0 = 4 \ 36 \ 52 \ .11$$

$$\varphi_0 = 11 \ 15 \ 51 \ .02$$

$$\log a_0 = 0.3881319$$

$$\mu_0 = 928''.55745.$$

} Ecliptic and Mean
Equinox 1860.0

From these elements we derive the following values:—

| Berlin Mean Time. | x_0 | y_0 | z_0 | $\log r_0$ |
|-------------------|-----------|-----------|-----------|------------|
| 1863 Dec. 12.0 | + 1.53616 | + 1.23012 | — 0.03312 | 0.294084, |
| 1864 Jan. 21.0 | 1.15097 | 1.59918 | 0.07369 | 0.294837, |
| March 1.0 | 0.69518 | 1.87033 | 0.10978 | 0.300674, |
| April 10.0 | + 0.19817 | 2.03141 | 0.13936 | 0.310864, |
| May 20.0 | — 0.31012 | 2.08092 | 0.16134 | 0.324298, |
| June 29.0 | 0.80326 | 2.02602 | 0.17523 | 0.339745, |
| Aug. 8.0 | 1.26055 | 1.87959 | 0.18122 | 0.356101, |
| Sept. 17.0 | 1.66729 | 1.65711 | 0.17990 | 0.372469, |
| Oct. 27.0 | 2.01414 | 1.37473 | 0.17209 | 0.388214, |
| Dec. 6.0 | 2.29597 | 1.04766 | 0.15870 | 0.402894, |
| 1865 Jan. 15.0 | — 2.51077 | + 0.68978 | — 0.14066 | 0.416240. |

The adopted interval is $\omega = 40$ days, and the co-ordinates are referred to the ecliptic and mean equinox of 1860.0. The first date, it will be observed, corresponds to $t_0 - \frac{1}{2}\omega$, and the integration is to commence at 1864 Jan. 1.0.

The places of *Jupiter* derived from the tables give the following values of the co-ordinates of that planet, with which we write also the distances of *Eurynome* from *Jupiter* computed by means of the formula

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

| Berlin Mean Time. | x' | y' | z' | $\log r'$ | $\log \rho$ |
|-------------------|-----------|-----------|-----------|-----------|-------------|
| 1863 Dec. 12.0 | — 4.09683 | — 3.55184 | + 0.10533 | 0.73425 | 0.86866, |
| 1864 Jan. 21.0 | 3.89630 | 3.76053 | 0.10152 | 0.73368 | 0.86713, |
| March 1.0 | 3.68416 | 3.95803 | 0.09744 | 0.73305 | 0.86292, |
| April 10.0 | — 3.46098 | — 4.14366 | + 0.09304 | 0.73237 | 0.85622, |

| Berlin Mean Time. | x' | y' | z' | $\log r'$ | $\log \rho$ |
|-------------------|-----------|-----------|-----------|-----------|-------------|
| 1864 May 20.0 | — 3.22739 | — 4.31684 | + 0.08839 | 0.73164 | 0.84732, |
| June 29.0 | 2.98405 | 4.47693 | 0.08346 | 0.73086 | 0.83656, |
| Aug. 8.0 | 2.73162 | 4.62343 | 0.07827 | 0.73003 | 0.82428, |
| Sept. 17.0 | 2.47085 | 4.75576 | 0.07284 | 0.72915 | 0.81077, |
| Oct. 27.0 | 2.20247 | 4.87345 | 0.06720 | 0.72823 | 0.79628, |
| Dec. 6.0 | 1.92728 | 4.97606 | 0.06134 | 0.72726 | 0.78098, |
| 1865 Jan. 15.0 | — 1.64600 | — 5.06301 | + 0.05531 | 0.72625 | 0.76498. |

These co-ordinates are also referred to the ecliptic and mean equinox of 1860.0.

If we neglect the mass of *Eurynome* and adopt for the mass of *Jupiter*

$$m' = \frac{1}{1047.879},$$

we obtain, in units of the seventh decimal place,

$$\omega^2 m' k^2 = 4518.27,$$

and the equations (9) become

$$\begin{aligned} \omega^2 \frac{d^2 \delta x}{dt^2} &= 4518.27 \left(\frac{x' - x_0}{\rho^3} - \frac{x'}{r^3} \right) + \frac{0.47346}{r_0^3} \left(3 \frac{x_0}{r_0} \delta r - \delta x \right), \\ \omega^2 \frac{d^2 \delta y}{dt^2} &= 4518.27 \left(\frac{y' - y_0}{\rho^3} - \frac{y'}{r^3} \right) + \frac{0.47346}{r_0^3} \left(3 \frac{y_0}{r_0} \delta r - \delta y \right), \\ \omega^2 \frac{d^2 \delta z}{dt^2} &= 4518.27 \left(\frac{z' - z_0}{\rho^3} - \frac{z'}{r^3} \right) + \frac{0.47346}{r_0^3} \left(3 \frac{z_0}{r_0} \delta r - \delta z \right). \end{aligned} \quad (40)$$

Substituting for the quantities in the first term of the second member of each of these equations the values already found, we obtain

| Argument. | Date. | $\omega^2 X_0$ | $\omega^2 Y_0$ | $\omega^2 Z_0$ |
|---------------|----------------|----------------|----------------|----------------|
| $a - \omega$ | 1863 Dec. 12.0 | + 53.00 | + 47.09 | — 1.43, |
| a | 1864 Jan. 21.0 | 53.71 | 46.31 | 0.91, |
| $a + \omega$ | March 1.0 | 54.23 | 45.18 | — 0.37, |
| $a + 2\omega$ | April 10.0 | 54.69 | 43.59 | + 0.22, |
| $a + 3\omega$ | May 20.0 | 55.23 | 41.51 | 0.70, |
| $a + 4\omega$ | June 29.0 | 56.06 | 38.96 | 1.19, |
| $a + 5\omega$ | Aug. 8.0 | 57.30 | 35.92 | 1.66, |
| $a + 6\omega$ | Sept. 17.0 | 59.09 | 32.47 | 2.08, |
| $a + 7\omega$ | Oct. 27.0 | 61.55 | 28.60 | 2.43, |
| $a + 8\omega$ | Dec. 6.0 | 64.85 | 24.34 | 2.69, |
| $a + 9\omega$ | 1865 Jan. 15.0 | + 69.09 | + 19.78 | + 2.83, |

which are expressed in units of the seventh decimal place.

We now, for a first approximation, regard the perturbations as

being equal to zero for the dates Dec. 12.0 and Jan. 21.0, and, in the case of the variation of x , we compute first

$$\begin{aligned} f'(a - \tfrac{1}{2}\omega) &= -\tfrac{1}{24}f''(a - \tfrac{1}{2}\omega) = -\tfrac{1}{24}(53.71 - 53.00) = -0.03, \\ f''(a - \omega) &= \tfrac{1}{24}f'(a) = +\frac{53.71}{24} = +2.24, \end{aligned}$$

and the approximate table of integration becomes

$$\begin{array}{rcl} f(a - \omega) & = + 53.00 & f'(a - \tfrac{1}{2}\omega) = -0.03 & f''(a - \omega) = + 2.24, \\ f(a) & = + 53.71 & f'(a) & = + 2.21. \end{array}$$

Then the formula (39), putting first $i = -1$, and then $i = 0$, gives

$$\begin{array}{rcl} \text{Dec. 12.0} & \delta x = + 2.24 + \frac{53.00}{12} & = + 6.66, \\ \text{Jan. 21.0} & \delta x = + 2.21 + \frac{53.71}{12} & = + 6.69. \end{array}$$

In a similar manner, we find

$$\begin{array}{rcl} \text{Dec. 12.0} & \delta y = + 5.85 & \delta z = - 0.16, \\ \text{Jan. 21.0} & \delta y = + 5.82 & \delta z = - 0.14. \end{array}$$

By means of these results we compute the complete values of the second members of equations (40), δr being found from

$$\delta r = \frac{x_0}{r_0} \delta x + \frac{y_0}{r_0} \delta y + \frac{z_0}{r_0} \delta z,$$

and thus we obtain

| Date. | $\omega^2 \frac{d^2 \delta x}{dt^2}$ | $\omega^2 \frac{d^2 \delta y}{dt^2}$ | $\omega^2 \frac{d^2 \delta z}{dt^2}$ | δr |
|-----------|--------------------------------------|--------------------------------------|--------------------------------------|------------|
| Dec. 12.0 | + 53.86 | + 47.76 | - 1.45 | + 8.85, |
| Jan. 21.0 | + 54.23 | + 47.25 | - 0.96 | + 8.63. |

We now commence anew the table of integration, namely,

| f | $\frac{x}{f}$ | f' | f'' | f | $\frac{y}{f}$ | f' | f'' | f | $\frac{z}{f}$ | f' | f'' |
|--------|---------------|-------|---------|--------|---------------|-------|---------|-------|---------------|--------|-------|
| +53.86 | - | 0.02 | + 2.26, | +47.76 | + | 0.02 | + 1.97, | -1.45 | -0.02 | -0.04, | |
| +54.23 | + | 54.21 | + 2.24, | +47.25 | + | 47.27 | + 1.99, | -0.96 | -0.98 | -0.06, | |
| | | | +56.45, | | | | +49.26, | | | -1.04, | |

the formation of which is made evident by what precedes.

We may next assume for approximate values of the differential coefficients, for the date March 1.0, + 54.6, + 46.7, and - 0.5, respectively; and these give, for this date,

$$\delta x = + 56.45 + \frac{54.6}{12} = + 61.00,$$

$$\delta y = + 49.26 + \frac{46.7}{12} = + 53.15,$$

$$\delta z = - 1.04 - \frac{0.5}{12} = - 1.08.$$

By means of these approximate values we obtain the following results:—

$$1864 \text{ March } 1.0 \quad \omega^2 \frac{d^2 \delta x}{dt^2} = + 55.01, \quad \omega^2 \frac{d^2 \delta y}{dt^2} = + 53.86, \quad \omega^2 \frac{d^2 \delta z}{dt^2} = - 1.00, \\ \delta r = + 71.03.$$

Introducing these into the table of integration, we find, for the corresponding values of the integrals,

$$\delta x = + 61.03, \quad \delta y = + 53.75, \quad \delta z = - 1.12.$$

These results differ so little from those already derived from the assumed values of the function that a repetition of the calculation is unnecessary. This repetition, however, gives

$$\omega^2 \frac{d^2 \delta x}{dt^2} = + 55.04, \quad \omega^2 \frac{d^2 \delta y}{dt^2} = + 53.91, \quad \omega^2 \frac{d^2 \delta z}{dt^2} = - 1.00.$$

Assuming, again, approximate values of the differential coefficients for April 10.0, and computing the corresponding values of δx , δy , and δz , we derive, for this date,

$$\omega^2 \frac{d^2 \delta x}{dt^2} = + 48.06, \quad \omega^2 \frac{d^2 \delta y}{dt^2} = + 63.19, \quad \omega^2 \frac{d^2 \delta z}{dt^2} = - 1.54.$$

Introducing these into the table of integration, and thus deriving approximate values of δx , δy , and δz for May 20, we carry the process one step further. In this manner, by successive approximations, we obtain the following results:—

| Date. | $\omega^2 \frac{d^2 \delta x}{dt^2}$ | $\omega^2 \frac{d^2 \delta y}{dt^2}$ | $\omega^2 \frac{d^2 \delta z}{dt^2}$ |
|----------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 1863 Dec. 12.0 | + 53.86 | + 47.76 | - 1.45, |
| 1864 Jan. 21.0 | 54.23 | 47.25 | 0.96, |
| March 1.0 | 55.04 | 53.91 | 1.00, |
| April 10.0 | 48.06 | 63.19 | 1.54, |
| May 20.0 | 32.85 | 65.40 | 2.07, |
| June 29.0 | 16.74 | 54.48 | 1.75, |
| Aug. 8.0 | 8.62 | 31.39 | - 0.36, |
| Sept. 17.0 | + 14.20 | + 2.09 | + 1.86, |

| Date. | | $\omega^2 \frac{d^2 \delta x}{dt^2}$ | $\omega^2 \frac{d^2 \delta y}{dt^2}$ | $\omega^2 \frac{d^2 \delta z}{dt^2}$ |
|-----------|------|--------------------------------------|--------------------------------------|--------------------------------------|
| 1864 Oct. | 27.0 | + 34.84 | — 26.32 | + 4.44, |
| Dec. | 6.0 | 68.79 | 47.87 | 6.86, |
| 1865 Jan. | 15.0 | + 112.64 | — 58.39 | + 8.68. |

The complete integration may now be effected, and we may use both equation (37) and equation (39), the former giving the integral for the dates Jan. 1.0, Feb. 10.0, March 21.0, &c., and the latter the integrals for the dates in the foregoing table of values of the function. The final results for the perturbations of the rectangular co-ordinates, expressed in units of the seventh decimal place, are thus found to be the following:—

| Berlin Mean Time. | | δx | δy | δz |
|-------------------|------|------------|------------|------------|
| 1863 Dec. | 12.0 | + 6.7 | + 5.9 | — 0.2, |
| 1864 Jan. | 1.0 | 0.0 | 0.0 | 0.0, |
| | 21.0 | + 6.8 | 5.9 | 0.1, |
| Feb. | 10.0 | 27.1 | 23.5 | 0.5, |
| March | 1.0 | 61.0 | 53.7 | 1.1, |
| | 21.0 | 108.9 | 97.4 | 2.0, |
| April | 10.0 | 169.7 | 155.7 | 3.1, |
| | 30.0 | 242.7 | 229.9 | 4.7, |
| May | 20.0 | 325.7 | 320.3 | 6.7, |
| June | 9.0 | 417.1 | 427.2 | 9.3, |
| | 29.0 | 514.6 | 549.1 | 12.3, |
| July | 19.0 | 616.1 | 684.9 | 15.7, |
| Aug. | 8.0 | 720.8 | 831.4 | 19.5, |
| | 28.0 | 827.4 | 986.0 | 23.4, |
| Sept. | 17.0 | 936.8 | 1144.6 | 27.0, |
| Oct. | 7.0 | 1049.4 | 1303.8 | 30.2, |
| | 27.0 | 1168.2 | 1460.0 | 32.6, |
| Nov. | 16.0 | 1295.4 | 1609.4 | 33.9, |
| Dec. | 6.0 | 1435.6 | 1749.6 | 33.8, |
| | 26.0 | 1592.8 | 1877.6 | 32.0, |
| 1865 Jan. | 15.0 | + 1772.6 | + 1992.3 | — 28.2. |

During the interval included by these perturbations, the terms of the second order of the disturbing forces will have no sensible effect; but to illustrate the application of the rigorous formulæ, let us commence at the date 1864 Sept. 17.0 to consider the perturbations of the second order.

In the first place, the components of the disturbing force must be computed by means of the equations

$$\begin{aligned}\omega^2 X &= \omega^2 m' k^2 \left(\frac{x' - x}{\rho^3} - \frac{x'}{r^3} \right), & \omega^2 Y &= \omega^2 m' k^2 \left(\frac{y' - y}{\rho^3} - \frac{y'}{r^3} \right), \\ \omega^2 Z &= \omega^2 m' k^2 \left(\frac{z' - z}{\rho^3} - \frac{z'}{r^3} \right).\end{aligned}$$

The approximate values of δx , δy , and δz for Sept. 17.0 given immediately by the table of integration extended to this date, will suffice to furnish the required values of the disturbed co-ordinates by means of

$$x = x_0 + \delta x, \quad y = y_0 + \delta y, \quad z = z_0 + \delta z;$$

and to find $\rho = \rho_0 + \delta \rho$, we have

$$\delta \rho = - \frac{x' - x}{\rho} \delta x - \frac{y' - y}{\rho} \delta y - \frac{z' - z}{\rho} \delta z,$$

or

$$\delta \log \rho = - \frac{\lambda_0}{\rho^2} ((x' - x) \delta x + (y' - y) \delta y + (z' - z) \delta z),$$

in which λ_0 is the modulus of the system of logarithms. Thus we obtain, for Sept. 17.0,

$$\begin{aligned}\delta \log \rho &= + 0.0000084, \\ \omega^2 X &= + 59.09, & \omega^2 Y &= + 32.48, & \omega^2 Z &= + 2.08,\end{aligned}$$

which require no further correction.

Next, we compute the values of

$$\frac{x_0 + \frac{1}{2} \delta x}{r_0^2}, \quad \frac{y_0 + \frac{1}{2} \delta y}{r_0^2}, \quad \frac{z_0 + \frac{1}{2} \delta z}{r_0^2},$$

which also will not require any further correction, and thus we form, according to (12), the equation

$$q = - 0.29996 \delta x + 0.29815 \delta y - 0.03237 \delta z.$$

The approximate values of δx , δy , and δz being substituted in this equation, we obtain

$$q = + 0.0000061,$$

corresponding to which Table XVII. gives

$$\log f = 0.477115.$$

Hence we derive

$$\begin{aligned}\frac{\omega^2 k^2}{r_0^3} (fqx - \delta x) &= - 44.87, & \frac{\omega^2 k^2}{r_0^3} (fqy - \delta y) &= - 30.40, \\ \frac{\omega^2 k^2}{r_0^3} (fqz - \delta z) &= - 0.21,\end{aligned}$$

and the equations (14) give

$$\omega^2 \frac{d^2 \delta x}{dt^2} = + 14.22, \quad \omega^2 \frac{d^2 \delta y}{dt^2} = + 2.08, \quad \omega^2 \frac{d^2 \delta z}{dt^2} = + 1.87.$$

These values being introduced into the table of integration, the resulting values of the integrals are changed so little that a repetition of the calculation is not required.

We now derive approximate values of δx , δy , and δz for Oct. 27.0, and in a similar manner we obtain the corrected values of the differential coefficients for this date; and thus by computing the forces for each place in succession from approximate values of the perturbations, and repeating the calculation whenever it may appear necessary, we may determine the perturbations rigorously for all powers of the masses. The results in the case under consideration are the following:—

| Date. | $\omega^2 \frac{d^2 \delta x}{dt^2}$ | $\omega^2 \frac{d^2 \delta y}{dt^2}$ | $\omega^2 \frac{d^2 \delta z}{dt^2}$ |
|-----------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 1864 Sept. 17.0 | + 14.22 | + 2.08 | + 1.87, |
| Oct. 27.0 | 34.84 | — 26.31 | 4.44, |
| Dec. 6.0 | 68.77 | 47.86 | 6.86, |
| 1865 Jan. 15.0 | + 112.60 | — 58.39 | + 8.68. |

Introducing these results into the table of integration, the integrals for Jan. 15.0 are found to be

$$\delta x = + 1772.6, \quad \delta y = + 1992.3, \quad \delta z = - 28.2,$$

agreeing exactly with those obtained when terms of the order of the square of the disturbing forces are neglected.

If the perturbations of the rectangular co-ordinates referred to the equator are required, we have, whatever may be the magnitude of the perturbations,

$$\begin{aligned} \delta x, &= \delta x, \\ \delta y, &= \cos \epsilon \delta y - \sin \epsilon \delta z, \\ \delta z, &= \sin \epsilon \delta y + \cos \epsilon \delta z, \end{aligned} \tag{41}$$

x, y, z , being the co-ordinates in reference to the equator as the fundamental plane. Thus we obtain, for 1865 Jan. 15.0,

$$\delta x, = + 1772.6, \quad \delta y, = + 1838.9, \quad \delta z, = + 767.2.$$

These values, expressed in seconds of arc of a circle whose radius is the unit of space, are

$$\delta x = + 36''.562, \quad \delta y, = + 37''.930, \quad \delta z, = + 15''.825.$$

The approximate geocentric place of the planet for the same date is

$$\alpha = 183^{\circ} 28', \quad \delta = -5^{\circ} 39', \quad \log \Delta = 0.3229,$$

and hence, neglecting terms of the second order, we derive, by means of the equations (3)₂, for the perturbations of the geocentric right ascension and declination,

$$\Delta \alpha = -17''.03, \quad \Delta \delta = +5''.67.$$

167. The values of δx , δy , and δz , computed by means of the co-ordinates referred to the ecliptic and mean equinox of the date t , must be added to the co-ordinates given by the undisturbed elements and referred to the same mean equinox. The co-ordinates referred to the ecliptic and mean equinox of t may be readily transformed into those referred to the ecliptic and mean equinox of another date t' . Thus, let θ denote the longitude of the descending node of the ecliptic of t' on that of t , measured from the mean equinox of t , and let η be the mutual inclination of these planes; then, if we denote by x' , y' , z' the co-ordinates referred to the ecliptic of t as the fundamental plane, the positive axis of x , however, being directed to the point whose longitude is θ , we shall have

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= -x \sin \theta + y \cos \theta, \\ z' &= z. \end{aligned} \tag{42}$$

Let us now denote by x'' , y'' , z'' the co-ordinates when the ecliptic of t is the plane of xy , the axis of x remaining the same as in the system of x' , y' , z' . Then we shall have

$$\begin{aligned} x'' &= x', \\ y'' &= y' \cos \eta - z' \sin \eta, \\ z'' &= y' \sin \eta + z' \cos \eta. \end{aligned} \tag{43}$$

Finally, transforming these so that the axis of z remains unchanged while the positive axis of x is directed to the mean equinox of t , and denoting the new co-ordinates by x , y , z , we get

$$\begin{aligned} x &= x'' \cos(\theta + p) - y'' \sin(\theta + p), \\ y &= x'' \sin(\theta + p) + y'' \cos(\theta + p), \\ z &= z'', \end{aligned} \tag{44}$$

in which p denotes the precession during the interval $t' - t$. Eliminating x'' , y'' , and z'' from these equations by means of (43) and (42), observing that, since η is very small, we may put $\cos \eta = 1$, we get

$$\begin{aligned}
 x, &= x \cos p - y \sin p + \frac{\eta}{s} z \sin (\theta + p), \\
 y, &= x \sin p + y \cos p - \frac{\eta}{s} z \cos (\theta + p), \\
 z, &= z - \frac{\eta}{s} x \sin \theta + \frac{\eta}{s} y \cos \theta,
 \end{aligned} \tag{45}$$

in which $s = 206264.8$, η being supposed to be expressed in seconds of arc. If we neglect terms of the order p^3 , these equations become

$$\begin{aligned}
 x, &= x - \frac{1}{2} \frac{p^2}{s^2} x - \frac{p}{s} y + \frac{\eta}{s} (\sin \theta + p \cos \theta) z, \\
 y, &= y - \frac{1}{2} \frac{p^2}{s^2} y + \frac{p}{s} x - \frac{\eta}{s} (\cos \theta - p \sin \theta) z, \\
 z, &= z - \frac{\eta}{s} x \sin \theta + \frac{\eta}{s} y \cos \theta.
 \end{aligned} \tag{46}$$

These formulæ give the co-ordinates referred to the ecliptic and mean equinox of one epoch when those referred to the ecliptic and mean equinox of another date are known. For the values of p , η , and θ , we have

$$\begin{aligned}
 p &= (50''.21129 + 0''.0002442966\tau) (t' - t), \\
 \eta &= (0''.48892 - 0''.000006143\tau) (t' - t), \\
 \theta &= 351^\circ 36' 10'' + 39''.79 (t - 1750) - 5''.21 (t' - t),
 \end{aligned}$$

in which $\tau = \frac{1}{2}(t' - t) - 1750$, t and t' being expressed in years from the beginning of the era. If we add the nutation to the value of p , the co-ordinates will be derived for the true equinox of t' .

The equations (45) and (46) serve also to convert the values of δx , δy , and δz belonging to the co-ordinates referred to the ecliptic and mean equinox of t into those to be applied to the co-ordinates referred to the ecliptic and mean equinox of t' . For this purpose it is only necessary to write δx , δy , and δz in place of x , y , and z respectively, and similarly for x , y , and z .

In the computation of the perturbations of a heavenly body during a period of several years, it will be convenient to adopt a fixed equinox and ecliptic throughout the calculation; but when the perturbations are to be applied to the co-ordinates, in the calculation of an ephemeris of the body taking into account the perturbations, it will be convenient to compute the co-ordinates directly for the ecliptic and mean equinox of the beginning of the year for which the ephemeris is required, and the values of δx , δy , and δz must be reduced, by means of the equations (45), as already explained, from the ecliptic and mean equinox to which they belong, to the ecliptic and mean equinox adopted in the case of the co-ordinates required.

In a similar manner we may derive formulæ for the transformation of the co-ordinates or of their variations referred to the mean equinox and equator of one date into those referred to the mean equinox and equator of another date; but a transformation of this kind will rarely be required, and, whenever required, it may be effected by first converting the co-ordinates referred to the equator into those referred to the ecliptic, reducing these to the equinox of t' by means of (45) or (46), and finally converting them into the values referred to the equator of t' . Since, in the computation of an ephemeris for the comparison of observations, the co-ordinates are generally required in reference to the equator as the fundamental plane, it would appear preferable to adopt this plane as the plane of xy in the computation of the perturbations, and in some cases this method is most advantageous. But, generally, since the elements of the orbit of the disturbed planet as well as the elements of the orbits of the disturbing bodies are referred to the ecliptic, the calculation of the perturbations will be most conveniently performed by adopting the ecliptic as the fundamental plane. The consideration of the change of the position of the fundamental plane from one epoch to another is thus also rendered more simple. Whenever an ephemeris giving the geocentric right ascension and declination is required, the heliocentric co-ordinates of the body referred to the mean equinox and equator of the beginning of the year will be computed by means of the osculating elements corrected for precession to that epoch, and the perturbations of the co-ordinates referred to the ecliptic and mean equinox of any other date will be first corrected according to the equations (46), and then converted into those to be applied to the co-ordinates referred to the mean equinox and equator. If the perturbations are not of considerable magnitude and the interval $t' - t$ is also not very large, the correction of δx , δy , and δz on account of the change of the position of the ecliptic and of the equinox will be insignificant; and the conversion of the values of these quantities referred to the ecliptic into the corresponding values for the equator, is effected with great facility.

In the determination of the perturbations of comets, ephemerides being required only during the time of describing a small portion of their orbits, it will sometimes be convenient to adopt the plane of the undisturbed orbit as the fundamental plane. In this case the positive axis of x should be directed to the ascending node of this plane on the ecliptic, and the subsequent change to the ecliptic and equinox, whenever it may be required, will be readily effected.

168. The perturbations of a heavenly body may thus be determined rigorously for a long period of time, provided that the osculating elements may be regarded as accurately known. The peculiar object, however, of such calculations is to facilitate the correction of the assumed elements of the orbit by means of additional observations according to the methods which have already been explained; and when the osculating elements have, by successive corrections, been determined with great precision, a repetition of the calculation of the perturbations may become necessary, since changes of the elements which do not sensibly affect the residuals for the given differential equations in the determination of the most probable corrections, may have a much greater influence on the accuracy of the resulting values of the perturbations.

When the calculation of the perturbations is carried forward for a long period, using constantly the same osculating elements,—and those which are supposed to require no correction,—the secular perturbations of the co-ordinates arising from the secular variation of the elements, and the perturbations of long period, will constantly affect the magnitude of the resulting values, so that δx , δy , and δz will not again become simultaneously equal to zero. Hence it appears that even when the adopted elements do not differ much from their mean values, the numerical amount of the perturbations may be very greatly increased by the secular perturbations and by the large perturbations of long period. But when the perturbations are large, the calculation of the complete values of $\frac{d^2\delta x}{dt^2}$, $\frac{d^2\delta y}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$ (which is effected indirectly) cannot be performed with facility, requiring often several repetitions in order to obtain the required accuracy, since any error in the value of the second differential coefficient produces, by the double integration, an error increasing proportionally to the time in the values of the integral. Errors, therefore, in the values of the second differential coefficients which for a moderate period would have no sensible effect, may in the course of a long period produce large errors in the values of the perturbations, and it is evident that, both for convenience in the numerical calculation and for avoiding the accumulation of error, it will be necessary from time to time to apply the perturbations to the elements in order that the integrals may, in the case of each of the co-ordinates, be again equal to zero. The calculation will then be continued until another change of the elements is required.

The transformation from a system of osculating elements for one epoch to that for another epoch is very easily effected by means of the values of the perturbations of the co-ordinates in connection with the corresponding values of the variations of the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$. The latter will be obtained from the values of the second differential coefficients by means of a single integration according to the equations (27) and (32). Thus, in the case of the example given, we obtain for the date 1865 Jan. 15.0, by means of (32), in units of the seventh decimal place,

$$40 \frac{d\delta x}{dt} = + 385.9, \quad 40 \frac{d\delta y}{dt} = + 214.6, \quad 40 \frac{d\delta z}{dt} = + 9.7.$$

The velocities in the case of the disturbed orbit will be given by the formulæ

$$\frac{dx}{dt} = \frac{dx_0}{dt} + \frac{d\delta x}{dt}, \quad \frac{dy}{dt} = \frac{dy_0}{dt} + \frac{d\delta y}{dt}, \quad \frac{dz}{dt} = \frac{dz_0}{dt} + \frac{d\delta z}{dt}. \quad (47)$$

To obtain the expressions for the components of the velocity resolved parallel to the co-ordinates, we have, according to the equations (6)₂,

$$\begin{aligned} \frac{dx}{dt} &= \sin a \sin (A + u) \frac{dr}{dt} + r \sin a \cos (A + u) \frac{dv}{dt}, \\ \frac{dy}{dt} &= \sin b \sin (B + u) \frac{dr}{dt} + r \sin b \cos (B + u) \frac{dv}{dt}, \\ \frac{dz}{dt} &= \sin c \sin (C + u) \frac{dr}{dt} + r \sin c \cos (C + u) \frac{dv}{dt}. \end{aligned}$$

These equations are applicable in the case of any fundamental plane, if the auxiliaries $\sin a$, $\sin b$, $\sin c$, A , B , and C are determined in reference to that plane. To transform them still further, we have

$$\begin{aligned} \frac{dr}{dt} &= \frac{k\sqrt{1+m}}{\sqrt{p}} e \sin (u - \omega), \\ r \frac{dv}{dt} &= \frac{k\sqrt{p(1+m)}}{r} = \frac{k\sqrt{1+m}}{\sqrt{p}} (1 + e \cos (u - \omega)), \end{aligned}$$

in which ω denotes the angular distance of the perihelion from the ascending node. Substituting these values, we obtain, by reduction,

$$\begin{aligned}\frac{dx}{dt} &= \frac{k\sqrt{1+m}}{\sqrt{p}} ((e \cos \omega + \cos u) \cos A - (e \sin \omega + \sin u) \sin A) \sin a, \\ \frac{dy}{dt} &= \frac{k\sqrt{1+m}}{\sqrt{p}} ((e \cos \omega + \cos u) \cos B - (e \sin \omega + \sin u) \sin B) \sin b, \\ \frac{dz}{dt} &= \frac{k\sqrt{1+m}}{\sqrt{p}} ((e \cos \omega + \cos u) \cos C - (e \sin \omega + \sin u) \sin C) \sin c.\end{aligned}$$

Let us now put

$$\begin{aligned}\frac{k\sqrt{1+m}}{\sqrt{p}} (e \sin \omega + \sin u) &= V \sin U, \\ \frac{k\sqrt{1+m}}{\sqrt{p}} (e \cos \omega + \cos u) &= V \cos U,\end{aligned}\tag{48}$$

and we have

$$\begin{aligned}\frac{dx}{dt} &= V \sin a \cos (A + U), \\ \frac{dy}{dt} &= V \sin b \cos (B + U), \\ \frac{dz}{dt} &= V \sin c \cos (C + U).\end{aligned}\tag{49}$$

These equations determine the components of the velocity of a heavenly body resolved in directions parallel to the co-ordinate axes, and for any fundamental plane to which the auxiliaries A , B , &c. belong. When the ecliptic is the fundamental plane, we have

$$\sin c = \sin i, \quad C = 0.$$

The sum of the squares of the equations (48) gives

$$V^2 = \frac{k^2(1+m)}{p} (1 + e^2 + 2e \cos(u - \omega)) = k^2(1+m) \left(\frac{2}{r} - \frac{1}{a} \right),$$

and hence it appears that V is the linear velocity of the body.

The determination of the osculating elements corresponding to any date for which the perturbations of the co-ordinates and of the velocities have been found, is therefore effected in the following manner:—

First, by means of the osculating elements to which the perturbations belong, we compute accurate values of r_0 , x_0 , y_0 , z_0 , and by means of the equations (48) and (49) we compute the values of $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, and $\frac{dz_0}{dt}$. Then we apply to these the values of the perturbations, and thus find x , y , z , $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$. These having been

found, the equations (32)₁ will furnish the values of Ω , i , and p ; and the remaining elements may be determined as explained in Art. 112. Thus, from

$$\begin{aligned} Vr \sin \psi_0 &= kV \sqrt{p(1+m)}, \\ Vr \cos \psi_0 &= x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt}, \end{aligned}$$

we obtain Vr and ψ_0 , and from

$$\begin{aligned} r \sin u &= (-x \sin \Omega + y \cos \Omega) \sec i, \\ r \cos u &= x \cos \Omega + y \sin \Omega, \end{aligned}$$

we derive r and u ; and hence V from the value of Vr . When i is not very small, we may use, instead of the preceding expression for $r \sin u$,

$$r \sin u = z \operatorname{cosec} i.$$

Next, we compute a from

$$2a - r = \frac{r}{\frac{2}{r} \cdot \frac{k^2(1+m)}{V^2} - 1},$$

and from

$$\begin{aligned} 2ae \sin \omega &= -(2a - r) \sin(2\psi_0 + u) - r \sin u, \\ 2ae \cos \omega &= -(2a - r) \cos(2\psi_0 + u) - r \cos u, \end{aligned}$$

we find ω and e . The mean daily motion and the mean anomaly or the mean longitude for the epoch will then be determined by means of the usual formulæ.

In the case of a very eccentric orbit, after r and u have been found, $\frac{dr}{dt}$ will be given by equations (48)₆, and the values of e and v will be given by the equations (49)₆. Then the perihelion distance will be found from

$$q = \frac{p}{1+e},$$

and the time of perihelion passage will be found from v and e by means of Table IX. or Table X.

In the numerical values of the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c., more decimals must be retained than in the values of the co-ordinates, and enough must be retained to secure the required accuracy of the solution. If it be considered necessary, the different parts of the calculation may be checked by means of various formulæ which have already been given. Thus, the values of Ω and i must satisfy the equation

$$z \cos i - y \sin i \cos \Omega + x \sin i \sin \Omega = 0.$$

We have, also,

$$\begin{aligned} V^2 &= \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2, \\ r^2 &= x^2 + y^2 + z^2, \\ z &= r \sin u \sin i, \end{aligned}$$

which must be satisfied by the resulting values of V , r , and u ; and the values of a and e must satisfy the equation

$$p = a(1 - e^2) = a \cos^2 \varphi.$$

169. When the plane of the undisturbed orbit is adopted as the fundamental plane, we obtain at once the perturbations

$$\delta(r \cos u), \quad \delta(r \sin u), \quad \delta z,$$

and from these the perturbations of the polar co-ordinates are easily derived. There are, however, advantages which may be secured by employing formulæ which give the perturbations of the polar co-ordinates directly, retaining the plane of the orbit for the date t_0 as the fundamental plane.

Let w denote the angle which the projection of the disturbed radius-vector on the plane of xy makes with the axis of x , and β the latitude of the body with respect to the plane of xy ; then we shall have

$$\begin{aligned} x &= r \cos \beta \cos w, \\ y &= r \cos \beta \sin w, \\ z &= r \sin \beta. \end{aligned} \tag{50}$$

Let us now denote by X , Y , and Z , respectively, the forces which are expressed by the second members of the equations (1), and the first two of these equations give

$$x \frac{dy}{dt} - y \frac{dx}{dt} = \int (Yx - Xy) dt + C,$$

C being the constant of integration. The equations (50) give

$$\begin{aligned} \frac{dx}{dt} &= \cos w \frac{d(r \cos \beta)}{dt} - r \cos \beta \sin w \frac{dw}{dt}, \\ \frac{dy}{dt} &= \sin w \frac{d(r \cos \beta)}{dt} + r \cos \beta \cos w \frac{dw}{dt}, \end{aligned}$$

and hence

$$x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \cos^2 \beta \frac{dw}{dt}.$$

Therefore we have

$$r^2 \cos^2 \beta \frac{dw}{dt} = \int (Yx - Xy) dt + C.$$

If we denote by S_0 the component of the disturbing force in a direction perpendicular to the disturbed radius-vector and parallel with the plane of xy , we shall have

$$X = -S_0 \sin w, \quad Y = S_0 \cos w,$$

and

$$Yx - Xy = S_0 r \cos \beta.$$

Therefore

$$r^2 \cos^2 \beta \frac{dw}{dt} = \int S_0 r \cos \beta dt + C.$$

In the undisturbed orbit we have $\beta = 0$, and

$$r_0^2 \frac{dw_0}{dt} = kV \sqrt{p_0(1+m)};$$

and thus the preceding equation becomes

$$r^2 \cos^2 \beta \frac{dw}{dt} = \int S_0 r \cos \beta dt + kV \sqrt{p_0(1+m)}. \quad (51)$$

The equations (1) also give

$$\frac{1}{r} \cdot \frac{xd^2x + yd^2y + zd^2z}{dt^2} + \frac{k^2(1+m)}{r^2} = X \frac{x}{r} + Y \frac{y}{r} + Z \frac{z}{r}. \quad (52)$$

If we denote by R the component of the disturbing force in the direction of the disturbed radius-vector, we have

$$R = X \frac{x}{r} + Y \frac{y}{r} + Z \frac{z}{r}. \quad (53)$$

We have, also,

$$\begin{aligned} xd^2x + yd^2y + zd^2z &= d(xdx + ydy + zdz) - (dx^2 + dy^2 + dz^2) \\ &= d(rdr) - (dr^2 + r^2dv^2) = rd^2r - r^2dv^2, \end{aligned}$$

v denoting the true anomaly in the disturbed orbit, or, since $dv^2 = \cos^2 \beta dw^2 + d\beta^2$,

$$xd^2x + yd^2y + zd^2z = rd^2r - r^2 \cos^2 \beta dw^2 - r^2 d\beta^2.$$

Hence the equation (52) becomes

$$\frac{d^2r}{dt^2} - r \cos^2 \beta \frac{dw^2}{dt^2} - r \frac{d\beta^2}{dt^2} + \frac{k^2(1+m)}{r^2} = R. \quad (54)$$

170. The equations (51) and (54), in connection with the last of equations (1), completely represent the motion of a heavenly body about the sun when acted upon by disturbing forces, and, when completely integrated, they will give the values of w , r , and z for any point of the orbit; but, since they cannot be integrated directly, we must, as in the case of the rectangular co-ordinates, find the equations which give by integration the values of δw , δr , and z . In the case of the undisturbed orbit, we have

$$\begin{aligned} r_0^2 \frac{dw_0}{dt} &= kV\overline{p_0(1+m)}, \\ \frac{d^2 r_0}{dt^2} - r_0 \frac{dw_0^2}{dt^2} + \frac{k^2(1+m)}{r_0^2} &= 0. \end{aligned} \quad (55)$$

If we denote by δw the variation of w arising from the action of the disturbing force, we have $w = w_0 + \delta w$; and hence we easily find, from (51),

$$\frac{d\delta w}{dt} = \frac{1}{r^2 \cos^2 \beta} \int S_0 r \cos \beta dt - \left(1 - \frac{r_0^2}{r^2 \cos^2 \beta}\right) \frac{kV\overline{p_0(1+m)}}{r_0^2}. \quad (56)$$

We have, further,

$$r^2 = r_0^2 + 2r_0 \delta r + \delta r^2,$$

which gives

$$\frac{r^2}{r_0^2} = 1 + 2 \frac{r_0 + \frac{1}{2} \delta r}{r_0^2} \delta r.$$

Let us now put

$$q' = \left(\frac{r_0 + \frac{1}{2} \delta r}{r_0^2} \cos^2 \beta - \frac{\sin^2 \beta}{2\delta r} \right) \delta r, \quad f'q' = 1 - \frac{r_0^2}{r^2 \cos^2 \beta} \quad (57)$$

and we have

$$f' = \frac{2}{1 + 2q'}. \quad (58)$$

The equation (56), therefore, becomes

$$\frac{d\delta w}{dt} = \frac{1}{r^2 \cos^2 \beta} \int S_0 r \cos \beta dt - g_0 f' q', \quad (59)$$

in which we put

$$g_0 = \frac{dw_0}{dt} = \frac{kV\overline{p_0(1+m)}}{r_0^2}. \quad (60)$$

If we substitute $r_0 + \delta r$ for r in equation (54), and combine the result with the second of equations (55), we get

$$\frac{d^2 \delta r}{dt^2} = R - g_0^2 r_0 + r \cos^2 \beta \frac{dw^2}{dt^2} + r \frac{d\beta^2}{dt^2} + k^2(1+m) \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right);$$

and if we put

$$q'' = \frac{r_0 + \frac{1}{2}\delta r}{r_0^2} \delta r, \quad f''q'' = 1 - \frac{r_0^2}{r^3}, \quad (61)$$

we have

$$f'' = \frac{2}{1 + 2q''} \quad (62)$$

and hence

$$\begin{aligned} \frac{d^2\delta r}{dt^2} = R + \frac{k^2(1+m)}{r_0^2} f''q'' + g_0^2\delta r + 2g_0r \frac{d\delta w}{dt} \\ - r \sin^2\beta \left(g_0 + \frac{d\delta w}{dt} \right)^2 + r \left(\frac{d\delta w}{dt} \right)^2 + r \left(\frac{d\beta}{dt} \right)^2. \end{aligned} \quad (63)$$

Finally, we have, from the last of equations (1),

$$\frac{d^2z}{dt^2} = Z - \frac{k^2(1+m)}{r^3} z, \quad (64)$$

by means of which the value of z may be found, since, in the case of the undisturbed motion, we have $z_0 = 0$.

The values of f' corresponding to different values of q' may be tabulated with the argument q' , and, since the equation (62) is of the same form as (58), the same table will give the value of f'' when q'' is used as the argument. Table XVII. gives the values of $\log f'$ or $\log f''$ corresponding to values of q' or q'' from -0.03 to $+0.03$. Beyond the limits of this table the required quantities may be computed directly.

171. When we consider only terms of the first order with respect to the disturbing force, we have

$$f'q' = f''q'' = \frac{2\delta r}{r_0},$$

and the equations become

$$\begin{aligned} \frac{d\delta w}{dt} &= \frac{1}{r_0^2} \int S_0 r_0 dt - \frac{2g_0}{r_0} \delta r, \\ \frac{d^2\delta r}{dt^2} &= R + \frac{2g_0}{r_0} \int S_0 r_0 dt + \left(\frac{2k^2(1+m)}{r_0^3} - 3g_0^2 \right) \delta r, \\ \frac{d^2z}{dt^2} &= Z - \frac{k^2(1+m)}{r_0^3} z. \end{aligned} \quad (65)$$

In determining the perturbations of a heavenly body, we first consider only the terms depending on the first power of the disturbing force, for which these equations will be applied. The value of δr

will be obtained from the second equation by an indirect process, as already illustrated for the case of the variation of the rectangular co-ordinates. Then δw will be obtained directly from the first equation, and, finally, z indirectly from the last equation. Each of the integrals is equal to zero for the date t_0 , to which the osculating elements belong.

When the magnitude of the perturbations is such that the terms depending on the squares and products of the masses must be considered, the general equations (59), (63), and (64) will be applied. The values of the perturbations for the dates preceding that for which the complete expressions are to be used, will at once indicate approximate values of δw , δr , and z ; and with the values

$$r = r_0 + \delta r, \quad w = w_0 + \delta w, \quad \sin \beta = \frac{z}{r}$$

the components of the disturbing force will be computed. We compute also q' from the first of equations (57), and q'' from the first of (61); then, by means of Table XVII., we derive the corresponding values of $\log f'$ and $\log f''$. The coefficients of δr in the expressions for q and q'' will be given with sufficient accuracy by means of the approximate values of δr and $\sin \beta$, and will not require any further correction. Then we compute $S_0 r \cos \beta$, and find the integral

$$\int S_0 r \cos \beta \, dt;$$

and the complete value of $\frac{d\delta w}{dt}$ will be given by (59). The value of $\frac{d^2\delta r}{dt^2}$ will then be given by equation (63). The term $r \left(\frac{d\beta}{dt} \right)^2$ will always be small, and, unless the inclination of the orbit of the disturbed body is large, it may generally be neglected. Whenever it shall be required, we may put it equal to $\frac{1}{r} \left(\frac{dz}{dt} \right)^2$. The corrected values of the differential coefficients being introduced into the table of integration, the exact or very approximate values of δw , δr , and z will be obtained. Should these results, however, differ much from the corresponding values already assumed, a repetition of the calculation may become necessary. In this manner, by computing each place separately, the terms depending on the squares, products, and higher powers of the disturbing forces may be included in the results. It will, however, be generally possible to estimate the values of δw , δr ,

and z for two or three intervals in advance to a degree of approximation sufficient for the computation of the forces for these dates.

In order that the quantity ω , representing the interval adopted in the calculation of the perturbations, may not appear in the integration, we should introduce it into the equations as in the case of the variation of the rectangular co-ordinates. Thus, in the determination of δw we compute the values of $\omega \frac{d\delta w}{dt}$, and since the second member of the equation contains the integral $\int S_0 r \cos \beta dt$, if we introduce the factor ω^2 under the sign of integration, this integral, omitting the factor ω in the formulæ of integration, will become $\omega \int S_0 r \cos \beta dt$, as required. The last term of the equation will be multiplied by ω .

In the case of δr , each term of the equation for $\frac{d^2 \delta r}{dt^2}$ must contain the factor ω^2 . If the second of equations (65) is employed, the first and third terms of the second member will be multiplied by ω^2 ; but since the value of S_0 is supposed to be already multiplied by ω^2 , the second term will only be multiplied by ω .

The perturbations may be conveniently determined either in units of the seventh decimal place, or expressed in seconds of arc of a circle whose radius is unity. If they are to be expressed in seconds, the factor $s = 206264.8$ must be introduced so as to preserve the homogeneity of the several terms, and finally δr and δz must be converted into their values in terms of the unit of space.

172. It remains yet to derive convenient formulæ for the determination of the forces S_0 , R , and Z . For this purpose, it first becomes necessary to determine the position of the orbit of the disturbing planet in reference to the fundamental plane adopted, namely, the plane defined by the osculating elements of the disturbed orbit at the instant t_0 . Let i' and Ω' denote the inclination and the longitude of the ascending node of the disturbing body with respect to the ecliptic, and let I denote the inclination of the orbit of the disturbing body with respect to the fundamental plane. Further, let N denote the longitude of its ascending node on the same plane measured from the ascending node of this plane on the ecliptic or from the point whose longitude is Ω_0 , and let N' be the angular distance between the ascending node of the orbit of the disturbing body on the ecliptic and the ascending node on the fundamental plane adopted. Then, from the spherical triangle formed by the intersection of the plane of the

ecliptic, the fundamental plane, and the plane of the orbit of the disturbing body with the celestial vault, we have

$$\begin{aligned}\sin \tfrac{1}{2} I \sin \tfrac{1}{2} (N + N') &= \sin \tfrac{1}{2} (\Omega' - \Omega_0) \sin \tfrac{1}{2} (i' + i_0), \\ \sin \tfrac{1}{2} I \cos \tfrac{1}{2} (N + N') &= \cos \tfrac{1}{2} (\Omega' - \Omega_0) \sin \tfrac{1}{2} (i' - i_0), \\ \cos \tfrac{1}{2} I \sin \tfrac{1}{2} (N - N') &= \sin \tfrac{1}{2} (\Omega' - \Omega_0) \cos \tfrac{1}{2} (i' + i_0), \\ \cos \tfrac{1}{2} I \cos \tfrac{1}{2} (N - N') &= \cos \tfrac{1}{2} (\Omega' - \Omega_0) \cos \tfrac{1}{2} (i' - i_0),\end{aligned}\quad (66)$$

from which to find N , N' , and I .

Let β' denote the heliocentric latitude of the disturbing planet with respect to the fundamental plane, w' its longitude in this plane measured from the axis of x , as in the case of w , and u'_0 the argument of the latitude with respect to this plane. Then, according to the equations (82), we have

$$\begin{aligned}\tan (w' - N) &= \tan u'_0 \cos I, \\ \tan \beta' &= \tan I \sin (w' - N).\end{aligned}\quad (67)$$

If u' denotes the argument of the latitude of the disturbing planet with respect to the ecliptic, we have

$$u'_0 = u' - N'. \quad (68)$$

This formula will give the value of u'_0 , and then w' and β' will be found from (67). We have, also,

$$\cos u'_0 = \cos \beta' \cos (w' - N),$$

which will serve to indicate the quadrant in which $w' - N$ must be taken.

The relations here derived are evidently applicable to the case in which the elements of the orbits of the disturbed and disturbing planets are referred to the equator, the signification of the quantities involved being properly considered.

The co-ordinates of the disturbing planet in reference to the plane of the disturbed orbit at the instant t_0 as the fundamental plane will be given by

$$\begin{aligned}x' &= r' \cos \beta' \cos w', \\ y' &= r' \cos \beta' \sin w', \\ z' &= r' \sin \beta'.\end{aligned}\quad (69)$$

To find the force R , we have

$$R = X \frac{x}{r} + Y \frac{y}{r} + Z \frac{z}{r},$$

and

$$\begin{aligned} X &= m'k^2 \left(\frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right), \\ Y &= m'k^2 \left(\frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right), \\ Z &= m'k^2 \left(\frac{z' - z}{\rho^3} - \frac{z'}{r'^3} \right). \end{aligned}$$

Substituting in these the values of x', y', z' given by (69), and the corresponding values of x, y, z given by (50), and putting

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3}, \quad (70)$$

we get

$$R = m'k^2 \left(h r' \cos \beta' \cos \beta \cos (w' - w) + h r' \sin \beta \sin \beta' - \frac{r}{\rho^3} \right). \quad (71)$$

The equation

$$S_0 r \cos \beta = Yx - Xy$$

gives

$$S_0 = m'k^2 h r' \cos \beta' \sin (w' - w), \quad (72)$$

from which to find S_0 . Finally, we have

$$Z = m'k^2 \left(h r' \sin \beta' - \frac{z}{\rho^3} \right), \quad (73)$$

from which to find Z .

When we determine the perturbations only with respect to the first power of the disturbing force, the expressions for R, S_0 , and Z become

$$\begin{aligned} R &= m'k^2 \left(h r' \cos \beta' \cos (w' - w_0) - \frac{r_0}{\rho_0^3} \right), \\ S_0 &= m'k^2 h r' \cos \beta' \sin (w' - w_0), \\ Z &= m'k^2 h r' \sin \beta'. \end{aligned} \quad (74)$$

To compute the distance ρ , we have

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2,$$

which gives

$$\rho^2 = r'^2 + r^2 - 2r r' \cos \beta \cos \beta' \cos (w' - w) - 2r r' \sin \beta \sin \beta', \quad (75)$$

and, if we neglect terms of the second order, we have

$$\rho_0^2 = r'^2 + r_0^2 - 2r_0 r' \cos \beta' \cos (w' - w_0). \quad (76)$$

If we put

$$\cos \gamma = \cos \beta \cos \beta' \cos (w' - w) + \sin \beta \sin \beta', \quad (77)$$

we have

$$\begin{aligned} \rho^2 &= r'^2 + r^2 - 2rr' \cos \gamma \\ &= r'^2 \sin^2 \gamma + (r - r' \cos \gamma)^2; \end{aligned}$$

and hence we may readily find ρ from

$$\begin{aligned}\rho \sin n &= r' \sin \gamma, \\ \rho \cos n &= r - r' \cos \gamma,\end{aligned}\tag{78}$$

the exact value of the angle n , however, not being required.

Introducing γ into the expression for R , it becomes

$$R = m'k^2 \left(h r' \cos \gamma - \frac{r}{\rho^3} \right),\tag{79}$$

by means of which R may be conveniently determined.

173. When we neglect the terms depending on the squares and higher powers of the masses in the computation of the perturbations, the forces R , S_0 , and Z will be computed by means of the equations (74), ρ_0 being found from (76) or from (78), when we put

$$\cos \gamma = \cos \beta' \cos (w' - w_0).$$

But when the terms of the order of the square of the disturbing force are to be taken into account, the complete equations must be used. Thus, we find ρ from (78), S_0 from (72), Z from (73), and R from (71) or (79). The values of δw , δr , and z , computed to the point at which it becomes necessary to consider the terms of the second order, will enable us at once to estimate the values of the perturbations for two or three intervals in advance to a degree of approximation sufficient for the calculation of the forces; and the values of R , S_0 , and Z thus found will not require any further correction.

When the places of the disturbing planet are to be derived from an ephemeris giving the heliocentric longitudes and latitudes, the values of Ω' and i' will be obtained from two places separated by a considerable interval, and then the values of u' will be determined by means of the first of equations (82)₁ or by means of (85)₁. When the inclination i' is very small, it will be sufficient to take

$$u' = l' - \Omega' + s \tan^2 \frac{1}{2} i' \sin 2(l' - \Omega'),$$

in which $s = 206264.8$. But when the tables give directly the longitude in the orbit, $u' + \Omega'$, by subtracting Ω' from each of these longitudes we obtain the required values of u' .

It should be observed, also, that the exact determination of the values of the forces requires that the actual disturbed values of r' , w' , and β' should be used. The disturbed radius-vector r' will be

given immediately by the tables of the motion of the disturbing body, but the determination of the actual values of w' and β' requires that we should use the actual values of N' , N , and I in the solution of the equations (68) and (67). Hence the disturbed values of Ω' and i' should be used in the determination of these quantities for each date by means of (66). It will, however, generally be the case that for a moderate period the variation of Ω' and i' may be neglected; and whenever the variation of either of these has a sensible effect, we may compute new values of N , N' , and I from time to time, by means of which the true values may be readily interpolated for each date. We may also determine the variations of N , N' , and I arising from the variation of Ω' and i' , by means of differential formulæ. Thus the relations will be similar to those given by the equations (71)₂, so that we have

$$\begin{aligned}\delta N' &= \frac{\sin N'}{\sin(\Omega' - \Omega_0)} \cos N \delta \Omega' - \frac{\sin N'}{\sin I} \cos I \delta i', \\ \delta N &= \frac{\sin N}{\sin(\Omega' - \Omega_0)} \cos N' \delta \Omega' - \frac{\sin N'}{\sin I} \delta i', \\ \delta I &= \sin N' \sin i' \delta \Omega' + \cos N' \delta i',\end{aligned}\tag{80}$$

from which to find $\delta N'$, δN , and δI .

When the perturbations are computed only in reference to the first power of the mass, the change of Ω' and i' may be entirely neglected; but when the perturbations are to be computed for a long period of time, and the terms depending on the squares and products of the disturbing forces are to be included, it will be advisable to take into account the values of δN , $\delta N'$, and δI , and, using also the value of u' in the actual orbit of the disturbing body, compute the actual values of w' and β' .

In the case of several disturbing bodies, the forces will be determined for each of these, and then, instead of R , S_0 , and Z , in the formulæ for the differential coefficients, ΣR , ΣS_0 , and ΣZ will be used.

174. By means of the values of δw , δr , and z , the heliocentric or the geocentric place of the disturbed planet may be readily found. Thus, let the positive axis of x be directed to the ascending node of the osculating orbit at the instant t_0 on the plane of the ecliptic; then, in the undisturbed orbit, we shall have

$$w_0 = u_0,$$

u denoting the argument of the latitude. Let x , y , z , be the co-or-

dinates of the body referred to a system of rectangular co-ordinates in which the ecliptic is the plane of xy , and in which the positive axis of x is directed to the vernal equinox. Then we shall have

$$\begin{aligned}x_1 &= x \cos \Omega_0 - y \cos i_0 \sin \Omega_0 + z \sin i_0 \sin \Omega_0, \\y_1 &= x \sin \Omega_0 + y \cos i_0 \cos \Omega_0 - z \sin i_0 \cos \Omega_0, \\z_1 &= y \sin i_0 + z \cos i_0,\end{aligned}$$

or, introducing the values of x and y given by (50),

$$\begin{aligned}x_1 &= r \cos \beta \cos w \cos \Omega_0 - r \cos \beta \sin w \cos i_0 \sin \Omega_0 + z \sin i_0 \sin \Omega_0, \\y_1 &= r \cos \beta \cos w \sin \Omega_0 + r \cos \beta \sin w \cos i_0 \cos \Omega_0 - z \sin i_0 \cos \Omega_0, \\z_1 &= r \cos \beta \sin w \sin i_0 + z \cos i_0.\end{aligned}\quad (81)$$

Introducing also the auxiliary constants for the ecliptic according to the equations (94)₁ and (96)₁, we obtain

$$\begin{aligned}x_1 &= r \cos \beta \sin a \sin (A + w) + z \cos a, \\y_1 &= r \cos \beta \sin b \sin (B + w) + z \cos b, \\z_1 &= r \cos \beta \sin i_0 \sin w + z \cos i_0,\end{aligned}\quad (82)$$

by means of which the heliocentric co-ordinates in reference to the ecliptic may be determined.

If the place of the disturbed body is required in reference to the equator, denoting the heliocentric co-ordinates by x_{11} , y_{11} , z_{11} , and the obliquity of the ecliptic by ϵ , we have

$$\begin{aligned}x_{11} &= x_1, \\y_{11} &= y_1 \cos \epsilon - z_1 \sin \epsilon, \\z_{11} &= y_1 \sin \epsilon + z_1 \cos \epsilon.\end{aligned}$$

Substituting for x_1 , y_1 , z_1 , their values given by (81), and introducing the auxiliary constants for the equator, according to the equations (99)₁ and (101)₁, we get

$$\begin{aligned}x_{11} &= r \cos \beta \sin a \sin (A + w) + z \cos a, \\y_{11} &= r \cos \beta \sin b \sin (B + w) + z \cos b, \\z_{11} &= r \cos \beta \sin c \sin (C + w) + z \cos c.\end{aligned}\quad (83)$$

The combination of the values derived from these equations with the corresponding values of the co-ordinates of the sun, will give the required geocentric places of the disturbed body. These equations are applicable to the case of any fundamental plane, provided that the auxiliary constants a , A , b , B , &c. are determined with respect to that plane. In the numerical application of the formulæ, the value of w will be found from

$$w = u_1 + \delta w,$$

u_0 being the argument of the latitude for the fundamental osculating elements, and care must be taken that the proper algebraic sign is assigned to $\cos a$, $\cos b$, and $\cos c$.

If the values of π_0 , Ω_0 , and i_0 used in the calculation of the perturbations are referred to the ecliptic and mean equinox of the date t_0' , and the rectangular co-ordinates of the disturbed body are required in reference to the ecliptic and mean equinox of the date t_0'' , the value of w must be found from

$$w = v_0 + \omega_0 + \delta w,$$

the value of ω_0 referred to the ecliptic of t_0' being reduced to that of t_0'' , by means of the first of equations (115)₁. Then Ω_0 and i_0 should be reduced from the ecliptic and mean equinox of t_0' to the ecliptic and mean equinox of t_0'' by means of the second and third of the equations (115)₁, and, using the values thus found in the calculation of the auxiliary constants for the ecliptic, the equations (82) will give the required values of the heliocentric co-ordinates. If the co-ordinates referred to the mean equinox and equator of the date t_0'' are to be determined, the proper corrections having been applied to Ω_0 and i_0 , the mean obliquity of the ecliptic for this date will be employed in the determination of the auxiliary constants a , A , &c. with respect to the equator, and the equations (83) will then give the required values of the co-ordinates.

If we differentiate the equations (83), we obtain, by reduction,

$$\begin{aligned} \frac{dx''}{dt} &= r \cos \beta \sin a \cos (A + w) \frac{dw}{dt} + \sec \beta \sin a \sin (A + w) \frac{dr}{dt} \\ &\quad + (\cos a - \tan \beta \sin a \sin (A + w)) \frac{dz}{dt}, \\ \frac{dy''}{dt} &= r \cos \beta \sin b \cos (B + w) \frac{dw}{dt} + \sec \beta \sin b \sin (B + w) \frac{dr}{dt} \\ &\quad + (\cos b - \tan \beta \sin b \sin (B + w)) \frac{dz}{dt}, \\ \frac{dz''}{dt} &= r \cos \beta \sin c \cos (C + w) \frac{dw}{dt} + \sec \beta \sin c \sin (C + w) \frac{dr}{dt} \\ &\quad + (\cos c - \tan \beta \sin c \sin (C + w)) \frac{dz}{dt}, \end{aligned} \quad (84)$$

by means of which the components of the velocity of the disturbed body in directions parallel to the co-ordinate axes may be determined.

The values of $\frac{d\delta r}{dt}$ and $\frac{d\delta z}{dt}$ will be obtained from $\frac{d^2\delta r}{dt^2}$ and $\frac{d^2\delta z}{dt^2}$ by a single integration, and then we have

$$\frac{dw}{dt} = \frac{k\sqrt{p_0(1+m)}}{r_0^2} + \frac{d\delta w}{dt}, \quad \frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p_0}} e_0 \sin v_0 + \frac{d\delta r}{dt}, \quad (85)$$

from which to find $\frac{dw}{dt}$ and $\frac{dr}{dt}$.

175. EXAMPLE.—In order to illustrate the calculation of the perturbations of r , w , and z , let us take the data given in Art. 166, and determine these perturbations instead of those of the rectangular coordinates.

In the first place, we derive from the tables of the motion of *Jupiter* the values

$$\Omega' = 98^\circ 58' 22''.7, \quad i' = 1^\circ 18' 40''.5,$$

which refer to the ecliptic and mean equinox of 1860.0. We find, also, from the data given by the tables the values of u' measured from the ecliptic of 1860.0. Then, by means of the formulæ (66), using the values of Ω_0 and i_0 given in Art. 166, we derive

$$N = 194^\circ 0' 49''.9, \quad N' = 301^\circ 38' 31''.7, \quad I = 5^\circ 9' 56''.4.$$

The value of u'_0 is given by equation (68), and then w' and β' are found from the equations (67). Thus we have

| Berlin Mean Time. | $\log r_0$ | $w_0 = u_0$ | | $\log r$ | w' | β' |
|-------------------|------------|-------------|-----------|----------|----------------|----------------|
| 1863 Dec. 12.0, | 0.294084 | 192° | 4' 24''.5 | 0.73425 | 14° 18' 54''.6 | — 0° 1' 38''.1 |
| 1864 Jan. 21.0, | 0.294837 | 207 | 40 52 .2 | 0.73368 | 17 21 44 .2 | 0 18 9 .1 |
| March 1.0, | 0.300674 | 223 | 3 5 .9 | 0.73305 | 20 25 5 .2 | 0 34 39 .9 |
| April 10.0, | 0.310864 | 237 | 51 38 .3 | 0.73237 | 23 28 59 .8 | 0 51 7 .6 |
| May 20.0, | 0.324298 | 251 | 52 47 .9 | 0.73164 | 26 33 32 .1 | 1 7 29 .7 |
| June 29.0, | 0.339745 | 264 | 59 30 .0 | 0.73086 | 29 38 44 .8 | 1 23 43 .5 |
| Aug. 8.0, | 0.356101 | 277 | 10 24 .6 | 0.73003 | 32 44 41 .2 | 1 39 46 .3 |
| Sept. 17.0, | 0.372469 | 288 | 28 4 .1 | 0.72915 | 35 51 24 .6 | 1 55 35 .2 |
| Oct. 27.0, | 0.388214 | 298 | 57 16 .3 | 0.72823 | 38 58 57 .5 | 2 11 7 .5 |
| Dec. 6.0, | 0.402894 | 308 | 43 48 .7 | 0.72726 | 42 7 23 .3 | 2 26 20 .3 |
| 1865 Jan. 15.0, | 0.416240 | 317 | 53 39 .1 | 0.72625 | 45 16 43 .9 | — 2 41 10 .6 |

The values of ρ_0 may be found from (76) or (78) as already given in Art. 166.

The forces R , S_0 , and Z may now be determined by means of the equations (74), h being found from (70), and if we introduce the factor ω^2 for convenience in the integration, as already explained, we obtain the following results:

| Date. | $\omega^2 R$ | $\omega^2 S_0 r_0$ | $\omega^2 Z$ | $\omega \int S_0 r_0 dt$ |
|-----------------|--------------|--------------------|--------------|--------------------------|
| 1863 Dec. 12.0, | + 1''.4608 | + 0''.1476 | + 0''.0009 | + 0''.0282 |
| 1864 Jan. 21.0, | + 1 .4223 | — 0 .6757 | + 0 .0101 | — 0 .2361 |

| Date. | $\omega^2 R$ | $\omega^2 S_0 r_0$ | $\omega^2 Z$ | $\omega \int S_0 r_0 dt$ |
|-----------------|--------------|--------------------|--------------|--------------------------|
| 1864 March 1.0, | + 1".2616 | - 1".4512 | + 0".0190 | - 1".3060 |
| April 10.0, | 1 .0018 | 2 .1226 | 0 .0273 | 3 .1035 |
| May 20.0, | 0 .6760 | 2 .6473 | 0 .0347 | 5 .5020 |
| June 29.0, | + 0 .3179 | 2 .9988 | 0 .0406 | 8 .3402 |
| Aug. 8.0, | - 0 .0452 | 3 .1650 | 0 .0449 | 11 .4378 |
| Sept. 17.0, | 0 .3944 | 3 .1437 | 0 .0470 | 14 .6076 |
| Oct. 27.0, | 0 .7180 | 2 .9392 | 0 .0466 | 17 .6640 |
| Dec. 6.0, | 1 .0097 | 2 .5586 | 0 .0432 | 20 .4273 |
| 1865 Jan. 15.0, | - 1 .2674 | - 2 .0081 | + 0 .0362 | - 22 .7245 |

The integral $\omega \int S_0 r_0 dt$ is obtained from the successive values of $\omega^2 S_0 r_0$, by means of the formula (32).

Next we compute the values of the differential coefficients by means of the formulæ (65). For the dates 1863 Dec. 12.0 and 1864 Jan. 21.0 we may first assume $\delta r = 0$, and, by a preliminary integration, having thus derived very approximate values of δr for these dates, the values of $\frac{d^2 \delta r}{dt^2}$ will be recomputed. Then, commencing anew the table of integration, we may at once derive an approximate value of δr for the date March 1.0 with which the last term of the expression for $\frac{d^2 \delta r}{dt^2}$ may be computed. Continuing this indirect process, as already illustrated in the case of the perturbations of the rectangular co-ordinates, we obtain the required values of the second differential coefficient. In a similar manner, the values of $\frac{d^2 z}{dt^2}$ will be obtained. The values of $\frac{d \delta w}{dt}$ will then be given directly by means of the first of equations (65); and the final integration will furnish the perturbations required. Thus we derive the following results:-

| Date. | $\omega \frac{d \delta w}{dt}$ | $\omega^2 \frac{d^2 \delta r}{dt^2}$ | $\omega^2 \frac{d^2 z}{dt^2}$ | δw | δr | z |
|-----------------|--------------------------------|--------------------------------------|-------------------------------|------------|------------|---------|
| 1863 Dec. 12.0, | - 0".0423 | + 1".4509 | + 0".0009 | - 0".00 | + 0".18 | + 0".00 |
| 1864 Jan. 21.0, | 0 .1086 | 1 .3405 | 0 .0101 | 0 .02 | 0 .17 | 0 .00 |
| Mar. 1.0, | 0 .7162 | + 0 .7829 | 0 .0183 | 0 .40 | 1 .47 | 0 .01 |
| Apr. 10.0, | 1 .6114 | - 0 .0455 | 0 .0251 | 1 .55 | 3 .53 | 0 .04 |
| May 20.0, | 2 .4795 | 0 .9344 | 0 .0300 | 3 .61 | 5 .54 | 0 .09 |
| June 29.0, | 3 .0807 | 1 .7333 | 0 .0326 | 6 .42 | 6 .62 | 0 .18 |
| Aug. 8.0, | 3 .2971 | 2 .3752 | 0 .0331 | 9 .64 | 5 .98 | 0 .29 |
| Sept. 17.0, | 3 .1080 | 2 .8533 | 0 .0311 | 12 .88 | + 2 .98 | 0 .44 |
| Oct. 27.0, | - 2 .5425 | - 3 .1872 | + 0 .0265 | - 15 .73 | - 2 .86 | + 0 .62 |

| Date. | $\omega \frac{d\delta w}{dt}$ | $\omega^2 \frac{d^2 \delta r}{dt^2}$ | $\omega^2 \frac{d^2 z}{dt^2}$ | δw | δr | z |
|-----------------|-------------------------------|--------------------------------------|-------------------------------|------------|------------|-----------|
| 1864 Dec. 6.0, | $-1''.6443$ | $-3''.4009$ | $+0''.0190$ | $-17''.85$ | $-11''.88$ | $+0''.83$ |
| 1865 Jan. 15.0, | -0.4511 | -3.5334 | $+0.0079$ | -18.92 | -24.29 | $+1.05$ |

It has already been found that, during the period included by these results, the perturbations arising from the squares and products of the disturbing forces are insensible, and hence the application of the complete equations for the forces and for the differential coefficients is not required. The equations (83) will give, by means of the results for $w = u_0 + \delta w$, $r = r_0 + \delta r$, and z , the values of the heliocentric co-ordinates of the disturbed body, and the combination of these with the co-ordinates of the sun will give the geocentric place.

When we neglect terms of the second order, we have, according to the equations (84),

$$\begin{aligned}\delta x_{,,} &= x_0 \cot(A + w) \delta w + \frac{x_0}{r_0} \delta r + z \cos a, \\ \delta y_{,,} &= y_0 \cot(B + w) \delta w + \frac{y_0}{r_0} \delta r + z \cos b, \\ \delta z_{,,} &= z_0 \cot(C + w) \delta w + \frac{z_0}{r_0} \delta r + z \cos c,\end{aligned}\tag{86}$$

the heliocentric co-ordinates x_0, y_0, z_0 being referred to the same fundamental plane as the auxiliary constants, a, b, A , &c. Thus, in the case of *Eurynome*, to find the perturbations of the rectangular co-ordinates, referred to the ecliptic and mean equinox of 1860.0, from 1864 Jan. 1.0 to 1865 Jan. 15.0, we have

$$\begin{aligned}A &= 296^\circ 34' 37''.5, & B &= 206^\circ 43' 34''.4, & C &= 0, \\ \log \cos a &= 8.557354_n, & \log \cos b &= 8.856746, & \log \cos c &= \log \cos i_0 = 9.998590, \\ \log x_0 &= 0.399807_n, & \log y_0 &= 9.838709, & \log z_0 &= 9.148170_n, \\ w &= w_0 + \delta w = 317^\circ 53' 20''.2,\end{aligned}$$

and hence, by means of (86), we derive

$$\delta x_{,,} = + 36''.559, \quad \delta y_{,,} = + 41''.083, \quad \delta z_{,,} = - 0''.588.$$

If we express these in parts of the unit of space, and in units of the seventh decimal place, we obtain

$$\delta x_{,,} = + 1772.4, \quad \delta y_{,,} = + 1991.8, \quad \delta z_{,,} = - 28.5,$$

agreeing with the results already obtained by the method of the variation of rectangular co-ordinates, namely,

$$\delta x_{,,} = + 1772.6, \quad \delta y_{,,} = + 1992.3, \quad \delta z_{,,} = - 28.2.$$

176. By using the complete formulæ, the perturbations of r , w , and z may be computed with respect to all powers of the disturbing force, and for a long series of years, using constantly the same fundamental osculating elements. But even when these elements are so accurate as not to require correction, on account of the effect of the large perturbations of long period upon the values of δw and δr , the numerical values of the perturbations will at length be such that a change of the osculating elements becomes desirable, so that the integration may again commence with the value zero for the variation of each of the co-ordinates. This change from one system of elements to another system may be readily effected when the values of the perturbations are known. Thus, having found the disturbed values of r , w , and z , we have

$$\frac{dv^2}{dt^2} = \cos^2 \beta \frac{dw^2}{dt^2} + \frac{d\beta^2}{dt^2} = \frac{k^2 p (1+m)}{r^4},$$

p being the semi-parameter of the instantaneous orbit of the disturbed body. In the undisturbed orbit we have

$$g_0 = \frac{dv_0}{dt} = \frac{k\sqrt{p_0(1+m)}}{r_0^2},$$

and hence we derive

$$p = \frac{p_0 r_0^4}{g_0^2 r_0^4} \cdot \frac{dv^2}{dt^2}.$$

Substituting for $\frac{dv}{dt}$ the value above given, there results

$$p = p_0 \frac{r^4}{r_0^4} \left(\cos^2 \beta \left(1 + \frac{1}{g_0} \cdot \frac{d\delta w}{dt} \right)^2 + \frac{1}{g_0^2} \cdot \frac{d\beta^2}{dt^2} \right), \quad (87)$$

by means of which p may be determined. To find $\frac{d\beta}{dt}$, we have

$$\frac{d\beta}{dt} = \frac{1}{r \cos \beta} \cdot \frac{dz}{dt} - \frac{\tan \beta}{r} \cdot \frac{dr}{dt}. \quad (88)$$

We have, also,

$$\frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v = \frac{k\sqrt{1+m}}{\sqrt{p_0}} e_0 \sin v_0 + \frac{d\delta r}{dt},$$

and if we put

$$\sqrt{\frac{p}{p_0}} = 1 + \alpha, \quad r = \frac{\sqrt{p}}{k\sqrt{1+m}} \cdot \frac{d\delta r}{dt}, \quad (89)$$

this equation becomes

$$e \sin v = e_0 \sin v_0 + \alpha e_0 \sin v_0 + \gamma. \quad (90)$$

We have, further,

$$e \cos v = \frac{p}{r} - 1,$$

and, putting

$$\frac{p}{p_0} \cdot \frac{r_0}{r} = 1 + \beta, \quad (91)$$

we obtain

$$e \cos v = e_0 \cos v_0 + \beta \frac{p_0}{r_0}.$$

This equation, combined with (90), gives

$$\begin{aligned} e \sin (v - v_0) &= \alpha e_0 \sin v_0 \cos v_0 + \gamma \cos v_0 - \frac{p_0}{r_0} \beta \sin v_0, \\ e \cos (v - v_0) &= e_0 + \alpha e_0 \sin^2 v_0 + \gamma \sin v_0 + \frac{p_0}{r_0} \beta \cos v_0, \end{aligned} \quad (92)$$

by means of which the values of e and v may be found, those of the auxiliaries α , β , γ , being found from (89) and (91). Then we have

$$\begin{aligned} e &= \sin \varphi, & \alpha &= p \sec^2 \varphi, \\ \mu &= \frac{kV\sqrt{1+m}}{\alpha^{\frac{3}{2}}}, & \tan \frac{1}{2}E &= \tan (45^\circ - \frac{1}{2}\varphi) \tan \frac{1}{2}v, \\ M &= E - e \sin E, \end{aligned}$$

by means of which φ , α , μ , and M may be determined. In the case of orbits of great eccentricity, we find the perihelion distance from

$$q = \frac{p}{1+e},$$

and the time of perihelion passage will be derived from e and v by means of Table IX. or Table X.

It remains yet to determine the values of Ω , i , and ω or π . Let θ_c denote the longitude of the ascending node of the instantaneous orbit on the plane of the osculating orbit, defined by Ω_0 and i_0 , measured from the origin of w , and let γ_0 denote its inclination to this plane. Then we have

$$\begin{aligned} \tan \gamma_0 \sin (w - \theta_0) &= \tan \beta, \\ \tan \gamma_0 \cos (w - \theta_0) \frac{dw}{dt} &= \sec^2 \beta \frac{d\beta}{dt}, \end{aligned} \quad (93)$$

and hence

$$\tan(w - \theta_0) = \frac{1}{2} \sin 2\beta \frac{g_0 + \frac{d\delta w}{dt}}{\frac{d\beta}{dt}}, \quad (94)$$

by means of which θ_0 may be found. The quadrant in which θ_0 is situated is determined by the condition that $\sin(w - \theta_0)$ and $\tan \beta$ must have the same sign. The value of η_0 will be found from the first or the second of equations (93).

If we denote by ζ the argument of the latitude of the disturbed body with respect to the adopted fundamental plane, we have

$$\tan \zeta = \frac{\tan(w - \theta_0)}{\cos \eta_0}, \quad (95)$$

and the angle ζ must be taken in the same quadrant as $w - \theta_0$. Then, from the spherical triangle formed by the intersection of the planes of the ecliptic and instantaneous orbit of the disturbed body, and the fundamental plane, with the celestial vault, we derive

$$\begin{aligned} \cos \frac{1}{2} i \sin \left(\frac{1}{2} (u - \zeta) + \frac{1}{2} (\Omega - \Omega_0) \right) &= \sin \frac{1}{2} \theta_0 \cos \frac{1}{2} (i_0 - \eta_0), \\ \cos \frac{1}{2} i \cos \left(\frac{1}{2} (u - \zeta) + \frac{1}{2} (\Omega - \Omega_0) \right) &= \cos \frac{1}{2} \theta_0 \cos \frac{1}{2} (i_0 + \eta_0), \\ \sin \frac{1}{2} i \sin \left(\frac{1}{2} (u - \zeta) - \frac{1}{2} (\Omega - \Omega_0) \right) &= \sin \frac{1}{2} \theta_0 \sin \frac{1}{2} (i_0 - \eta_0), \\ \sin \frac{1}{2} i \cos \left(\frac{1}{2} (u - \zeta) - \frac{1}{2} (\Omega - \Omega_0) \right) &= \cos \frac{1}{2} \theta_0 \sin \frac{1}{2} (i_0 + \eta_0). \end{aligned} \quad (96)$$

These equations will furnish the values of i , $u - \zeta$, and $\Omega - \Omega_0$, and hence, since ζ and Ω_0 are given, those of Ω and u . The value of v having been already found, we have, finally,

$$\begin{aligned} \omega &= u - v, \\ \pi &= u - v + \Omega, \end{aligned}$$

and the elements are completely determined. These elements will be referred to the ecliptic and mean equinox to which Ω_0 and i_0 are referred, and they may be reduced to the equinox and ecliptic of any other date by means of the formulæ which have already been given.

The elements of the instantaneous orbit of the disturbed body may also be determined by first computing the values of $x_{\text{''}}$, $y_{\text{''}}$, $z_{\text{''}}$, in reference to the fundamental plane to which Ω and i are to be referred, by means of the equations (83), and also those of $\frac{dx_{\text{''}}}{dt}$, $\frac{dy_{\text{''}}}{dt}$, $\frac{dz_{\text{''}}}{dt}$ by means of (85) and (84), and then determining the elements from the co-ordinates and velocities, as already explained.

It should be observed that when the factor ω^2 , or the square of the

adopted interval, is introduced into the expressions for the forces and differential coefficients, the first integrals will be

$$\omega \frac{d\delta r}{dt}, \quad \omega \frac{d\delta w}{dt}, \quad \omega \frac{d\delta z}{dt},$$

and that when these quantities are expressed in seconds of arc, they must be converted into their values in parts of the unit of space whenever they are to be combined with quantities which are not expressed in seconds. In other words, the homogeneity of the several terms must be carefully attended to in the actual application of the formulæ.

When the elements which correspond to given values of the perturbations have been determined, if we compute the heliocentric longitude and latitude of the body for the instant to which the elements belong, the results should agree with those obtained by computing the heliocentric place from the fundamental osculating elements and adding the perturbations.

177. The computation of the indirect terms when the perturbations of the co-ordinates r , w , and z are determined, is effected with greater facility than in the case of the rectangular co-ordinates, although the final results are not so convenient for the calculation of an ephemeris for the comparison of observations. This indirect calculation, which, when the perturbations of any system of three co-ordinates are to be computed, cannot in any case be avoided without impairing the accuracy of the results, may be further simplified by determining, in a peculiar form, the perturbations of the mean anomaly, the radius-vector, and the co-ordinate z perpendicular to the fundamental plane adopted.

Let the motion of the disturbed body be, at each instant, referred to the plane of its instantaneous orbit; then we shall have $\beta = 0$, and the equations (51) and (54) become

$$\begin{aligned} r^2 \frac{dw}{dt} &= \int S r dt + k \sqrt{p_0(1+m)}, \\ \frac{d^2 r}{dt^2} - r \frac{dw^2}{dt^2} + \frac{k^2(1+m)}{r^2} &= R, \end{aligned} \tag{97}$$

in which R denotes the component of the disturbing force in the direction of the disturbed radius-vector, and S the component in the plane of the disturbed orbit and perpendicular to the disturbed radius-vector, being positive in the direction of the motion. The effect of

the components R and S is to vary the form of the orbit and the angular distance of the perihelion from the node. If we denote by Z the component of the disturbing force perpendicular to the plane of the instantaneous orbit, the effect of this will be to change the position of the plane of the orbit, and hence to vary the elements which depend on the position of this plane.

Let us take a fixed line in the plane of the instantaneous orbit, and suppose it to be directed from the centre of the sun to a point whose angular distance back from the place of the ascending node is σ , and let the value of σ be so taken that, so long as the position of the plane of the orbit is unchanged, we shall have

$$\sigma = \Omega.$$

The line thus taken in the plane of the orbit may be regarded as fixed during all changes in the position of this plane. Let χ denote the angle between this fixed line and the semi-transverse axis; then will

$$\chi = \omega + \sigma, \quad (98)$$

and when the position of the plane of the orbit is unchanged, we have

$$\chi = \pi.$$

But if, on account of the action of the component Z , the position of the plane of the orbit is changed, we have, according to the equations (72)₂, the relations

$$\begin{aligned} d\sigma &= \cos i \, d\Omega, \\ d\omega &= d\chi - \cos i \, d\Omega, \\ d\pi &= d\chi + (1 - \cos i) \, d\Omega = d\chi + 2 \sin^2 \frac{1}{2} i \, d\Omega. \end{aligned} \quad (99)$$

We have, further,

$$u = v + \chi - \sigma, \quad (100)$$

v being the true anomaly in the instantaneous orbit.

The two components of the disturbing force which act in the plane of the disturbed orbit will only vary χ and the elements which determine the dimensions of the conic section. We have, therefore, in the case of the osculating elements, for the instant t_0 ,

$$\chi_0 = \omega_0 + \Omega_0 = \pi_0.$$

Let us now suppose λ to denote the true longitude in the orbit, so that we have

$$\lambda = v + \pi = v + \omega + \Omega,$$

or

$$\lambda = v + \chi - (\sigma - \Omega); \quad (101)$$

then, since χ is equal to π when the position of the plane of the orbit is unchanged, it follows that $\sigma - \Omega$ represents the variation of the true longitude in the orbit arising from the action of the component Z of the disturbing force. The elements may refer to the ecliptic or the equator, or to any other fundamental plane which may be adopted.

178. For the instant t we have, in the case of the disturbed motion, the following relations:—

$$\begin{aligned} E - e \sin E &= M + \mu(t - t_0), \\ r \cos v &= a \cos E - ae, \\ r \sin v &= a\sqrt{1 - e^2} \sin E, \\ \lambda &= v + \chi - (\sigma - \Omega). \end{aligned} \quad (102)$$

Let us first consider only the perturbations arising from the action of the two components of the disturbing force in the plane of the disturbed orbit, and let us put

$$\lambda_1 = v + \chi. \quad (103)$$

Further, let $M_0 + \mu_0(t - t_0) + \delta M$ be the mean anomaly which, by means of a system of equations identical in form with the preceding, but in which the values of a_0 , e_0 , χ_0 are used instead of the instantaneous values a , e , and χ , gives the same longitude λ_1 , so that we have

$$\begin{aligned} E_1 - e_0 \sin E_1 &= M_0 + \mu_0(t - t_0) + \delta M, \\ r_1 \cos v_1 &= a_0 \cos E_1 - a_0 e_0, \\ r_1 \sin v_1 &= a_0 \sqrt{1 - e_0^2} \sin E_1, \\ \lambda_1 &= v_1 + \chi_0 = v_1 + \pi_0. \end{aligned} \quad (104)$$

If, therefore, we determine the value of δM so as to satisfy the condition that $\lambda_1 = v + \chi$, the disturbed value of the true longitude in the orbit, neglecting the effect of the component Z of the disturbing force, will be known. The value of r_1 will generally differ from that of the disturbed radius-vector r , and hence it becomes necessary to introduce another variable in order to consider completely the effect of the components R and S . Thus, we may put

$$r = r_1(1 + \nu), \quad (105)$$

and ν will always be a very small quantity. When δM and ν have been found, the effect of the disturbing force perpendicular to the plane of the instantaneous orbit may be considered, and thus the complete perturbations will be obtained.

In the equations (97), $\frac{1}{2}r^2 \frac{dw}{dt}$ expresses the areal velocity in the instantaneous orbit, and it is evident that, since the true anomaly is not affected by the force Z perpendicular to the plane of the actual orbit, $\frac{1}{2}r^2 \frac{dv}{dt}$ must also represent this areal velocity, and hence the equations become

$$\begin{aligned} r^2 \frac{dv}{dt} &= \int S r dt + k\sqrt{p_0(1+m)}, \\ \frac{d^2r}{dt^2} - r \left(\frac{dv}{dt} \right)^2 + \frac{k^2(1+m)}{r^2} &= R. \end{aligned} \quad (106)$$

179. If we differentiate each of the equations (104), we get

$$\begin{aligned} (1 - e_0 \cos E_0) \frac{dE_0}{dt} &= \mu_0 + \frac{d\delta M}{dt}, \\ \cos v_0 \frac{dr_0}{dt} - r_0 \sin v_0 \frac{dv_0}{dt} &= -a_0 \sin E_0 \frac{dE_0}{dt}, \\ \sin v_0 \frac{dr_0}{dt} + r_0 \cos v_0 \frac{dv_0}{dt} &= a_0 \sqrt{1 - e_0^2} \cos E_0 \frac{dE_0}{dt}, \\ \frac{d\lambda_0}{dt} &= \frac{dv_0}{dt}. \end{aligned} \quad (107)$$

From the second and the third of these equations we easily derive

$$r_0 \frac{dr_0}{dt} = (a_0 \sqrt{1 - e_0^2} r_0 \sin v_0 \cos E_0 - a_0 r_0 \cos v_0 \sin E_0) \frac{dE_0}{dt}.$$

Substituting in this the values of $r_0 \sin v_0$, $r_0 \cos v_0$, and $\frac{dE_0}{dt}$, and reducing, we get

$$\begin{aligned} r_0 \frac{dr_0}{dt} &= a_0^2 e_0 \sin E_0 \left(\mu_0 + \frac{d\delta M}{dt} \right), \\ \text{or} \quad \frac{dr_0}{dt} &= \frac{k\sqrt{1+m}}{\sqrt{p_0}} e_0 \sin v_0 \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right). \end{aligned} \quad (108)$$

From the same equations, eliminating $\frac{dr_0}{dt}$, we get

$$r_0^2 \frac{dv_0}{dt} = (a_0 \sqrt{1 - e_0^2} r_0 \cos v_0 \cos E_0 + a_0 r_0 \sin v_0 \sin E_0) \frac{dE_0}{dt},$$

which reduces to

$$r_0^2 \frac{dv_0}{dt} = k\sqrt{p_0(1+m)} \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right), \quad (109)$$

or

$$r^2 \frac{dv}{dt} = k\sqrt{p_0(1+m)} (1+\nu)^2 \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt}\right).$$

Combining this with the first of equations (106), we get

$$\frac{d\delta M}{dt} = \mu_0 \left(\frac{1}{(1+\nu)^2} - 1 \right) + \frac{\mu_0}{(1+\nu)^2} \cdot \frac{1}{k\sqrt{p_0(1+m)}} \int Sr dt, \quad (110)$$

from which δM may be found as soon as ν is known.

The equation (105) gives

$$\begin{aligned} \frac{dr}{dt} &= (1+\nu) \frac{dr}{dt} + r, \frac{d\nu}{dt}, \\ \frac{d^2 r}{dt^2} &= (1+\nu) \frac{d^2 r}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\nu}{dt} + r, \frac{d^2 \nu}{dt^2}. \end{aligned} \quad (111)$$

Differentiating equation (108) and substituting for $\frac{dv}{dt}$, its value already found, we obtain

$$\frac{d^2 r}{dt^2} = \frac{k^2(1+m)e_0 \cos v}{r^2} \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt}\right)^2 + \frac{k\sqrt{1+m}e_0 \sin v}{\mu_0 \sqrt{p_0}} \cdot \frac{d^2 \delta M}{dt^2},$$

and the last of the preceding equations becomes

$$\begin{aligned} \frac{d^2 r}{dt^2} &= r, \frac{d^2 \nu}{dt^2} + \frac{k^2(1+m)e_0 \cos v}{r^2} (1+\nu) \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt}\right)^2 \\ &\quad + \frac{k\sqrt{1+m}e_0 \sin v}{\sqrt{p_0}} \left(\frac{1+\nu}{\mu_0} \cdot \frac{d^2 \delta M}{dt^2} + 2 \frac{d\nu}{dt} + \frac{2}{\mu_0} \cdot \frac{d\nu}{dt} \cdot \frac{d\delta M}{dt} \right). \end{aligned}$$

The equation (110) gives

$$\begin{aligned} \frac{1}{\mu_0} \cdot \frac{d^2 \delta M}{dt^2} + \frac{2}{(1+\nu)^3} \cdot \frac{d\nu}{dt} + \frac{2}{(1+\nu)^3} \cdot \frac{d\nu}{dt} \cdot \frac{1}{k\sqrt{p_0(1+m)}} \int Sr dt \\ = \frac{1}{(1+\nu)^2} \cdot \frac{Sr}{k\sqrt{p_0(1+m)}}, \end{aligned}$$

which is easily reduced to

$$\frac{1+\nu}{\mu_0} \cdot \frac{d^2 \delta M}{dt^2} + 2 \frac{d\nu}{dt} + \frac{2}{\mu_0} \cdot \frac{d\nu}{dt} \cdot \frac{d\delta M}{dt} = \frac{1}{1+\nu} \cdot \frac{Sr}{k\sqrt{p_0(1+m)}};$$

and hence we derive

$$\frac{d^2 r}{dt^2} = r, \frac{d^2 \nu}{dt^2} + \frac{k^2(1+m)e_0 \cos v}{r^2} (1+\nu) \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt}\right)^2 + \frac{e_0 \sin v}{p_0(1+\nu)} Sr. \quad (112)$$

The equation (109) gives

$$r, \left(\frac{dv_r}{dt} \right)^2 = \frac{k^2 p_0 (1+m)}{r^3} \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2,$$

and, since

$$r, = \frac{r}{1+\nu}, \quad \frac{p_0}{r,} = 1 + e_0 \cos v,,$$

this becomes

$$\begin{aligned} r, \left(\frac{dv_r}{dt} \right)^2 &= \frac{k^2 (1+m)}{r,^2} (1+\nu) \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 \\ &+ \frac{k^2 (1+m) e_0 \cos v,}{r,^2} (1+\nu) \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2. \end{aligned} \quad (113)$$

Combining equations (112) and (113) with the second of equations (106), we get

$$\begin{aligned} \frac{d^2 \nu}{dt^2} &= \frac{1+\nu}{r} R + \frac{k^2 (1+m) (1+\nu)^4}{r^3} \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 \\ &- \frac{e_0 \sin v,}{p_0} S - \frac{k^2 (1+m) (1+\nu)}{r^3}. \end{aligned} \quad (114)$$

From (110) we derive

$$\begin{aligned} (1+\nu)^4 \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 &= 1 + \frac{2}{k\sqrt{p_0(1+m)}} \int Sr dt \\ &+ \left(\frac{1}{k\sqrt{p_0(1+m)}} \int Sr dt \right)^2, \end{aligned}$$

and the preceding equation becomes

$$\begin{aligned} \frac{d^2 \nu}{dt^2} &= \frac{R}{r,} + 2 \frac{k^2 (1+m)}{r^3} \cdot \frac{1}{k\sqrt{p_0(1+m)}} \int Sr dt - \frac{e_0 \sin v,}{p_0} S \\ &- \frac{k^2 (1+m)}{r^3} \nu + \frac{k^2 (1+m)}{r^3} \left(\frac{1}{k\sqrt{p_0(1+m)}} \int Sr dt \right)^2, \end{aligned} \quad (115)$$

which is the complete expression for the determination of ν .

180. It remains now to consider the effect of the component of the disturbing force which is perpendicular to the plane of the disturbed orbit. Let x, y, z , denote the co-ordinates of the body referred to the fundamental plane to which the elements belong, and x, y the co-ordinates in the plane of the instantaneous orbit. Further, let α denote the cosine of the angle which the axis of x makes with that of $x,$, and β the cosine of the angle which the axis of y makes with that of $y,$, and we shall have

$$z, = \alpha x + \beta y. \quad (116)$$

If the position of the plane of the orbit remained unchanged, these

cosines α and β would be constant; but on account of the action of the force perpendicular to the plane of the orbit, these quantities are functions of the time. Now, the co-ordinate z , is subject to two distinct variations: if the elements remain constant, it varies with the time; and, in the case of the disturbed orbit, it is also subject to a variation arising from the change of the elements themselves. We shall, therefore, have

$$\frac{dz}{dt} = \left(\frac{dz_1}{dt} \right) + \left[\frac{dz_1}{dt} \right],$$

in which $\left(\frac{dz_1}{dt} \right)$ expresses the velocity resulting from the constant elements, and $\left[\frac{dz_1}{dt} \right]$ that part of the actual velocity which is due to the change of the elements by the action of the disturbing force. But during the element of time dt the elements may be regarded as constant, and hence the velocity $\frac{dz_1}{dt}$ in a direction parallel to the axis of z , may be regarded as constant during the same time, and as receiving an increment only at the end of this instant. Hence we shall have

$$\frac{dz_1}{dt} = \left(\frac{dz_1}{dt} \right) \quad \left[\frac{dz_1}{dt} \right] = 0.$$

Differentiating equation (116), regarding α and β as constant, we get

$$\left(\frac{dz_1}{dt} \right) = \frac{dz_1}{dt} = \alpha \frac{dx}{dt} + \beta \frac{dy}{dt}, \quad (117)$$

and differentiating the same equation, regarding x and y as constant, we get

$$\left[\frac{dz_1}{dt} \right] = x \frac{d\alpha}{dt} + y \frac{d\beta}{dt} = 0. \quad (118)$$

Differentiating equation (117), regarding all the quantities involved as variable, the result is

$$\frac{d^2 z_1}{dt^2} = \frac{d\alpha}{dt} \cdot \frac{dx}{dt} + \frac{d\beta}{dt} \cdot \frac{dy}{dt} + \alpha \frac{d^2 x}{dt^2} + \beta \frac{d^2 y}{dt^2}. \quad (119)$$

Now, we have

$$Z_1 = \alpha X + \beta Y + Z \cos i, \quad (120)$$

in which Z_1 denotes the component of the disturbing force parallel to the axis of z , and i the inclination of the instantaneous orbit to

the fundamental plane. Substituting for X and Y their values given by the equations (1), and reducing by means of (116), we obtain

$$Z = \alpha \frac{d^2x}{dt^2} + \beta \frac{d^2y}{dt^2} + k^2(1+m) \frac{z}{r^3} + Z \cos i,$$

or

$$\frac{d^2z}{dt^2} = \alpha \frac{d^2x}{dt^2} + \beta \frac{d^2y}{dt^2} + Z \cos i.$$

Comparing this with (119), there results

$$\frac{d\alpha}{dt} \cdot \frac{dx}{dt} + \frac{d\beta}{dt} \cdot \frac{dy}{dt} = Z \cos i. \quad (121)$$

181. The equation (120) gives

$$\frac{d^2z}{dt^2} = -\frac{k^2(1+m)}{r^3} z + Z \cos i + \alpha X + \beta Y. \quad (122)$$

The component of the disturbing force perpendicular to the plane of the disturbed orbit does not affect the radius-vector r ; and hence, when we neglect the effect of this component, and consider only the components R and S which act in the plane of the orbit, we have

$$\frac{d^2z_0}{dt^2} = -\frac{k^2(1+m)}{r^3} z_0 + \alpha_0 X + \beta_0 Y, \quad (123)$$

in which z_0 denotes the value of z , obtained when we put $Z=0$. Let us now denote by δz , that part of the change in the value of z , which arises from the action of the force perpendicular to the plane of the disturbed orbit, so that we shall have

$$z = z_0 + \delta z, \quad \alpha = \alpha_0 + \delta \alpha, \quad \beta = \beta_0 + \delta \beta.$$

Substituting these in equation (122) and then subtracting equation (123) from the result, we get

$$\frac{d^2\delta z}{dt^2} = -\frac{k^2(1+m)}{r^3} \delta z + Z \cos i + X\delta \alpha + Y\delta \beta. \quad (124)$$

The equations (116) and (117) give

$$\delta z = x\delta \alpha + y\delta \beta, \quad \frac{d\delta z}{dt} = \frac{dx}{dt} \delta \alpha + \frac{dy}{dt} \delta \beta.$$

If we eliminate $\delta \beta$ between these equations, there results

$$\delta \alpha \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \frac{dy}{dt} \delta z - y \frac{d\delta z}{dt},$$

and since the factor of $\delta\alpha$ in this equation is double the areal velocity in the disturbed orbit, we have

$$\delta\alpha = \frac{1}{k\sqrt{p(1+m)}} \left(\frac{dy}{dt} \delta z, - y \frac{d\delta z,}{dt} \right). \quad (125)$$

Eliminating $\delta\alpha$ from the same equations, we obtain, in a similar manner,

$$\delta\beta = \frac{1}{k\sqrt{p(1+m)}} \left(x \frac{d\delta z,}{dt} - \frac{dx}{dt} \delta z, \right). \quad (126)$$

Substituting these values in equation (124), it becomes

$$\begin{aligned} \frac{d^2\delta z,}{dt^2} = & -\frac{k^2(1+m)}{r^3} \delta z, + Z \cos i \\ & + \frac{1}{k\sqrt{p(1+m)}} \left(\left(X \frac{dy}{dt} - Y \frac{dx}{dt} \right) dz, + (Yx - Xy) \frac{d\delta z,}{dt} \right). \end{aligned} \quad (127)$$

If we introduce the components R and S of the disturbing force, we have

$$X = R \frac{x}{r} - S \frac{y}{r}, \quad Y = R \frac{y}{r} + S \frac{x}{r},$$

and hence

$$\begin{aligned} X \frac{dy}{dt} - Y \frac{dx}{dt} &= \frac{R}{r} k\sqrt{p(1+m)} - S \frac{dr}{dt}, \\ Yx - Xy &= Sr. \end{aligned}$$

Therefore the equation (127) becomes

$$\begin{aligned} \frac{d^2\delta z,}{dt^2} = & -\frac{k^2(1+m)}{r^3} \delta z, + Z \cos i \\ & + \left(\frac{R}{r} - \frac{S}{k\sqrt{p(1+m)}} \cdot \frac{dr}{dt} \right) \delta z, + \frac{Sr}{k\sqrt{p(1+m)}} \cdot \frac{d\delta z,}{dt}. \end{aligned} \quad (128)$$

We have, further,

$$\frac{dr}{dt} = (1 + \nu) \frac{dr,}{dt} + r, \frac{d\nu}{dt},$$

which, by means of the equations (108) and (109), gives

$$\frac{dr}{dt} = \frac{e_0 \sin \nu,}{p_0(1+\nu)} r^2 \frac{d\nu,}{dt} + r, \frac{d\nu}{dt} = \frac{k\sqrt{p(1+m)}}{p_0(1+\nu)} e_0 \sin \nu, + r, \frac{d\nu}{dt}.$$

Substituting this value in the equation (128), we obtain

$$\frac{d^2 \delta z}{dt^2} = -\frac{k^2(1+m)}{r^3} \delta z + Z \cos i + \left(\frac{R}{r} - \frac{e_0 \sin v}{p_0} S \right) \frac{\delta z}{1+\nu} + \frac{Sr}{kVp(1+m)} \left(\frac{d\delta z}{dt} - \frac{\delta z}{1+\nu} \cdot \frac{d\nu}{dt} \right), \quad (129)$$

which is the complete expression for the determination of δz .

182. The equations (110), (115), and (129) determine the complete perturbations of the disturbed body. The value of ν must first be obtained by an indirect process from the equation (115), and then δM is given directly by means of (110). The value of δz will also be determined by an indirect process by means of (129).

In order to obtain the expressions for the forces R , S , and Z , let w denote the longitude of the disturbed body measured in the plane of the instantaneous orbit from its ascending node on the fundamental plane to which Ω and i are referred, it being the argument of the latitude in the case of the disturbed motion. Let w' denote the longitude of the disturbing body measured from the same origin and in the plane of the orbit of the disturbed body, and let β' denote its latitude in reference to this plane. Finally, let N , N' , I , and u_0' have the same signification in reference to the plane of the instantaneous orbit that they have in reference to the plane of the undisturbed orbit in the case of the equations (66). Then we shall have

$$\begin{aligned} \sin \frac{1}{2} I \sin \frac{1}{2} (N + N') &= \sin \frac{1}{2} (\Omega' - \Omega) \sin \frac{1}{2} (i' + i), \\ \sin \frac{1}{2} I \cos \frac{1}{2} (N + N') &= \cos \frac{1}{2} (\Omega' - \Omega) \sin \frac{1}{2} (i' - i), \\ \cos \frac{1}{2} I \sin \frac{1}{2} (N - N') &= \sin \frac{1}{2} (\Omega' - \Omega) \cos \frac{1}{2} (i' + i), \\ \cos \frac{1}{2} I \cos \frac{1}{2} (N - N') &= \cos \frac{1}{2} (\Omega' - \Omega) \cos \frac{1}{2} (i' - i), \end{aligned} \quad (130)$$

from which to determine N , N' , and I . We have, also,

$$\begin{aligned} u_0' &= u' - N', \\ \tan (w' - N) &= \tan u_0' \cos I, \\ \tan \beta' &= \tan I \sin (w' - N), \end{aligned} \quad (131)$$

from which to find w' and β' , u' being the argument of the latitude of the disturbing body in reference to the plane to which Ω and i are referred.

Since, when the motion of the disturbed body is referred to the plane of its instantaneous orbit, $\beta = 0$, the equations (71), (72), and (73) become

$$\begin{aligned} R &= m' k^2 \left(h r' \cos \beta' \cos (w' - w) - \frac{r}{\rho^3} \right), \\ S &= m' k^2 h r' \cos \beta' \sin (w' - w), \\ Z &= m' k^2 h r' \sin \beta', \end{aligned} \quad (132)$$

by means of which the required components of the disturbing force may be found, the value of h being given by

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3}.$$

To find ρ , we have

$$\rho^2 = r'^2 + r^2 - 2rr' \cos \beta' \cos (w' - w), \quad (133)$$

or, putting

$$\cos \gamma = \cos \beta' \cos (w' - w),$$

the equations

$$\begin{aligned} \rho \sin n &= r' \sin \gamma, \\ \rho \cos n &= r - r' \cos \gamma. \end{aligned} \quad (134)$$

The values of r' and u' for the actual places of the disturbing body will be given by the tables of its motion, and the actual values of Ω' and i' will also be obtained by means of the tables. The determination of the actual values of r and w requires that the perturbations shall be known. Thus, when δM and ν have been found, we compute, by means of the mean anomaly $M_0 + \mu_0(t - t_0) + \delta M$ and the elements a_0, e_0 , the values of v , and r . Then, since $v + \chi = v + \pi_0$, we have, according to (100),

$$w = v + \pi_0 - \sigma. \quad (135)$$

We have, also,

$$r = (1 + \nu) r_0.$$

In the case of the fundamental osculating elements, we have

$$\sigma_0 = \Omega_0,$$

which may be used as an approximate value of σ ; but the complete determination of w requires that $\sigma = \Omega_0 + \delta\sigma$ shall also be determined. The exact determination of the forces also requires that the actual values of Ω and i as well as those of Ω' and i' , shall be used in the determination of N, N' , and I for each instant. When these have been found, it will be sufficient to compute the actual values of N, N' , and I at intervals during the entire period for which the perturbations are required, and to interpolate their values for the intermediate dates. The variations of these quantities arising from the variations of Ω, i, Ω' , and i' may also be determined by means of differential formulæ. Thus, from the differential relations of the parts of the spherical triangle from which the equations (130) are derived, we easily find

$$\begin{aligned}
 dN' &= \frac{\sin i}{\sin I} \cos N \, d(\Omega' - \Omega) - \frac{\sin N'}{\sin I} \cos I \, di' + \frac{\sin N}{\sin I} \, di, \\
 dN &= \frac{\sin i'}{\sin I} \cos N' \, d(\Omega' - \Omega) - \frac{\sin N'}{\sin I} \, di' + \frac{\sin N}{\sin I} \cos I \, di, \\
 dI &= \cos N' \, di' - \cos N \, di + \sin i \sin N \, d(\Omega' - \Omega).
 \end{aligned} \tag{136}$$

When i and I are very small, it will be better to use

$$\frac{\sin i}{\sin I} = \frac{\sin N'}{\sin(\Omega' - \Omega)}, \quad \frac{\sin i'}{\sin I} = \frac{\sin N}{\sin(\Omega' - \Omega)}, \tag{137}$$

in finding the numerical values of these coefficients. By means of these formulæ we may derive the values of δN , $\delta N'$, and δI corresponding to given values of $\delta \Omega$, δi , $\delta \Omega'$, and $\delta i'$. The formulæ by means of which $\delta \sigma$, $\delta \Omega$, and δi may be obtained directly, will be presently considered.

The results for δN , $\delta N'$, and δI being applied to the quantities to which they belong, we may compute the actual values of w' and β' . The value of r will be found from the given value of ν , and that of w will be given by means of equation (135). Then, by means of the formulæ (132), the forces R , S , and Z will be obtained. The perturbations will first be computed in reference only to terms depending on the first power of the disturbing force, and, whenever it becomes necessary to consider the terms of the second order, the results already obtained will enable us to estimate the values of the perturbations for two or more intervals in advance with sufficient accuracy for the determination of the three required components of the disturbing force; and when there are two or more disturbing bodies to be considered, the forces for each of these may be computed at once, and the values of each component for the several disturbing bodies may be united into a single sum, thus using ΣR , ΣS , and ΣZ in place of R , S , and Z respectively. The approximate values of the perturbations will also facilitate the indirect calculation in the determination of the complete values of the required differential coefficients.

183. When only the perturbations due to the first power of the disturbing force are required, the osculating elements Ω_0 and i_0 will be used in finding N , N' , and I , and r_0 , w_0 will be used instead of r and w in the calculation of the values of R , S , and Z . The equations for the determination of the perturbations δM , ν , and δz , neglecting terms of the second order, are, according to the equations (110), (115), and (129), the following:—

$$\begin{aligned}
\frac{d\delta M}{dt} &= \mu_0 \frac{1}{k\sqrt{p_0(1+m)}} \int S r_0 dt - 2\mu_0\nu, \\
\frac{d^2\nu}{dt^2} &= \frac{R}{r_0} + \frac{2k^2(1+m)}{r_0^3} \cdot \frac{1}{k\sqrt{p_0(1+m)}} \int S r_0 dt - \frac{e_0 \sin v_0}{p_0} S - \frac{k^2(1+m)}{r_0^3} \nu, \\
\frac{d^2\delta z_r}{dt^2} &= Z \cos i_0 - \frac{k^2(1+m)}{r_0^3} \delta z_r.
\end{aligned} \tag{138}$$

The value of ν is first found by integration from the results given by the second of these equations, and then δM is found from the first equation. Finally, δz_r is found by means of the last equation. The integrals are in each case equal to zero for the dates to which the fundamental osculating elements belong, and the process of integration is analogous, in all respects, to that already illustrated in the case of the variation of the rectangular co-ordinates. It will be observed, however, that the expression for $\frac{d^2\nu}{dt^2}$ involves only one indirect term, the coefficient of which is small, and the same is true in the case of $\frac{d^2\delta z_r}{dt^2}$, while $\frac{d\delta M}{dt^2}$ is given directly. When the perturbations have been found for a few dates, the values for the following date can be estimated so closely that a repetition of the calculation will rarely or never be required; and the actual value of r may be used instead of the approximate value r_0 in these expressions for the differential coefficients. Neglecting terms of the second order, we have

$$\log r = \log r_0 + \lambda_0 \nu,$$

wherein λ_0 denotes the modulus of the system of logarithms. We may also use v , instead of v_0 ; but in this case, since r , and v , depend on δM , only the quantities required for two or three places may be computed in advance of the integration.

A comparison of the equations (138) with the complete equations (110), (115), and (129) shows that, if the values of β' and w' are known to a sufficient degree of approximation, we may, with very little additional labor, consider the terms depending on the squares and higher powers of the masses. It will, however, appear from what follows, that when we consider the perturbations due to the higher powers of the disturbing forces, the consideration of the effect of the variation of z_r in the determination of the heliocentric place of the disturbed body, becomes much more difficult than when the terms of the second order are neglected; and hence it will be found advisable to determine new osculating elements whenever the consideration of these terms becomes troublesome.

The results may be conveniently expressed in seconds of arc, and afterwards ν and δz , may be converted into their values expressed in units of the seventh decimal place, or, giving proper attention to the homogeneity of the several terms of the equations, in the numerical operations, δM may be expressed in seconds of arc, while ν and δz , are obtained directly in units of the seventh decimal place. It will be advisable, also, to introduce the interval ω into the formulæ in such a manner that this quantity may be omitted in the case of the formulæ of integration.

184. In the case of orbits of great eccentricity, the mean anomaly and the mean daily motion cannot be conveniently used in the numerical application of the formulæ. Instead of these we must employ the time of perihelion passage and the elements q and e . Thus, let T_0 be the time of perihelion passage for the osculating elements for the date t_0 , and let $T_0 + \delta T$ be the time of perihelion passage to be used in the formulæ in the place of T_0 and in connection with the elements q_0 and e_0 in the determination of the values of r , and v , so that we have

$$v + \chi = v_0 + \pi_0.$$

In the case of parabolic motion we have, neglecting the mass of the disturbed body,

$$\frac{k(t - (T_0 + \delta T))}{\sqrt{2} q_0^{\frac{3}{2}}} = \tan \frac{1}{2} v_0 + \frac{1}{3} \tan^3 \frac{1}{2} v_0, \quad (139)$$

the solution of which to find v , is effected by means of Table VI. as already explained. To find r , we have

$$r_0 = q_0 \sec^2 \frac{1}{2} v_0.$$

For the other cases in which the elements M_0 and μ_0 cannot be employed, the solution must be effected by means of Table IX. or Table X. Thus, when Table IX. is used, we compute M from

$$M = (t - (T_0 + \delta T)) \frac{C_0}{q_0^{\frac{3}{2}}} \sqrt{\frac{1 + e_0}{2}},$$

wherein $\log C_0 = 9.9601277$, and with this as the argument we derive from Table VI. the corresponding value of V . Then, having found

$i = \frac{1 - e_0}{1 + e_0}$, by means of Table IX. we derive the coefficients required in the equation

$$v_0 = V + A(100i) + B(100i)^2 + C(100i)^3, \quad (140)$$

from which v , will be determined. Finally, r , will be found from

$$r, = \frac{q_0(1 + e_0)}{1 + e_0 \cos v,} \quad (141)$$

When Table X. is used, we proceed as explained in Art. 41, using the elements $T = T_0 + \delta T$, q_0 , and e_0 , and thus we obtain the required values of v , and r .

It is evident, therefore, that, for the determination of the perturbations, only the formula for finding the value of δM requires modification in the case of orbits of great eccentricity, and this modification is easily effected. The expression

$$M_0 + \mu_0(t - t_0) + \delta M = M,$$

gives

$$\mu_0(t_0 - T_0) + \mu_0(t - t_0) + \delta M = \mu_0(t - (T_0 + \delta T)),$$

or, simply,

$$\delta M = -\mu_0 \delta T,$$

and the equation (110) becomes

$$\frac{d\delta T}{dt} = 1 - \frac{1}{(1 + \nu)^2} - \frac{1}{(1 + \nu)^2} \cdot \frac{1}{kV/p_0(1+m)} \int Sr dt, \quad (142)$$

by means of which the value δT required in the solution of the equations for r , and v , may be found.

If we denote by t , the time for which the true anomaly and the radius-vector computed by means of the fundamental osculating elements have the values which have been designated by v , and r , respectively, we have

$$\delta M = \mu_0(t, -t), \quad 1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} = \frac{dt}{dt},$$

and the equation (110) becomes

$$\frac{dt}{dt} = \frac{1}{(1 + \nu)^2} + \frac{1}{(1 + \nu)^2} \cdot \frac{1}{kV/p_0(1+m)} \int Sr dt, \quad (143)$$

or, putting $t, = t + \delta t$,

$$\frac{d\delta t}{dt} = \frac{1}{(1 + \nu)^2} - 1 + \frac{1}{(1 + \nu)^2} \cdot \frac{1}{kV/p_0(1+m)} \int Sr dt. \quad (144)$$

If we determine δt by means of this equation, the values of the radius-vector and true anomaly will be found for the time $t + \delta t$ instead of t , according to the methods for the different conic sections,

using the fundamental osculating elements. The results thus obtained are the required values of r , and v , respectively.

185. When the values of the perturbations ν , δz , and δM , δT , or δt have been determined, it remains to find the place of the disturbed body. The heliocentric longitude and latitude will be given by

$$\begin{aligned}\cos b \cos (l - \Omega) &= \cos (\lambda - \Omega), \\ \cos b \sin (l - \Omega) &= \sin (\lambda - \Omega) \cos i, \\ \sin b &= \sin (\lambda - \Omega) \sin i,\end{aligned}$$

or, since $\lambda = \lambda - \sigma + \Omega$,

$$\begin{aligned}\cos b \cos (l - \Omega) &= \cos (\lambda - \sigma), \\ \cos b \sin (l - \Omega) &= \sin (\lambda - \sigma) \cos i, \\ \sin b &= \sin (\lambda - \sigma) \sin i,\end{aligned}\tag{145}$$

in which $\lambda = v + \pi_0$. If we multiply the first of these equations by $\cos (\Omega - h)$, and the second by $-\sin (\Omega - h)$, in which h may have any value whatever, and add the results; then multiply the first by $(\sin \Omega - h)$, and the second by $\cos (\Omega - h)$, and add, we get

$$\begin{aligned}\cos b \cos (l - h) &= \cos (\lambda - \sigma) \cos (\Omega - h) - \sin (\lambda - \sigma) \sin (\Omega - h) \cos i, \\ \cos b \sin (l - h) &= \cos (\lambda - \sigma) \sin (\Omega - h) + \sin (\lambda - \sigma) \cos (\Omega - h) \cos i, \\ \sin b &= \sin (\lambda - \sigma) \sin i.\end{aligned}$$

But, since $\lambda - \sigma = (\lambda - \Omega_0) - (\sigma - \Omega_0)$, these equations may be written

$$\begin{aligned}\cos b \cos (l - h) &= \cos (\lambda - \Omega_0) (\cos (\sigma - \Omega_0) \cos (\Omega - h) + \sin (\sigma - \Omega_0) \sin (\Omega - h) \cos i) \\ &\quad + \sin (\lambda - \Omega_0) (\sin (\sigma - \Omega_0) \cos (\Omega - h) - \cos (\sigma - \Omega_0) \sin (\Omega - h) \cos i), \\ \cos b \sin (l - h) &= \cos (\lambda - \Omega_0) (\cos (\sigma - \Omega_0) \sin (\Omega - h) - \sin (\sigma - \Omega_0) \cos (\Omega - h) \cos i) \\ &\quad + \sin (\lambda - \Omega_0) (\sin (\sigma - \Omega_0) \sin (\Omega - h) + \cos (\sigma - \Omega_0) \cos (\Omega - h) \cos i), \\ \sin b &= \sin (\lambda - \Omega_0) \cos (\sigma - \Omega_0) \sin i - \cos (\lambda - \Omega_0) \sin (\sigma - \Omega_0) \sin i.\end{aligned}\tag{146}$$

Let us now conceive a spherical triangle to be formed, of which two of the sides are $\sigma - \Omega_0$ and $\Omega - h$, respectively, and let the angle included by these sides be i . Since h is entirely arbitrary, we may assign to it a value such that the other angle adjacent to the side $\sigma - \Omega_0$ will be equal to i_0 . Let the third side be designated by $h_0 - \Omega_0$, and the angle opposite to $\sigma - \Omega_0$ by η' . The auxiliary triangle thus formed gives the following relations:—

$$\begin{aligned}
\cos(h_0 - \Omega_0) &= \cos(\sigma - \Omega_0) \cos(\Omega - h) + \sin(\sigma - \Omega_0) \sin(\Omega - h) \cos i, \\
\sin(h_0 - \Omega_0) \sin i_0 &= \sin(\Omega - h) \sin i, \\
\sin(h_0 - \Omega_0) \cos i_0 &= \sin(\sigma - \Omega_0) \cos(\Omega - h) - \cos(\sigma - \Omega_0) \sin(\Omega - h) \cos i, \\
\sin(h_0 - \Omega_0) \cos \eta' &= \cos(\sigma - \Omega_0) \sin(\Omega - h) - \sin(\sigma - \Omega_0) \cos(\Omega - h) \cos i.
\end{aligned} \tag{147}$$

Combining these with the preceding equations, we easily derive

$$\begin{aligned}
\cos b \cos(l - h) &= \cos(\lambda, - \Omega_0) \cos(h_0 - \Omega_0) + \sin(\lambda, - \Omega_0) \sin(h_0 - \Omega_0) \cos i_0, \\
\cos b \sin(l - h) &= \sin(\lambda, - \Omega_0) \cos(h_0 - \Omega_0) \cos i_0 - \cos(\lambda, - \Omega_0) \sin(h_0 - \Omega_0) \\
&\quad + \cos(\lambda, - \Omega_0) \sin(h_0 - \Omega_0) (1 + \cos \eta') \\
&\quad + \sin(\lambda, - \Omega_0) ((\cos i - \cos i_0) \cos(h_0 - \Omega_0) + \sin(\sigma - \Omega_0) \sin(\Omega - h) \sin^2 i), \\
\sin b &= \sin i_0 \sin(\lambda, - \Omega_0) + (\cos(\sigma - \Omega_0) \sin i - \sin i_0) \sin(\lambda, - \Omega_0) \\
&\quad - \cos(\lambda, - \Omega_0) \sin(\sigma - \Omega_0) \sin i.
\end{aligned} \tag{148}$$

Since the action of the component of the disturbing force perpendicular to the plane of the disturbed orbit does not change the radius-vector, we have

$$r \sin b = r \sin i_0 \sin(\lambda, - \Omega_0) + \delta z,$$

and hence the last of these equations gives

$$\begin{aligned}
\frac{\delta z}{r} &= \sin(\lambda, - \Omega_0) (\cos(\sigma - \Omega_0) \sin i - \sin i_0) \\
&\quad - \cos(\lambda, - \Omega_0) \sin(\sigma - \Omega_0) \sin i.
\end{aligned} \tag{149}$$

From the relation of the parts of the auxiliary spherical triangle, we have

$$\begin{aligned}
\sin i \sin(\sigma - \Omega_0) &= \sin \eta' \sin(h_0 - \Omega_0), \\
\sin i \cos(\sigma - \Omega_0) &= \sin \eta' \cos(h_0 - \Omega_0) \cos i_0 + \cos \eta' \sin i_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\delta z}{r} &= \sin(\lambda, - \Omega_0) (\cos i_0 \cos(h_0 - \Omega_0) \sin \eta' - \sin i_0 (1 - \cos \eta')), \\
&\quad - \cos(\lambda, - \Omega_0) \sin(h_0 - \Omega_0) \sin \eta',
\end{aligned} \tag{150}$$

and

$$\begin{aligned}
\frac{\delta z}{r} \cdot \frac{\sin \eta'}{1 - \cos \eta'} &= \sin(\lambda, - \Omega_0) (\cos i_0 \cos(h_0 - \Omega_0) (1 + \cos \eta') - \sin i_0 \sin \eta') \\
&\quad - \cos(\lambda, - \Omega_0) \sin(h_0 - \Omega_0) (1 + \cos \eta').
\end{aligned} \tag{151}$$

We have, further, from the auxiliary spherical triangle,

$$\cos i = \sin i_0 \sin \eta' \cos(h_0 - \Omega_0) - \cos i_0 \cos \eta',$$

from which we get

$$\cos i - \cos i_0 = \sin i_0 \cos(h_0 - \Omega_0) \sin \eta' - \cos i_0 (1 + \cos \eta').$$

We have, also,

$$\begin{aligned}
\sin(\sigma - \Omega_0) \sin i &= \sin \eta' \sin(h_0 - \Omega_0), \\
\sin(\Omega - h) \sin i &= \sin i_0 \sin(h_0 - \Omega_0),
\end{aligned}$$

or

$$\sin(\sigma - \Omega_0) \sin(\Omega - h) \sin^2 i = \sin^2(h_0 - \Omega_0) \sin i_0 \sin \eta'.$$

Hence we derive

$$\begin{aligned} (\cos i - \cos i_0) \cos(h_0 - \Omega_0) + \sin(\sigma - \Omega_0) \sin(\Omega - h) \sin^2 i &= \sin i_0 \sin \eta' \\ &- (1 + \cos \eta') \cos i_0 \cos(h_0 - \Omega_0). \end{aligned}$$

Combining this and the equation (151) with the equations (148), we obtain

$$\begin{aligned} \cos b \cos(l - h) &= \cos(\lambda, -\Omega_0) \cos(h_0 - \Omega_0) + \sin(\lambda, -\Omega_0) \sin(h_0 - \Omega_0) \cos i_0, \\ \cos b \sin(l - h) &= \sin(\lambda, -\Omega_0) \cos(h_0 - \Omega_0) \cos i_0 - \cos(\lambda, -\Omega_0) \sin(h_0 - \Omega_0) \\ &\quad - \frac{\sin \eta'}{1 - \cos \eta'} \cdot \frac{\delta z_r}{r}, \\ \sin b &= \sin(\lambda, -\Omega_0) \sin i_0 + \frac{\delta z_r}{r}. \end{aligned}$$

If we multiply the first of these equations by $\cos(h_0 - \Omega_0)$, and the second by $-\sin(h_0 - \Omega_0)$, and add the results; then multiply the first by $\sin(h_0 - \Omega_0)$, and the second by $\cos(h_0 - \Omega_0)$, and add, we get

$$\begin{aligned} \cos b \cos(l - \Omega_0 - (h - h_0)) &= \cos(\lambda, -\Omega_0) + \sin(h_0 - \Omega_0) \frac{\sin \eta'}{1 - \cos \eta'} \cdot \frac{\delta z_r}{r}, \\ \cos b \sin(l - \Omega_0 - (h - h_0)) &= \sin(\lambda, -\Omega_0) \cos i_0 - \cos(h_0 - \Omega_0) \frac{\sin \eta'}{1 - \cos \eta'} \cdot \frac{\delta z_r}{r}, \\ \sin b &= \sin(\lambda, -\Omega_0) \sin i_0 + \frac{\delta z_r}{r}. \end{aligned} \quad (152)$$

Let us now put

$$\begin{aligned} p' &= \sin(\sigma - \Omega_0) \sin i, \\ q' &= \cos(\sigma - \Omega_0) \sin i - \sin i_0, \end{aligned} \quad (153)$$

and there results, from (149),

$$\frac{\delta z_r}{r} = q' \sin(\lambda, -\Omega_0) - p' \cos(\lambda, -\Omega_0). \quad (154)$$

Comparing this with equation (150), we observe that

$$\begin{aligned} p' &= \sin \eta' \sin(h_0 - \Omega_0), \\ q' &= \sin \eta' \cos(h_0 - \Omega_0) \cos i_0 - \sin i_0 (1 - \cos \eta'). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\sin \eta'}{1 - \cos \eta'} \sin(h_0 - \Omega_0) &= \frac{p'}{1 - \cos \eta'}, \\ \frac{\sin \eta'}{1 - \cos \eta'} \cos(h_0 - \Omega_0) &= \tan i_0 + \frac{q'}{\cos i_0 (1 - \cos \eta')}, \end{aligned}$$

and, if we put $\Gamma = h - h_0$, the equations (152) become

$$\begin{aligned}\cos b \cos (l - \Omega_0 - \Gamma) &= \cos (\lambda, -\Omega_0) + \frac{p'}{1 - \cos \gamma'} \cdot \frac{\delta z_1}{r}, \\ \cos b \sin (l - \Omega_0 - \Gamma) &= \sin (\lambda, -\Omega_0) \cos i_0 - \left(\tan i_0 + \frac{q'}{\cos i_0 (1 - \cos \gamma')} \right) \frac{\delta z_1}{r}, \\ \sin b &= \sin (\lambda, -\Omega_0) \sin i_0 + \frac{\delta z_1}{r}.\end{aligned}\quad (155)$$

As soon as Γ , p' , q' , and γ' are known, these equations will furnish the exact values of l and b , those of λ , and r being found by means of the perturbations ν and δM .

186. The value of Γ may be expressed in terms of p' and q' . Thus, if we differentiate the first of equations (147) and reduce by means of the remaining equations of the same group, we get

$$d(h_0 - \Omega_0) = \cos \gamma' d(\Omega - h) + \cos i_0 d\sigma + \sin i_0 \sin (\sigma - \Omega_0) di,$$

and if we interchange $\Omega - h$ and $h_0 - \Omega_0$ in this equation, we must also interchange i and i_0 , which are the angles opposite to these sides, respectively, in the auxiliary spherical triangle, so that we shall have

$$d(\Omega - h) = \cos \gamma' d(h_0 - \Omega_0) + \cos i d\sigma,$$

i_0 being constant. Adding these equations, observing that Ω_0 is also constant, we get

$$(1 - \cos \gamma') d(\Omega - h + h_0) = \sin i_0 \sin (\sigma - \Omega_0) di + (\cos i + \cos i_0) d\sigma; \quad (156)$$

and since $d\sigma = \cos i d\Omega$, this becomes

$$\begin{aligned}(1 - \cos \gamma') d(h - h_0) &= -\sin i_0 \sin (\sigma - \Omega_0) di \\ &\quad + (\sin^2 i - \cos \gamma' - \cos i \cos i_0) \frac{d\sigma}{\cos i},\end{aligned}$$

which, since

$$\cos \gamma' = \sin i \sin i_0 \cos (\sigma - \Omega_0) - \cos i \cos i_0, \quad (157)$$

may be written

$$(1 - \cos \gamma') d\Gamma = -\sin i_0 \sin (\sigma - \Omega_0) di + \tan i (\sin i - \sin i_0 \cos (\sigma - \Omega_0)) d\sigma. \quad (158)$$

The differentiation of the equations (153) gives

$$\begin{aligned}dp' &= \sin (\sigma - \Omega_0) \cos i di + \sin i \cos (\sigma - \Omega_0) d\sigma, \\ dq' &= \cos (\sigma - \Omega_0) \cos i di - \sin i \sin (\sigma - \Omega_0) d\sigma,\end{aligned}$$

from which we derive

$$\begin{aligned} q'dp' - p'dq' &= \sin^2 i \, d\sigma - \sin i_0 \, dp' \\ &= \cos i \, (-\sin i_0 \sin(\sigma - \Omega_0) \, di + \tan i (\sin i - \sin i_0 \cos(\sigma - \Omega_0)) \, d\sigma). \end{aligned}$$

Combining this with equation (158), we get

$$\cos i (1 - \cos \eta') \, d\Gamma = q'dp' - p'dq',$$

and hence

$$\Gamma = \int \frac{q' \frac{dp'}{dt} - p' \frac{dq'}{dt}}{\cos i (1 - \cos \eta')} \, dt, \quad (159)$$

the integral being equal to zero for the instant to which the fundamental osculating elements belong. It is evident from the equations (153) that p' and q' are of the order of the first power of the disturbing forces, and hence, since η' differs but little from $180^\circ - (i + i_0)$, it follows that, so long as i is not very large, Γ is at least of the second order.

The last of equations (145) gives

$$z, = r \sin i \sin \lambda, \cos \sigma - r \sin i \cos \lambda, \sin \sigma,$$

and since

$$x = r \cos \lambda,, \quad y = r \sin \lambda,,$$

this becomes

$$z, = -x \sin i \sin \sigma + y \sin i \cos \sigma.$$

Comparing this with equation (116), it appears that

$$\alpha = -\sin i \sin \sigma, \quad \beta = \sin i \cos \sigma, \quad (160)$$

and hence, by means of (153), we derive

$$\begin{aligned} p' &= -\alpha \cos \Omega_0 - \beta \sin \Omega_0, \\ q' &= -\alpha \sin \Omega_0 + \beta \cos \Omega_0 - \sin i_0, \end{aligned}$$

and also

$$\begin{aligned} \frac{dp'}{dt} &= -\cos \Omega_0 \frac{d\alpha}{dt} - \sin \Omega_0 \frac{d\beta}{dt}, \\ \frac{dq'}{dt} &= -\sin \Omega_0 \frac{d\alpha}{dt} + \cos \Omega_0 \frac{d\beta}{dt}. \end{aligned} \quad (161)$$

From the equations (118) and (121), observing that

$$x \frac{dy}{dt} - y \frac{dx}{dt} = k\sqrt{p(1+m)},$$

we derive, by elimination,

$$\frac{d\alpha}{dt} = -\frac{r \sin \lambda, \cos i}{k\sqrt{p(1+m)}} Z, \quad \frac{d\beta}{dt} = \frac{r \cos \lambda, \cos i}{k\sqrt{p(1+m)}} Z.$$

Therefore we shall have

$$\begin{aligned}\frac{dp'}{dt} &= \frac{r \cos i \sin (\lambda, - \Omega_0)}{kV \overline{p(1+m)}} Z, \\ \frac{dq'}{dt} &= \frac{r \cos i \cos (\lambda, - \Omega_0)}{kV \overline{p(1+m)}} Z,\end{aligned}\tag{162}$$

by means of which p' and q' may be found by integration, the integral in each case being zero for the date t_0 at which the determination of the perturbations begins.

When the value of δz , has already been found by means of the equation (129), if we compute the value of q' , that of p' will be given by means of (154), or

$$p' = q' \tan (\lambda, - \Omega_0) - \frac{\delta z,}{r \cos (\lambda, - \Omega_0)},$$

and if p' is determined, q' will be given by

$$q' = p' \cot (\lambda, - \Omega_0) + \frac{\delta z,}{r \sin (\lambda, - \Omega_0)}.$$

If both p' and q' are found from the equations (162), δz , may be determined directly from (154); but the value thus obtained will be less accurate than that derived by means of equation (129).

Since the formula for $\frac{d^2 \delta z,}{dt^2}$ completely determines the perturbations due to the action of the component Z perpendicular to the plane of the instantaneous orbit, instead of determining p' and q' by an independent integration by means of the results given by the equations (162), it will be preferable to derive them directly from δz , and $\frac{d\delta z,}{dt}$. The equations (161) give

$$p' = -\cos \Omega_0 \delta \alpha - \sin \Omega_0 \delta \beta, \quad q' = -\sin \Omega_0 \delta \alpha + \cos \Omega_0 \delta \beta.$$

Substituting for $\delta \alpha$ and $\delta \beta$ their values given by (125) and (126), and putting

$$x'' = x \cos \Omega_0 + y \sin \Omega_0, \quad y'' = -x \sin \Omega_0 + y \cos \Omega_0,$$

we obtain

$$\begin{aligned}p' &= \frac{1}{kV \overline{p(1+m)}} \left(y'' \frac{d\delta z,}{dt} - \delta z, \frac{dy''}{dt} \right), \\ q' &= \frac{1}{kV \overline{p(1+m)}} \left(x'' \frac{d\delta z,}{dt} - \delta z, \frac{dx''}{dt} \right).\end{aligned}\tag{163}$$

Substituting further the values

$$\text{and also} \quad x'' = r \cos (\lambda, - \Omega_0), \quad y'' = r \sin (\lambda, - \Omega_0),$$

$$\frac{d\lambda}{dt} = \frac{k\sqrt{p(1+m)}}{r^2},$$

$$\frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v = \frac{k\sqrt{p(1+m)}}{r} \cdot \frac{e \sin v}{1 + e \cos v},$$

we easily find, since $\lambda, -v = \chi$,

$$\begin{aligned} p' &= -(\cos (\lambda, - \Omega_0) + e \cos (\chi - \Omega_0)) \frac{\delta z}{p} + \frac{r \sin (\lambda, - \Omega_0)}{k\sqrt{p(1+m)}} \cdot \frac{d\delta z}{dt}, \\ q' &= +(\sin (\lambda, - \Omega_0) + e \sin (\chi - \Omega_0)) \frac{\delta z}{p} + \frac{r \cos (\lambda, - \Omega_0)}{k\sqrt{p(1+m)}} \cdot \frac{d\delta z}{dt}, \end{aligned} \quad (164)$$

which may be used for the determination of p' and q' . These equations require, for their exact solution, that the disturbed values e , χ , and p shall be known, but it is evident that the error will be slight, especially when e is small, if we use the undisturbed values e_0 , p_0 , and $\chi_0 = \pi_0$. The actual values of λ , and r are obtained directly from the values of the perturbations.

When p' and q' have been found, it remains only to find $\cos i$, and $1 - \cos \gamma'$, in order to be able to obtain Γ by means of the equation (159). From (153) we get

$$\text{and hence} \quad p'^2 + q'^2 = \sin^2 i - \sin^2 i_0 - 2q' \sin i_0,$$

$$\cos i = \sqrt{1 - p'^2 - (q' + \sin i_0)^2}, \quad (165)$$

from which $\cos i$ may be found. The equation (157) gives

$$1 - \cos \gamma' = \cos i_0 (\cos i_0 + \cos i) - q' \sin i_0, \quad (166)$$

by means of which the value of $1 - \cos \gamma'$ will be obtained.

If we substitute the values of p' , q' , $\frac{dp'}{dt}$, and $\frac{dq'}{dt}$ given by the equations (153) and (162) in (159), it is easily reduced to

$$\Gamma = \int \frac{\delta z}{(1 - \cos \gamma') k\sqrt{p(1+m)}} Z dt, \quad (167)$$

which may be used for the determination of Γ . When we neglect terms of the order of the cube of the disturbing force, in finding Γ we may use p_0 in place of p and put $1 - \cos \gamma' = 2 \cos^2 i_0$, so that the formula becomes

$$r = \frac{1}{2 \cos^2 i_0 k \sqrt{p_0(1+m)}} \int \delta z, Z dt. \quad (168)$$

187. By means of the formulæ which have thus been derived, we may find the values of all the quantities required in the solution of the equations (155), in order to obtain the values of l and b for the disturbed motion. From r , l , and b the corresponding geocentric place may be found. The heliocentric longitude and latitude may also be determined directly by means of the equations (145), provided that Ω , σ , and i are known; and the required formulæ for the determination of these elements may be readily derived. Thus, the equations (160) give, by differentiation,

$$\begin{aligned} \frac{d\alpha}{dt} &= -\sin \sigma \cos i \frac{di}{dt} - \sin i \cos \sigma \frac{d\sigma}{dt}, \\ \frac{d\beta}{dt} &= \cos \sigma \cos i \frac{di}{dt} - \sin i \sin \sigma \frac{d\sigma}{dt}, \end{aligned}$$

whence

$$\begin{aligned} \sin i \frac{d\sigma}{dt} &= -\cos \sigma \frac{d\alpha}{dt} - \sin \sigma \frac{d\beta}{dt}, \\ \cos i \frac{di}{dt} &= -\sin \sigma \frac{d\alpha}{dt} + \cos \sigma \frac{d\beta}{dt}. \end{aligned}$$

Introducing the values of $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ already found into these equations, and putting

$$\sigma = \sigma_0 + \delta\sigma = \Omega_0 + \delta\sigma, \quad i = i_0 + \delta i, \quad \Omega = \Omega_0 + \delta\Omega,$$

we obtain

$$\begin{aligned} \frac{d\delta\sigma}{dt} &= \frac{1}{k\sqrt{p(1+m)}} \cot i \sin(\lambda, -\sigma) rZ, \\ \frac{d\delta i}{dt} &= \frac{1}{k\sqrt{p(1+m)}} \cos(\lambda, -\sigma) rZ, \end{aligned} \quad (169)$$

and also, since $d\sigma = \cos i d\Omega$,

$$\frac{d\delta\Omega}{dt} = \frac{1}{k\sqrt{p(1+m)}} \cdot \frac{\sin(\lambda, -\sigma)}{\sin i} rZ, \quad (170)$$

by means of which the variations of σ , i , and Ω due to the action of the disturbing forces, may be determined. The integral is in each case equal to zero at the initial date t_0 to which the fundamental oscillating elements belong and at which the integration is to commence.

If we find i , and then $\sigma - \Omega$ from

$$\Omega - \sigma = \int \frac{\tan \frac{1}{2}i}{kV^p(1+m)} \sin(\lambda, -\sigma) rZ dt, \quad (171)$$

the true longitude in the orbit will be obtained from

$$\lambda = \lambda, + \Omega - \sigma.$$

It is evident that since the expressions for $\frac{d\delta i}{dt}$, $\frac{d\delta\sigma}{dt}$, and $\frac{d\delta\Omega}{dt}$ require, for an accurate solution, that the disturbed values i , σ , and p shall be known, and require, besides, that three separate integrations shall be performed, unless the perturbations are computed only in reference to the first power of the disturbing force, in which case we use i_0 , p_0 , and Ω_0 in place of i , p , and σ , respectively, in the equations (169) and (170), the action of the component Z can be considered in the most advantageous manner by means of the variation of z , arising from this component alone; and even when only the perturbations of the first order are to be determined it will still be preferable to derive δz , by the indirect process from the expression for $\frac{d^2\delta z}{dt^2}$, and to determine the heliocentric place by means of the equations (155). When we neglect the terms of the second order, these equations become

$$\begin{aligned} \cos b \cos(l - \Omega_0) &= \cos(\lambda, -\Omega_0), \\ \cos b \sin(l - \Omega_0) &= \sin(\lambda, -\Omega_0) \cos i_0 - \tan i_0 \frac{\delta z}{r}, \\ \sin b &= \sin(\lambda, -\Omega_0) \sin i_0 + \frac{\delta z}{r}, \end{aligned} \quad (172)$$

by means of which l and b are determined immediately from the perturbations δM , ν , and δz . The peculiar advantage of determining the effect of the action of the component Z by means of the partial variation of z , is apparent when we observe that the expressions for $\frac{d\delta\sigma}{dt}$ and $\frac{d\delta\Omega}{dt}$ involve $\sin i$ as a divisor; and in the case of orbits whose inclination is small, this divisor may be the source of a considerable amount of error.

188. The determination of the perturbations so as to include the higher powers of the masses is readily effected by means of the complete expressions for $\frac{d\delta M}{dt}$, $\frac{d^2\nu}{dt^2}$, and $\frac{d^2\delta z}{dt^2}$, when the correct values of R , S , Z , i , and p are known. The corrected values of i and p —

which are required only in the case of δz ,—may be easily estimated with sufficient accuracy, since we require only $\cos i$, while \sqrt{p} appears as the divisor of a term whose numerical value is generally insignificant. To obtain the actual values of R , S , and Z , the corrections to be applied to N , N' , and I must first be determined by means of the formulæ (136). The values of $\delta i'$ and $\delta \Omega'$ will be found by means of the data furnished by the tables of the motion of the disturbing body, and the corresponding corrections for N , N' , and I having been found by means of the terms of (136) involving di' and $d\Omega'$, there remain the corrections due to δi and $\delta \Omega$ to be applied. These may be found in terms of the quantities p' and q' already introduced. Thus, the equations

$$\begin{aligned} dp' &= \cos i \sin(\sigma - \Omega_0) di + \sin i \cos(\sigma - \Omega_0) d\sigma, \\ dq' &= \cos i \cos(\sigma - \Omega_0) di - \sin i \sin(\sigma - \Omega_0) d\sigma, \end{aligned}$$

give

$$\begin{aligned} \cos i di &= \sin(\sigma - \Omega_0) dp' + \cos(\sigma - \Omega_0) dq', \\ \sin i d\sigma &= \cos(\sigma - \Omega_0) dp' - \sin(\sigma - \Omega_0) dq'. \end{aligned}$$

The equations (136) give, observing that $d\sigma = \cos i d\Omega$,

$$\begin{aligned} dI &= -\cos N di - \tan i \sin N d\sigma, \\ dN' &= + \frac{\sin N}{\sin I} di - \frac{\tan i}{\sin I} \cos N d\sigma, \end{aligned}$$

and, substituting the preceding values of di and $d\sigma$, these become

$$\begin{aligned} dI &= -\frac{\sin(N + \sigma - \Omega_0)}{\cos i} dp' - \frac{\cos(N + \sigma - \Omega_0)}{\cos i} dq', \\ dN' &= -\frac{\cos(N + \sigma - \Omega_0)}{\sin I \cos i} dp' + \frac{\sin(N + \sigma - \Omega_0)}{\sin I \cos i} dq'. \end{aligned}$$

If we neglect the perturbations of the third order, these equations give

$$\begin{aligned} \delta I &= -\sin N \frac{p'}{\cos i_0} - \cos N \frac{q'}{\cos i_0}, \\ \delta N' &= -\operatorname{cosec} I \left(\cos N \frac{p'}{\cos i_0} - \sin N \frac{q'}{\cos i_0} \right), \end{aligned} \tag{173}$$

by means of which δI and δN may be determined, p' and q' being found by means of the equations (164), using e_0 , π_0 , and p_0 in place of e , χ , and p . The results for δI and $\delta N'$ obtained from (173) being applied to the values of I' and N' as already corrected on account of $\delta i'$ and $\delta \Omega'$, give the required values of these quantities.

When we consider only di and $d\Omega$, since

$$\sin i' \cos N' = \cos i \sin I + \sin i \cos I \cos N,$$

we easily find

$$\delta N = \cos I \delta N' - \delta \sigma, \quad (174)$$

and if we add the quantity $\cos I \delta N'$ to the value of N already corrected on account of $\delta i'$ and $\delta \Omega'$, and denote the result by N_1 , the required value of N will be $N_1 - \delta \sigma$. Then, according to (131), we may compute $w' + \delta \sigma$ and β' by means of the formulæ

$$\begin{aligned} u'_0 &= w' - N', \\ \tan ((w' + \delta \sigma) - N_1) &= \tan u'_0 \cos I, \\ \tan \beta' &= \tan I \sin ((w' + \delta \sigma) - N_1), \end{aligned} \quad (175)$$

using the values of N' and I as finally corrected. We have, further, according to (135),

$$w + \delta \sigma = v_1 + \pi_0 - \Omega_0,$$

by means of which we may compute the value of $w + \delta \sigma$; then the value of $w' - w$ required in the equations (132), and also in finding the value of ρ , will be given by

$$w' - w = (w' + \delta \sigma) - (w + \delta \sigma),$$

and the forces R , S , and Z may be accurately determined.

By thus determining the correct values of R , S , and Z from date to date, the perturbations δM , ν , and δz , may be determined in reference to the higher powers of the disturbing forces according to the process already explained. The only difficulty to be encountered is that which arises from the quantities Γ , p' , and q' , required in the determination of the heliocentric place of the disturbed body by means of the equations (155). If an exact ephemeris for a short period is required, by means of the complete perturbations we may determine new osculating elements, and by means of these the required heliocentric or geocentric places.

189. EXAMPLE.—We will now illustrate the application of the formulæ for the determination of the perturbations δM , ν , and δz , by a numerical example; and for this purpose let it be required to determine the perturbations of *Eurynome* ²⁹ arising from the action of *Jupiter* from 1864 Jan. 1.0 to 1865 Jan. 15.0, Berlin mean

time, the fundamental osculating elements being those given in Art. 166.

In the first place, by means of the formulæ (130), using the values

$$\begin{aligned}\Omega &= 206^\circ 39' 5''.7, & i &= 4^\circ 36' 52''.1, \\ \Omega' &= 98 \ 58 \ 22 \ .7, & i' &= 1 \ 18 \ 40 \ .5,\end{aligned}$$

which refer to the ecliptic and mean equinox of 1860.0, we obtain

$$N = 194^\circ 0' 49''.9, \quad N' = 301^\circ 38' 31''.7, \quad I = 5^\circ 9' 56''.4.$$

Then, by means of the data furnished by the *Tables of Jupiter*, we find the values of w' , the argument of the latitude of *Jupiter* in reference to the ecliptic of 1860.0, and from the equations (131) we derive w' and β' . The values of r' are given by the *Tables of Jupiter*, and the values of r_0 and v_0 are found from the elements given in Art. 166. The results thus obtained are the following:—

| Berlin Mean Time. | | $\log r_0$ | v_0 | $\log r'$ | w' | β |
|-------------------|-------|------------|------------------|-----------|-----------------|----------------|
| 1863 Dec. | 12.0, | 0.294084 | 354° 26' 18'' .0 | 0.73425 | 14° 18' 54'' .6 | —0° 1' 38'' .1 |
| 1864 Jan. | 21.0, | 0.294837 | 10 2 45 .7 | 0.73368 | 17 21 44 .2 | 0 18 9 .1 |
| March | 1.0, | 0.300674 | 25 24 59 .4 | 0.73305 | 20 25 5 .2 | 0 34 39 .9 |
| April | 10.0, | 0.310864 | 40 13 31 .8 | 0.73237 | 23 28 59 .8 | 0 51 7 .6 |
| May | 20.0, | 0.324298 | 54 14 41 .4 | 0.73164 | 26 33 32 .1 | 1 7 29 .7 |
| June | 29.0, | 0.339745 | 67 21 23 .5 | 0.73086 | 29 38 44 .8 | 1 23 43 .5 |
| Aug. | 8.0, | 0.356101 | 79 32 18 .1 | 0.73003 | 32 44 41 .2 | 1 39 46 .3 |
| Sept. | 17.0, | 0.372469 | 90 49 57 .6 | 0.72915 | 35 51 24 .6 | 1 55 35 .2 |
| Oct. | 27.0, | 0.388214 | 101 19 9 .8 | 0.72823 | 38 58 57 .5 | 2 11 7 .5 |
| Dec. | 6.0, | 0.402894 | 111 5 42 .2 | 0.72726 | 42 7 23 .3 | 2 26 20 .3 |
| 1865 Jan. | 15.0, | 0.416240 | 120 15 32 .6 | 0.72625 | 45 16 43 .9 | —2 41 10 .6 |

The value of w for each date is now found from

$$w = v_0 + \pi_0 - \Omega_0 = v_0 + 197^\circ 38' 6''.5,$$

and the components of the disturbing force are determined by means of the formulæ (132), ρ being found from (133) or (134), and h from (70). The adopted value of the mass of *Jupiter* is

$$m' = \frac{1}{1047.879},$$

and the results for the components R , S , and Z are expressed in units of the seventh decimal place. The factor ω^2 is introduced for convenience in the integration, ω being the interval in days between the successive dates for which the forces are to be determined. Thus we obtain the following results:—

| | Date. | | $\omega^2 R$ | | $\omega^2 S r_0$ | $\omega^2 Z \cos i_0$ | $\omega \int S r_0 dt$ | | |
|------|-------------|---|--------------|---|------------------|-----------------------|------------------------|---|---------|
| 1863 | Dec. 12.0, | + | 70.82 | + | 7.16 | + | 0.04 | + | 1.37 |
| 1864 | Jan. 21.0, | | 68.95 | — | 32.76 | | 0.49 | — | 11.45 |
| | March 1.0, | | 61.16 | | 70.38 | | 0.92 | | 63.32 |
| | April 10.0, | | 48.57 | | 102.91 | | 1.32 | | 150.48 |
| | May 20.0, | | 32.77 | | 128.34 | | 1.68 | | 266.75 |
| | June 29.0, | + | 15.41 | | 145.39 | | 1.96 | | 404.35 |
| | Aug. 8.0, | — | 2.19 | | 153.44 | | 2.17 | | 554.54 |
| | Sept. 17.0, | | 19.12 | | 152.41 | | 2.29 | | 708.21 |
| | Oct. 27.0, | | 34.81 | | 142.50 | | 2.25 | | 856.39 |
| | Dec. 6.0, | | 48.95 | | 124.04 | | 2.09 | | 990.36 |
| 1865 | Jan. 15.0, | — | 61.45 | — | 97.36 | + | 1.75 | — | 1101.73 |

The single integration to find $\omega \int S r_0 dt$ is effected by means of the formula (32).

The equations for the determination of the required differential coefficients are

$$\omega \frac{d\delta M}{dt} = \mu_0 \left(\frac{1}{kV p_0} \omega \int S r_0 dt - 2\omega\nu \right),$$

$$\omega^2 \frac{d^2\nu}{dt^2} = \frac{\omega^2 R}{r_0} + \frac{2\omega k^2}{r_0^3} \cdot \frac{1}{kV p_0} \omega \int S r_0 dt - \frac{e_0 \sin v_0}{p_0} \omega^2 S - \frac{\omega^2 k^2}{r_0^3} \nu,$$

$$\omega^2 \frac{d^2\delta z_r}{dt^2} = \omega^2 Z \cos i_0 - \frac{\omega^2 k^2}{r_0^3} \delta z_r.$$

Substituting in these the results already obtained, and also

$$\log \mu_0 = 2.967809, \quad \log p_0 = 0.371237, \quad \log e_0 = 9.290776,$$

we obtain first, by an indirect process, as illustrated in the case of the direct determination of the perturbations of the rectangular co-ordinates, the values of $\omega^2 \frac{d^2\nu}{dt^2}$ and $\omega^2 \frac{d^2\delta z_r}{dt^2}$, and then, having found ν ,

$\omega \frac{d\delta M}{dt}$ is given directly by the first of these equations. The integration of the results thus derived, by the formulæ for mechanical quadrature, furnishes the required values of ν , δM , and δz_r . The calculation of the indirect terms in the determination of ν and δz_r , there being but one such term in each case, is, on account of the smallness of the coefficient, effected with very great facility.

The final results are the following:—

| Date. | $\omega \frac{d\delta M}{dt}$ | $\omega^2 \frac{d^2\nu}{dt^2}$ | $\omega^2 \frac{d^2\delta z}{dt^2}$ | δM | ν | δz |
|-----------------|-------------------------------|--------------------------------|-------------------------------------|------------|----------|------------|
| 1863 Dec. 12.0, | — 0''.028 + | 36.16 | + 0.04 | + 0''.01 | + 4.41 | + 0.02 |
| 1864 Jan. 21.0, | 0 .072 | 33.61 | 0.49 | — 0 .01 | 4.31 | 0.04 |
| March 1.0, | 0 .499 | 22.55 | 0.89 | 0 .27 | 37.11 | 0.54 |
| April 10.0, | 1 .213 + | 5.58 | 1.21 | 1 .11 | 91.96 | 1.93 |
| May 20.0, | 2 .070 — | 13.52 | 1.45 | 2 .75 | 152.22 | 4.52 |
| June 29.0, | 2 .902 | 31.59 | 1.53 | 5 .24 | 199.05 | 8.54 |
| Aug. 8.0, | 3 .546 | 46.65 | 1.60 | 8 .49 | 214.54 | 14.10 |
| Sept. 17.0, | 3 .858 | 57.88 | 1.52 | 12 .22 | 183.69 | 21.24 |
| Oct. 27.0, | 3 .723 | 65.19 | 1.28 | 16 .05 + | 95.29 | 29.90 |
| Dec. 6.0, | 3 .056 | 68.83 | 0.92 | 19 .49 — | 58.00 | 39.82 |
| 1865 Jan. 15.0, | — 1 .800 — | 69.19 | + 0.40 | — 21 .97 | — 279.84 | + 50.64 |

Since, during the period included by these results, the perturbations of the second order are insensible, we have, for the perturbations of *Eurynome* arising from the action of Jupiter from 1864 Jan. 1.0 to 1865 Jan. 15.0,

$$\delta M = -21''.97, \quad \nu = -0.00002798, \quad \delta z = +0.00000506.$$

It is to be observed that δz , is not the complete variation of the co-ordinate z , perpendicular to the ecliptic, but only that part of this variation which is due to the action of the component Z alone; and hence the results for δz , differ from the complete values obtained when we compute directly the variations of the rectangular co-ordinates.

Let us now determine the heliocentric longitude and latitude for 1865 Jan. 15.0, Berlin mean time, including the perturbations thus derived. From the equations

$$\begin{aligned} M, &= M_0 + \mu_0 (t - t_0) + \delta M, \\ E, - e_0 \sin E, &= M, \\ r, &= a_0 (1 - e_0 \cos E,), \\ \sin \frac{1}{2} (v, - E,) &= \sin \frac{1}{2} \varphi_0 \sin E, \sqrt{\frac{a_0}{r,}}, \\ \lambda, &= v, + \pi_0, \quad r = r, (1 + \nu), \end{aligned}$$

we obtain

$$\begin{aligned} M, &= 99^\circ 29' 35''.51, & E, &= 110^\circ 0' 33''.75, \\ \log r, &= 0.4162304, & v, &= 120 \ 15 \ 13 \ .80, \\ \log r &= 0.4162183, & \lambda, &= 164 \ 32 \ 25 \ .97. \end{aligned}$$

The calculation of the values of r , and v , from the values of M , a_0 , and e_0 , may be effected by means of the various formulæ for the

determination of the radius-vector and true anomaly from given elements. If we substitute these results for λ , r , and δz , in the equations (172), we get

$$l = 164^\circ 37' 59''.05, \quad b = -3^\circ 5' 32''.54,$$

which are referred to the ecliptic and mean equinox of 1860.0, and from these we may derive the geocentric place of the disturbed body. If the place of the body is required in reference to the equinox and ecliptic of any other date, it is only necessary to reduce the elements π_0 , Ω_0 , and i_0 to the equinox and ecliptic of that date; and then, having computed λ , and r , we obtain by means of the equations (172) the required values of l and b . In the determination of the perturbations it will be convenient to adopt a fixed equinox and ecliptic throughout the calculation; and afterwards, when the heliocentric or geocentric places are determined, the proper corrections for precession and nutation may be applied.

In order to compare the results obtained from the perturbations δM , ν , and δz , with those derived by the method of the variation of rectangular co-ordinates, we have, for the date 1865 Jan. 15.0,

$$x_0 = -2.5107584, \quad y_0 = +0.6897713, \quad z_0 = -0.1406590;$$

and for the perturbations of these co-ordinates we have found

$$\delta x = +0.0001773, \quad \delta y = +0.0001992, \quad \delta z = -0.0000028.$$

Hence we derive

$$x = -2.5105811, \quad y = +0.6899705, \quad z = -0.1406618,$$

and from these the corresponding polar co-ordinates, namely,

$$\log r = 0.4162182, \quad l = 164^\circ 37' 59''.05, \quad b = -3^\circ 5' 32''.54,$$

from which it appears that the agreement of the results obtained by the two methods is complete.

190. When the perturbations become so large that the terms of the second order must be retained, the approximate values which may be obtained for several intervals in advance by extending the columns of differences, will serve to enable us to consider the neglected terms partially or even completely, and thus derive the complete perturbations for a very long period. But on account of the increasing difficulties which present themselves, arising both from the consideration

of the perturbations due to the action of the component Z in computing the place of the body, and from the magnitude of the numerical values of the perturbations, it will be advantageous to determine, from time to time, new osculating elements corresponding to the values of the perturbations for any particular epoch, and thus commencing the integrals again with the value zero, only the terms of the first order will at first be considered, and the indirect part of the calculation will, on account of the smallness of the terms, be effected with great facility. The mode of effecting the calculation when the higher powers of the masses are taken into account has already been explained, and it will present no difficulty beyond that which is inseparably connected with the problem. The determination of Γ , p' , and q' may be effected from the results for $\frac{d\Gamma}{dt}$, $\frac{dp'}{dt}$, and $\frac{dq'}{dt}$ by means of the formulæ for integration by mechanical quadrature, as already illustrated, or we may find Γ by a direct integration, and the values of p' and q' by means of the equations (164), $\frac{d\delta z}{dt}$ being found from $\frac{d^2\delta z}{dt^2}$ by a single integration. The other quantities required for the complete solution of the equations for the perturbations will be obtained according to the directions which have been given; and in the numerical application of the formulæ, particular attention should be given to the homogeneity of the several terms, especially since, for convenience, we express some of the quantities in units of the seventh decimal place, and others in seconds of arc.

The magnitude of the perturbations will at length be such that, however completely the terms due to the squares and higher powers of the disturbing forces may be considered, the requirements of the numerical process will render it necessary to determine new osculating elements; and we therefore proceed to develop the formulæ for this purpose.

191. The single integration of the values of $\omega^2 \frac{d^2\nu}{dt^2}$ and $\omega^2 \frac{d^2\delta z}{dt^2}$, will give the values of $\omega \frac{d\nu}{dt}$ and $\omega \frac{d\delta z}{dt}$, and hence those of $\frac{d\nu}{dt}$ and $\frac{d\delta z}{dt}$, which, in connection with $\frac{d\delta M}{dt}$, are required in the determination of the new system of osculating elements. Since $r^2 \frac{d\nu}{dt}$ represents double the areal velocity in the disturbed orbit, we have

$$\frac{dv}{dt} = \frac{k\sqrt{p(1+m)}}{r^2}.$$

The equation (109) gives

$$\frac{dv}{dt} = \frac{k\sqrt{p_0(1+m)}}{r_0^2} \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right).$$

Hence, since $r = r_0(1 + \nu)$, we obtain

$$p = p_0 \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 (1 + \nu)^4, \quad (176)$$

by means of which we may derive p . This formula will furnish at once the value of p , which appears in the complete equation for $\frac{d^2\delta z}{dt^2}$, and also in the equations (164); and the value of $\cos i$ may be determined by means of (165).

In the disturbed orbit we have

$$\frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v,$$

and the equations (108) and (111) give

$$\frac{dr}{dt} = \frac{k\sqrt{1+m}}{\sqrt{p_0}} e_0 \sin v, \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right) (1 + \nu) + r, \frac{d\nu}{dt}.$$

Therefore we obtain

$$\sqrt{p_0} e \sin v = \sqrt{p} e_0 \sin v, \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right) (1 + \nu) + \frac{r\sqrt{pp_0}}{k\sqrt{1+m}} \cdot \frac{d\nu}{dt},$$

which, by means of (176), becomes

$$e \sin v = e_0 \sin v, \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 (1 + \nu)^3 + \frac{r\sqrt{p}}{k\sqrt{1+m}} \cdot \frac{d\nu}{dt}. \quad (177)$$

The relation between r and r_0 gives

$$\frac{p}{1 + e \cos v} = \frac{p_0}{1 + e_0 \cos v}, (1 + \nu),$$

and, substituting in this the value of p already found, we get

$$e \cos v = (1 + e_0 \cos v_0) \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt} \right)^2 (1 + \nu)^3 - 1. \quad (178)$$

Let us now put

$$\alpha = \left(1 + \frac{1}{\mu_0} \cdot \frac{d\delta M}{dt}\right)^2 (1 + \nu)^2 - 1,$$

$$\beta = \frac{r\sqrt{p}}{kV\sqrt{1+m}} \cdot \frac{d\nu}{dt}, \quad (179)$$

α and β being small quantities of the order of the disturbing force, and the equations (177) and (178) become

$$e \sin v = e_0 \sin v, + \alpha e_0 \sin v, + \beta,$$

$$e \cos v = e_0 \cos v, + \alpha e_0 \cos v, + \alpha.$$

These equations give, observing that $r, (\cos v, + e_0) = p_0 \cos E,$

$$e \sin (v, - v) = \alpha \sin v, - \beta \cos v,,$$

$$e \cos (v, - v) = e_0 + \frac{\alpha p_0}{r,} \cos E, + \beta \sin v,, \quad (180)$$

from which $e, v, -v$, and v may be found; and thus, since

$$\chi = \pi_0 + (v, - v), \quad (181)$$

we obtain the values of the only remaining unknown quantities in the second members of the equations (164). The determination of p' and q' may now be rigorously effected, and the corresponding value of $\cos i$ being found from (165), $\frac{dp'}{dt}$ and $\frac{dq'}{dt}$ will be given by (162). Then, having found also $1 - \cos \gamma'$ by means of (166), Γ may be determined rigorously by the equation (159), and not only the complete values of the perturbations in reference to all powers of the masses, but also the corresponding heliocentric or geocentric places of the body, may be found.

If we put

$$\gamma' = \alpha \sin v, - \beta \cos v,,$$

$$\delta' = \frac{\alpha p_0}{r,} \cos E, + \beta \sin v,, \quad (182)$$

and neglect terms of the third order, the equations (180) give

$$e = e_0 + \delta' + \frac{\gamma'^2}{2e_0},$$

$$v, - v = \frac{\gamma'}{e_c} s - \frac{\gamma' \delta'}{e_0^2} s, \quad (183)$$

in which $s = 206264''.8$. These equations are convenient for the

determination of e and v , — v , and hence χ by means of (181), when the neglected terms are insensible.

The values of p , e , and v having been found, we have

$$\begin{aligned}\sin \varphi &= e, & a &= p \sec^2 \varphi, & \mu &= \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}, \\ \tan \frac{1}{2} E &= \tan (45^\circ - \frac{1}{2} \varphi) \tan \frac{1}{2} v, & M &= E - e \sin E,\end{aligned}\quad (184)$$

from which to find the elements φ , a , μ , and M . The mean anomaly thus found belongs to the date t , and it may be reduced to any other epoch denoted by t_0 by adding to it the quantity $\mu(t_0 - t)$. When we neglect the terms of the third order, we have

$$\varphi - \varphi_0 = \frac{\sin \varphi - \sin \varphi_0}{\cos \varphi_0 - \frac{1}{2}(\varphi - \varphi_0) \sin \varphi_0},$$

and if we substitute for $\sin \varphi - \sin \varphi_0 = e - e_0$ the value given by the first of equations (183), the result is

$$\varphi - \varphi_0 = \frac{2\delta' \sin \varphi_0 + \gamma'^2}{2 \sin \varphi_0 \cos \varphi_0 - \delta' \sin \varphi_0 \tan \varphi_0},$$

from which we get

$$\varphi = \varphi_0 + \frac{\delta'}{\cos \varphi_0} s + \frac{\delta'^2 \sin \varphi_0}{2 \cos^3 \varphi_0} s + \frac{\gamma'^2}{2 \sin \varphi_0 \cos \varphi_0} s, \quad (185)$$

by means of which φ may be found directly, terms of the third order being neglected.

In the case of the orbits of comets for which e differs but little from unity, instead of δM we compute by means of the formula (142) the value of δT , and since we have

$$\frac{d\delta T}{dt} = -\frac{1}{\mu_0} \cdot \frac{d\delta M}{dt},$$

the equation for p becomes

$$p = p_0 \left(1 - \frac{d\delta T}{dt} \right)^2 (1 + \nu)^4; \quad (186)$$

and for a we have

$$a = \left(1 - \frac{d\delta T}{dt} \right)^2 (1 + \nu)^3 - 1. \quad (187)$$

Then e , v , and q will be found by means of the equations

$$\begin{aligned}
 e \sin (v, -v) &= \alpha \sin v, -\beta \cos v,, \\
 e \cos (v, -v) &= e_0 + \alpha (\cos v, + e_0) + \beta \sin v,, \\
 q &= \frac{p}{1+e},
 \end{aligned}
 \tag{188}$$

and the time of perihelion passage will be derived from e and v by means of Table IX. or Table X.

There remain yet to be found the elements σ , Ω , and i , which determine the position of the plane of the disturbed orbit in space. The values of p' and q' will be found from the equations (164), and Γ , whenever it may be required, will be determined as already explained. Then we shall have

$$\begin{aligned}
 \sin i \sin (\sigma - \Omega_0) &= p', \\
 \sin i \cos (\sigma - \Omega_0) &= q' + \sin i_0,
 \end{aligned}
 \tag{189}$$

from which to find i and σ . When we neglect the terms of the third order, these equations give

$$\sin i - \sin i_0 = q' + \frac{p'q'}{\sin i_0},$$

and hence

$$\begin{aligned}
 \sigma &= \Omega_0 + \frac{p'}{\sin i_0} s - \frac{p'q'}{\sin^2 i_0} s, \\
 i &= i_0 + \frac{q'}{\cos i_0} s + \frac{q'^2 \sin i_0}{2 \cos^3 i_0} s + \frac{p'^2}{2 \sin i_0 \cos i_0} s,
 \end{aligned}
 \tag{190}$$

in which $s = 206264''.8$. The auxiliary spherical triangle which we have employed in the derivation of the equations (155) gives directly

$$\frac{\cos \frac{1}{2} (i + i_0)}{\cos \frac{1}{2} (i - i_0)} = \frac{\tan \frac{1}{2} (\sigma - \Omega_0)}{\tan \frac{1}{2} (\Omega - h + h_0 - \Omega_0)},$$

and since $h - h_0 = \Gamma$, we have

$$\tan \frac{1}{2} (\Omega - \Omega_0 - \Gamma) = \frac{\cos \frac{1}{2} (i - i_0)}{\cos \frac{1}{2} (i + i_0)} \tan \frac{1}{2} (\sigma - \Omega_0), \tag{191}$$

by means of which the value of Ω may be found. This equation gives, when we neglect terms of the third order,

$$\Omega = \Omega_0 + \Gamma + \frac{\sigma - \Omega_0}{\cos i_0} + \frac{\sin i_0}{2 \cos^3 i_0} (i - i_0) (\sigma - \Omega_0). \tag{192}$$

Substituting in this the values of $\sigma - \Omega_0$ and $i - i_0$ given by (190), we get

$$\Omega = \Omega_0 + \frac{p'}{\sin i_0 \cos i_0} s - \frac{1 - \frac{3}{2} \sin^2 i_0}{\sin^2 i_0 \cos^3 i_0} p'q's + \Gamma, \tag{193}$$

Γ being expressed in seconds of arc. Finally, for the longitude of the perihelion, we have

$$\pi = \chi + \Omega - \sigma, \quad (194)$$

and the elements of the instantaneous orbit are completely determined. When we neglect terms of the third order, this equation, substituting the values given by (190) and (192), becomes

$$\pi = \chi + \frac{\tan \frac{1}{2} i_0}{\cos i_0} p's + \frac{\tan^2 \frac{1}{2} i_0 (1 + 2 \cos i_0)}{2 \cos^3 i_0} p'q's + \Gamma. \quad (195)$$

It should also be observed that the inclination i which appears in these formulæ is supposed to be susceptible of any value from 0° to 180° , and hence when i exceeds 90° and the elements are given in accordance with the distinction of retrograde motion, they are to be changed to the general form by using $180^\circ - i$ instead of i , and $2\Omega - \pi$ instead of π .

The accuracy of the numerical process may be checked by computing the heliocentric place of the body for the date to which the new elements belong by means of these elements, and comparing the results with those obtained directly by means of the equations (155). We may remark, also, that when the inclination does not differ much from 90° , the reduction of the longitudes to the fundamental plane becomes uncertain, and Γ may be very large, and hence, instead of the ecliptic, the equator must be taken as the fundamental plane to which the elements and the longitudes are referred.

192. Although, by means of the formulæ which have been given, the complete perturbations may be determined for a very long period of time, using constantly the same osculating elements, yet, on account of the ease with which new elements may be found from δM , ν , δz , $\frac{d\delta M}{dt}$, $\frac{d\nu}{dt}$, and $\frac{d\delta z}{dt}$, and on account of the facility afforded in the calculation of the indirect terms in the equations for the differential coefficients so long as the values of the perturbations are small, it is evident that the most advantageous process will be to compute δM , ν , and δz , only with respect to the first power of the disturbing force, and determine new osculating elements whenever the terms of the second order must be considered. Then the integration will again commence with zero, and will be continued until, on account of the terms of the second order, another change of the elements is required. The frequency of this transformation will necessarily de-

pend on the magnitude of the disturbing force; and if the disturbed body is so near the disturbing body that a very frequent change of the elements becomes necessary, it may be more convenient either to include the terms of the second order directly in the computation of the values of δM , ν , and δz , or to adopt one of the other methods which have been given for the determination of the perturbations of a heavenly body. In the case of the asteroid planets, the consideration of the terms of the second order in this manner will only require a change of the osculating elements after an interval of several years, and whenever this transformation shall be required, the equations for φ , i , Ω , and π , in which the terms of the third order are neglected, may be employed. It should be observed, however, that the perturbations of some of the elements are much greater than the perturbations of the co-ordinates, and hence when terms depending on the squares and higher powers of the masses have been neglected in the computation of these perturbations, it may still be necessary to include the values of the terms of the second order in the incomplete equations referred to. No general criterion can be given as to the time at which a change of the osculating elements will be required; but when, on account of the magnitude of the values of δM , ν , and δz , it appears probable that the perturbations of the second order ought to be included in the results, by computing a single place, taking into account the neglected terms, we may at once determine whether such is the case and whether new elements are required.

193. We have already found the expressions for the variations of Ω and i due to the action of the disturbing forces, and we shall now consider those for the variation of the other elements of the orbit directly. Let x , y , z be the co-ordinates of the body at any given time referred to any fixed system of co-ordinates. These will be known functions of the six elements of the orbit and of the time. If the body were not subject to the action of the disturbing forces, these six elements would be rigorously constant, and the co-ordinates would vary only with the time; but on account of the action of these forces the elements must be regarded as continuously varying in order that the relation between the elements and the co-ordinates at any instant shall be expressed by equations of the same form as in the case of the undisturbed motion. The co-ordinates will, therefore, in the disturbed motion, be subject to two distinct variations: that which results from considering the time alone to vary, and that which

results from the variation of the elements themselves. Let these two kinds of partial variations be symbolized respectively by $\left(\frac{dx}{dt}\right)$ and $\left[\frac{dx}{dt}\right]$, and similarly in the case of the other co-ordinates; then will the total variations be given by

$$\begin{aligned}\frac{dx}{dt} &= \left(\frac{dx}{dt}\right) + \left[\frac{dx}{dt}\right], & \frac{dy}{dt} &= \left(\frac{dy}{dt}\right) + \left[\frac{dy}{dt}\right], \\ \frac{dz}{dt} &= \left(\frac{dz}{dt}\right) + \left[\frac{dz}{dt}\right].\end{aligned}\tag{196}$$

But if we differentiate twice in succession the equations which express the values of x , y , and z as functions of the elements and of the time, regarding both the elements and the time as variable, the substitution of the results in the general equations for the motion of the disturbed body will furnish three equations for the determination of the variations of the elements. There are, however, six unknown quantities to be determined; and hence we may assign arbitrarily three other equations of condition. The supposition which affords the required facility in the solution of the problem is that

$$\left[\frac{dx}{dt}\right] = 0, \quad \left[\frac{dy}{dt}\right] = 0, \quad \left[\frac{dz}{dt}\right] = 0, \tag{197}$$

and hence that

$$\frac{dx}{dt} = \left(\frac{dx}{dt}\right), \quad \frac{dy}{dt} = \left(\frac{dy}{dt}\right), \quad \frac{dz}{dt} = \left(\frac{dz}{dt}\right).$$

It thus appears that in order that the integrals of the equations (1) shall be of the same form as those of the equations (3),—the arbitrary constants of integration which result from the integration of the latter being regarded as variable when the disturbing forces are considered,—the first differential coefficients of the co-ordinates with respect to the time have the same form in the disturbed and undisturbed orbits. But since $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ are the velocities of the disturbed body in directions parallel to the co-ordinate axes respectively, it follows that during the element of time dt the velocity of the body must be regarded as constant, and as receiving an increment only at the end of this instant. The equations (197) show also that if we differentiate any co-ordinate, rectangular or polar, referred to a

fixed plane and measured from a fixed origin, with respect to the elements alone considered as variable, the first differential coefficient must be put equal to zero, and this enables us at once to effect the solution of the problem under consideration. It is to be observed, further, that the functions whose first differential coefficients with respect to the time when only the elements are regarded as variable are thus put equal to zero, must not involve directly the motion of the disturbed body, since the second differential coefficients of the co-ordinates have not the same form in the case of the disturbed motion as in that of the undisturbed motion.

194. If we suppose the disturbing force to be resolved into three components, namely, R in the direction of the disturbed radius-vector, S in a direction perpendicular to the radius-vector and in the plane of disturbed orbit, positive in the direction of the motion, and Z perpendicular to the plane of the instantaneous orbit, the latter will only vary Ω and i and the longitude of the perihelion so far as it is affected by the change of the place of the node, while the forces R and S will cause the elements M , π , e , and a to vary without affecting Ω and i .

Let us now differentiate the equation

$$V^2 = k^2 (1 + m) \left(\frac{2}{r} - \frac{1}{a} \right),$$

regarding the elements as variable, and we get

$$\frac{2}{r^2} \left[\frac{dr}{dt} \right] = -\frac{1}{a^2} \cdot \frac{da}{dt} + \frac{2V}{k^2 (1 + m)} \cdot \frac{dV}{dt} = 0,$$

or

$$\frac{da}{dt} = \frac{2a^2 V}{k^2 (1 + m)} \cdot \frac{dV}{dt}.$$

The differential coefficient $\frac{dV}{dt}$ is here the increment of the accelerating force, in the direction of the tangent to the orbit at the given point, due to the action of the disturbing force; and if we designate the angle which the tangent makes with the prolongation of the radius-vector by ϕ_0 , we shall have

$$\frac{dV}{dt} = R \cos \phi_0 + S \sin \phi_0.$$

Substituting this value in the preceding equation, we obtain

$$\frac{ua}{dt} = \frac{2a^2}{k^2(1+m)} (RV \cos \phi_0 + SV \sin \phi_0).$$

But we have, according to the equations (50)₆,

$$V \cos \phi_0 = \left(\frac{dr}{dt} \right) = \frac{k\sqrt{1+m}}{\sqrt{p}} e \sin v,$$

$$V \sin \phi_0 = r \left(\frac{dv}{dt} \right) = \frac{k\sqrt{p(1+m)}}{r},$$

in which v denotes the true anomaly in the instantaneous orbit; and hence there results

$$\frac{da}{dt} = \frac{2a^2}{k\sqrt{p(1+m)}} \left(e \sin v R + \frac{p}{r} S \right), \quad (198)$$

by means of which the variation of a may be found.

If we introduce the mean daily motion μ , we shall have

$$\frac{d\mu}{dt} = -\frac{\frac{3}{2}\mu}{a} \cdot \frac{da}{dt}, \quad (199)$$

and hence

$$\frac{d\mu}{dt} = -\frac{3a\mu}{k\sqrt{p(1+m)}} \left(e \sin v R + \frac{p}{r} S \right), \quad (200)$$

for the determination of $\delta\mu$.

The first of the equations (97) gives

$$\frac{d}{dt} \left(r^2 \frac{dv}{dt} \right) = Sr;$$

and hence we obtain

$$\frac{d(\sqrt{p})}{dt} = \frac{Sr}{k\sqrt{1+m}},$$

or

$$\frac{dp}{dt} = \frac{2\sqrt{p} \cdot r}{k\sqrt{1+m}} S. \quad (201)$$

The equation $p = a(1 - e^2)$ gives

$$\frac{dp}{dt} = \frac{p}{a} \cdot \frac{da}{dt} - 2ae \frac{de}{dt}.$$

Equating these values of $\frac{dp}{dt}$, and introducing the value of $\frac{da}{dt}$ already found, we get

$$\frac{de}{dt} = \frac{1}{k\sqrt{p(1+m)}} \left(p \sin v R + \frac{p}{e} \left(\frac{p}{r} - \frac{r}{a} \right) S \right), \quad (202)$$

and since

$$\frac{p}{r} = 1 + e \cos v, \quad \frac{r}{a} = 1 - e \cos E,$$

E being the eccentric anomaly in the instantaneous orbit, this becomes

$$\frac{de}{dt} = \frac{1}{k\sqrt{p(1+m)}} (p \sin v R + p (\cos v + \cos E) S), \quad (203)$$

which will give the variation of e . If we introduce the angle of eccentricity φ , we shall have

$$\frac{de}{dt} = \cos \varphi \frac{d\varphi}{dt}, \quad p = a \cos^2 \varphi,$$

and hence

$$\frac{d\varphi}{dt} = \frac{1}{k\sqrt{p(1+m)}} (a \cos \varphi \sin v R + a \cos \varphi (\cos v + \cos E) S). \quad (204)$$

195. When we consider only the components R and S of the disturbing force, the longitude in the orbit will be

$$\lambda, = v + \chi.$$

We have, therefore,

$$\frac{p}{r} = 1 + e \cos (\lambda, - \chi),$$

the differentiation of which, regarding the elements as variable, gives

$$\begin{aligned} \frac{dp}{dt} - \frac{p}{r} \left[\frac{dr}{dt} \right] &= r \cos (\lambda, - \chi) \frac{de}{dt} - er \sin (\lambda, - \chi) \left[\frac{d\lambda,}{dt} \right] \\ &\quad + er \sin (\lambda, - \chi) \frac{d\chi}{dt}, \end{aligned}$$

or

$$\frac{dp}{dt} = r \cos v \frac{de}{dt} + er \sin v \frac{d\chi}{dt}.$$

Therefore

$$\frac{d\chi}{dt} = \frac{1}{k\sqrt{p(1+m)}} \cdot \frac{1}{e} (-p \cos v R + \frac{p}{\sin v} (2 - \cos^2 v - \cos v \cos E) S),$$

and, since $p \cos E = r (\cos v + e)$, we have

$$p (1 - \cos v \cos E) = r \sin^2 v,$$

so that the equation becomes

$$\frac{d\chi}{dt} = \frac{1}{k\sqrt{p(1+m)}} \cdot \frac{1}{e} (-p \cos v R + (p+r) \sin v S), \quad (205)$$

from which the value of $\frac{d\chi}{dt}$ may be derived.

If we introduce the element ω , or the angular distance of the perihelion from the ascending node, it will be necessary to consider also the component Z ; and, since $\omega = \chi - \sigma$, we shall have

$$\frac{d\omega}{dt} = \frac{d\chi}{dt} - \frac{d\sigma}{dt} = \frac{d\chi}{dt} - \cos i \frac{d\Omega}{dt},$$

and hence

$$\frac{d\omega}{dt} = \frac{1}{k\sqrt{p(1+m)}} \cdot \frac{1}{e} (-p \cos v R + (p+r) \sin v S) - \cos i \frac{d\Omega}{dt}. \quad (206)$$

In the case of the longitude of the perihelion, we have

$$\frac{d\pi}{dt} = \frac{d\omega}{dt} + \frac{d\Omega}{dt},$$

and therefore

$$\begin{aligned} \frac{d\pi}{dt} = \frac{1}{k\sqrt{p(1+m)}} \cdot \frac{1}{e} (-p \cos v R + (p+r) \sin v S) \\ + 2 \sin^2 \frac{1}{2} i \frac{d\Omega}{dt}. \end{aligned} \quad (207)$$

The first of the equations (15)₂ gives

$$\left[\frac{dr}{dt} \right] = a \tan \varphi \sin v \left(\frac{dM_0}{dt} + (t-t_0) \frac{d\mu}{dt} \right) - \frac{2r}{3\mu} \cdot \frac{d\mu}{dt} - a \cos v \frac{de}{dt} = 0,$$

in which M_0 denotes the mean anomaly at the epoch, which is usually adopted as one of the elements in the case of an elliptic orbit. Sub-

stituting for $\frac{d\mu}{dt}$ and $\frac{de}{dt}$ the values already found, we get

$$\begin{aligned} \frac{dM_0}{dt} = \frac{1}{k\sqrt{p(1+m)}} \{ (p \cot \varphi \cos v - 2r \cos \varphi) R \\ - \frac{p}{\sin v} (2 - \cos^2 v - \cos v \cos E) \cot \varphi S \} - (t-t_0) \frac{d\mu}{dt}, \end{aligned}$$

or

$$\begin{aligned} \frac{dM_0}{dt} = \frac{1}{k\sqrt{p(1+m)}} \{ (p \cot \varphi \cos v - 2r \cos \varphi) R - (p+r) \cot \varphi \sin v S \} \\ - (t-t_0) \frac{d\mu}{dt}. \end{aligned} \quad (208)$$

The equation (205) gives

$$\frac{1}{k\sqrt{p(1+m)}}(p+r)\cot\varphi\sin vS = \frac{1}{k\sqrt{p(1+m)}}p\cot\varphi\cos vR \\ + \cos\varphi\frac{d\chi}{dt},$$

by means of which (208) reduces to

$$\frac{dM_0}{dt} = -\cos\varphi\frac{d\chi}{dt} - \frac{2r\cos\varphi}{k\sqrt{p(1+m)}}R - (t-t_0)\frac{d\mu}{dt}, \quad (209)$$

which will determine the variation of the mean anomaly at the epoch.

Since the equations for the determination of the place of the body in the case of the disturbed motion are of the same form as those for the undisturbed motion, the mean anomaly at the time t will be given by

$$M = M_0 + \delta M_0 + (t-t_0)(\mu_0 + \delta\mu),$$

in which μ_0 denotes the mean daily motion at the instant t_0 . Therefore we shall have

$$M = M_0 + \int \frac{dM_0}{dt} dt + \mu_0(t-t_0) + (t-t_0) \int \frac{d\mu}{dt} dt,$$

the integrals being taken between the limits t_0 and t . The quantity

$$M_0 + \mu_0(t-t_0)$$

expresses the mean anomaly at the time t in the undisturbed orbit; and if we designate by δM the correction to be applied to this in order to obtain the mean anomaly in the disturbed orbit, so that

$$\delta M = \int_{t_0}^t \frac{dM}{dt} dt,$$

we shall have

$$M = M_0 + \mu_0(t-t_0) + \int \frac{dM}{dt} dt,$$

and hence

$$\int \frac{dM}{dt} dt = \int \frac{dM_0}{dt} dt + (t-t_0) \int \frac{d\mu}{dt} dt.$$

Differentiating this with respect to t , we get

$$\frac{dM}{dt} = \frac{dM_0}{dt} + (t-t_0)\frac{d\mu}{dt} + \int \frac{d\mu}{dt} dt.$$

Substituting in this the value of $\frac{dM_0}{dt}$ from (209), the result is

$$\frac{dM}{dt} = -\cos \varphi \frac{d\chi}{dt} - \frac{2r \cos \varphi}{kV_p(1+m)} R + \int \frac{d\mu}{dt} dt, \quad (210)$$

which does not involve the factor $t - t_0$ explicitly, and by means of which the mean anomaly in the disturbed orbit, at any instant t , may be found directly from that for the same instant in the undisturbed orbit.

To find the variation of the mean longitude L , we have

$$\frac{dL}{dt} = \frac{dM}{dt} + \frac{d\pi}{dt} = \frac{d\chi}{dt} + \frac{dM}{dt} + (1 - \cos i) \frac{d\Omega}{dt},$$

and therefore

$$\frac{dL}{dt} = 2 \sin^2 \frac{1}{2} \varphi \frac{d\chi}{dt} + 2 \sin^2 \frac{1}{2} i \frac{d\Omega}{dt} - \frac{2r \cos \varphi}{kV_p(1+m)} R + \int \frac{d\mu}{dt} dt. \quad (211)$$

To find the variations of Ω and i , since

$$u = \lambda, -\sigma,$$

u denoting the argument of the latitude in the disturbed orbit, we have, according to the equations (169) and (170),

$$\begin{aligned} \frac{d\Omega}{dt} &= \frac{1}{kV_p(1+m)} \cdot \frac{r \sin u}{\sin i} Z, \\ \frac{di}{dt} &= \frac{1}{kV_p(1+m)} r \cos u Z. \end{aligned} \quad (212)$$

The inclination i may have any value from 0° to 180° ; and whenever the elements are given in accordance with the distinction of retrograde motion, they must be converted into those of the general form by taking $180^\circ - i$ in place of the given value of i , and $2\Omega - \pi$ in place of the given value of π , before applying the formulæ which involve these elements.

196. In the case of the orbits of comets in which the eccentricity differs but little from that of the parabola, the perturbations of the perihelion distance q and of the time of perihelion passage T will be determined instead of those of the elements M and a or μ .

The equation

$$p = q(1 + e)$$

gives

$$\frac{dq}{dt} = \frac{1}{1+e} \cdot \frac{dp}{dt} - \frac{q}{1+e} \cdot \frac{de}{dt},$$

and substituting in this the value of $\frac{dp}{dt}$ already found, and neglecting the mass of the comet, which is always inconsiderable, we get

$$\frac{dq}{dt} = \frac{2qr}{k\sqrt{p}} S - \frac{q}{1+e} \cdot \frac{de}{dt}, \quad (213)$$

by means of which the variation of q may be found. In the case of elliptic motion the value of $\frac{de}{dt}$ may be found by means of (202) or (203); but in the case of hyperbolic motion the equation (202) will be employed. It should be observed, also, that when the general formulæ for the ellipse are applied to the hyperbola, the semi-transverse axis a must be considered negative.

When the orbit is a parabola, the equation (202) becomes

$$\frac{de}{dt} = \frac{1}{k\sqrt{p}} (p \sin v R + 2p \cos^2 \frac{1}{2} v S), \quad (214)$$

and for the value of $\frac{dq}{dt}$ we have

$$\frac{dq}{dt} = \frac{2qr}{k\sqrt{p}} S - \frac{1}{2} q \frac{de}{dt}. \quad (215)$$

It remains now to find the formula for the variation of the time of perihelion passage. The relation between T and M_0 is expressed by

$$360^\circ - M_0 = \mu (T - t_0),$$

the differentiation of which gives

$$-\frac{dM_0}{dt} = (T - t_0) \frac{d\mu}{dt} + \mu \frac{dT}{dt};$$

and, substituting for $\frac{dM_0}{dt}$ the value given by equation (209), we get

$$\frac{dT}{dt} = \frac{2ar}{k^2} R + \frac{a\sqrt{p}}{k} \cdot \frac{d\chi}{dt} + (t - T) \frac{1}{\mu} \cdot \frac{d\mu}{dt}.$$

Substituting further the values of $\frac{d\chi}{dt}$ and $\frac{d\mu}{dt}$ given by the equations (205) and (199), the result is

$$\begin{aligned} \frac{dT}{dt} = & \frac{aR}{k^2} \left(2r - \frac{p}{e} \cos v - \frac{3k(t-T)}{\sqrt{p}} e \sin v \right) \\ & + \frac{aS}{k^2} \left(\frac{p+r}{e} \sin v - \frac{3k(t-T)}{\sqrt{p}} \cdot \frac{p}{r} \right), \end{aligned} \quad (216)$$

which may be employed to determine the variation of T whenever the eccentricity is not very nearly equal to unity. It is obvious, however, that when a is very large this equation will not be convenient for numerical calculation, and hence a further transformation of it is desirable. Thus, if we derive the expressions for $\frac{dr}{de}$ and $\frac{dv}{de}$ from the equations (24)₂ and (23)₂, we easily obtain

$$\begin{aligned} \frac{2p}{1+e} \cdot \frac{dr}{de} &= a \left(2r - \frac{p}{e} \cos v - \frac{3k(t-T)}{\sqrt{p}} e \sin v \right) + \frac{p^2}{e(1+e)^2} \cos v, \\ \frac{2p}{1+e} r \frac{dv}{de} &= a \left(\frac{p+r}{e} \sin v - \frac{3k(t-T)}{\sqrt{p}} \cdot \frac{p}{r} \right) - \frac{p^2}{e(1+e)^2} \left(1 + \frac{r}{p} \right) \sin v. \end{aligned}$$

By means of these results the equation (216) is transformed into

$$\frac{dT}{dt} = \frac{qR}{k^2} \left(2 \frac{dr}{de} - \frac{q}{e} \cos v \right) + \frac{qS}{k^2} \left(2r \frac{dv}{de} + \frac{q}{e} \left(1 + \frac{r}{p} \right) \sin v \right), \quad (217)$$

which may be used for the determination of $\frac{dT}{dt}$, the values of $\frac{dr}{de}$ and $\frac{dv}{de}$ being found by means of the various formulæ developed in Art. 50. When a is very large, its reciprocal denoted by f may often be conveniently introduced as one of the elements, and, for the determination of the variation of f , we derive from equation (198)

$$\frac{df}{dt} = - \frac{2}{k\sqrt{p}} (e \sin v R + \frac{p}{r} S). \quad (218)$$

In the case of parabolic motion we have $e=1$, and $p=2q$; and if we substitute in (217) for $\frac{dr}{de}$ and $\frac{dv}{de}$ the values given by the equations (33)₂ and (30)₂, the result is

$$\begin{aligned} \frac{dT}{dt} = & \frac{q^2}{1 + \tan^2 \frac{1}{2} v} \left(\frac{R}{k^2} \left(-1 + 3 \tan^2 \frac{1}{2} v + \tan^4 \frac{1}{2} v + \frac{1}{5} \tan^6 \frac{1}{2} v \right) \right. \\ & \left. + \frac{S}{k^2} \left(4 \tan \frac{1}{2} v - \frac{4}{3} \tan^3 \frac{1}{2} v \right) \right). \end{aligned} \quad (219)$$

197. Instead of the elements usually employed, it may be desirable, in rare and special cases, to introduce other combinations of the elements or constants which determine the circumstances of the undisturbed motion, and the relation between the new elements adopted and those for which the expressions for the differential coefficients have been given, will furnish immediately the necessary formulæ. In the case of the periodic comets, it will often be desired to determine the alteration of the periodic time arising from the action of the disturbing planets. Let us, therefore, suppose that a comet has been identified at two successive returns to the perihelion, and let τ denote the elapsed interval. The observations at each appearance of the comet, however extended they may be, will not indicate with certainty the semi-transverse axis of the orbit, and hence the periodic time. But when τ is known, by eliminating the effect of the disturbing forces, we may determine with accuracy the value of the semi-transverse axis a at each epoch, and, from this and the observed places, the other elements of the orbit according to the process already explained.

Let μ_0 be the mean daily motion at the first epoch, and we shall have

$$\mu_0\tau + \int \frac{dM}{dt} dt = 2\pi,$$

in which π denotes the semi-circumference of a circle whose radius is unity. Hence we obtain

$$\mu_0 = \frac{2\pi - \int \frac{dM}{dt} dt}{\tau}, \quad (220)$$

by means of which to determine μ_0 . Then, to find the mean daily motion μ at the instant of the second return to the perihelion, we have

$$\mu = \mu_0 + \int \frac{d\mu}{dt} dt, \quad (221)$$

the integral being taken between the limits 0 and τ . The provisional value of the mean motion as given by the observed interval τ will be sufficiently accurate for the calculation of the variations of M and μ during this interval. The semi-transverse axis will now be derived by means of the formula

$$a = \sqrt[3]{\frac{k^2}{\mu^3}},$$

from the values of μ for the two epochs. Let τ' denote the interval which must elapse before the next succeeding perihelion passage of the comet, and we have

$$2\pi = \mu\tau' + \int \frac{dM}{dt} dt,$$

and consequently

$$\tau' = \frac{2\pi - \int \frac{dM}{dt} dt}{\mu}, \quad (222)$$

the integral being taken between the limits $t=0$, corresponding to the beginning of the interval, and $t=\tau'$. We have, therefore,

$$\delta\tau = -\frac{1}{\mu} \int \frac{dM}{dt} dt, \quad (223)$$

for the change of the periodic time due to the action of the disturbing forces.

198. The calculation of the values of the components R , S , and Z of the disturbing force will be effected by means of the formulæ given in Art. 182. It will be observed, however, that not only these components of the disturbing force, but also their coefficients in the expressions for the differential coefficients, involve the variable elements, and hence the perturbations which are sought. But if we consider only the perturbations of the first order, the fundamental osculating elements may be employed in place of the actual variable elements, and whenever the perturbations of the second order have a sensible influence, the elements must be corrected for the terms of the first order already obtained. Then, commencing the integration anew at the instant to which the corrected elements belong, the calculation may be continued until another change of the elements becomes necessary. The several quantities required in the computation of the forces may also be corrected from time to time as the elements are changed.

The frequency with which the elements must be changed in order to include in the results all the terms which have a sensible influence in the determination of the place of the disturbed body, will depend entirely on the circumstances of each particular case. In the case of the asteroid planets this change will generally be required only after an interval of about a year; but when the planet approaches very near to Jupiter, the interval may necessarily be much shorter. The

magnitude of the resulting values of the perturbations will suggest the necessity of correcting the elements whenever it exists; and if we apply the proper corrections and commence anew the integration for one or more intervals preceding the last date for which the perturbations of the first order have been found, it will appear at once, by a comparison of the results, whether the elements have too long been regarded as constant.

The intervals at which the differential coefficients must be computed directly, will also depend on the relation of the motion of the disturbing body to that of the disturbed body; and although the interval may be greater than in the case of the variations of the co-ordinates which require an indirect calculation, still it must not be so large that the places of both the disturbing and the disturbed body, as well as the values of the several functions involved, cannot be interpolated with the requisite accuracy for all intermediate dates. In the case of the asteroid planets a uniform interval of about forty days will generally be preferred; but in the case of the comets, which rapidly approach the disturbing body and then again rapidly recede from it, the magnitude of the proper interval for quadrature will be very different at different times, and the necessity of shortening the interval, or the admissibility of extending it, will be indicated, as the numerical calculation progresses, by the manner in which the several functions change value.

If we compute the forces for several disturbing bodies by using ΣR , ΣS , and ΣZ in the formulæ in place of R , S , and Z , respectively, the total perturbations due to the combined action of all of these bodies may be computed at once. But, although the numerical process is thus somewhat abbreviated, yet, if the adopted values of the masses of some of the disturbing bodies are uncertain, and it is desired subsequently to correct the results by means of corrected values of these masses, it will be better to compute the perturbations due to each disturbing body separately, and, since a large part of the numerical process remains unchanged, the additional labor will not be very considerable, especially when, for some of the disturbing bodies, the interval of quadrature may be extended. The successive correction of the elements in order to include in the results the perturbations due to the higher powers of the masses, must, however, involve the perturbations due to all the disturbing bodies considered.

The differential coefficients should be multiplied by the interval ω , so that the formulæ of integration, omitting this factor, will furnish directly the required integrals; and whenever a change of the inter-

val is introduced, the proper caution must be observed in regard to the process of integration. The quantity $s = 206264''.8$ should be introduced into the formulæ in such a manner that the variations of the elements which are expressed in angular measure will be obtained directly in seconds of arc; and the variations of the other elements will be conveniently determined in units of the n th decimal place. It should be observed, also, that if the constants of integration are put equal to zero at the beginning of the integration, the integrals obtained will be the required perturbations of the elements.

199. EXAMPLE.—We shall now illustrate the calculation of the perturbations of the elements by a numerical example, and for this purpose we shall take that which has already been solved by the other methods which have been given. From 1864 Jan. 1.0 to 1865 Jan. 15.0 the perturbations of the second order are insensible, and hence during the entire period it will be sufficient to use the values of r , v , and E given by the osculating elements for 1864 Jan. 1.0.

The calculation of the forces R , S , and Z is effected precisely as already illustrated in Art. 189, and from the results there given we obtain the following values of the forces, with which we write also the values of E_0 :—

| Berlin Mean Time. | | 40R | 40S | 40Z | E_0 |
|-------------------|--|------------|------------|-------------|----------------|
| 1863 Dec. 12.0, | | + 0''.0365 | + 0''.0019 | + 0''.00002 | 355° 26' 8''.2 |
| 1864 Jan. 21.0, | | 0 .0356 | — 0 .0086 | 0 .00025 | 8 14 57 .8 |
| March 1.0, | | 0 .0315 | 0 .0182 | 0 .00047 | 20 57 55 .1 |
| April 10.0, | | 0 .0250 | 0 .0259 | 0 .00068 | 33 26 47 .6 |
| May 20.0, | | 0 .0169 | 0 .0314 | 0 .00087 | 45 35 25 .3 |
| June 29.0, | | + 0 .0079 | 0 .0343 | 0 .00101 | 57 20 3 .8 |
| Aug. 8.0, | | — 0 .0011 | 0 .0349 | 0 .00112 | 68 39 14 .6 |
| Sept. 17.0, | | 0 .0099 | 0 .0333 | 0 .00117 | 79 33 13 .1 |
| Oct. 27.0, | | 0 .0179 | 0 .0301 | 0 .00116 | 90 3 23 .2 |
| Dec. 6.0, | | 0 .0252 | 0 .0253 | 0 .00108 | 100 11 49 .1 |
| 1865 Jan. 15.0, | | — 0 .0317 | — 0 .0193 | + 0 .00090 | 110 0 54 .3 |

We compute the values of the required differential coefficients by means of the equations

$$\frac{d\delta\Omega}{dt} = \frac{1}{k\sqrt{p}} \cdot \frac{r \sin u}{\sin i} Z, \quad \frac{d\delta i}{dt} = \frac{1}{k\sqrt{p}} r \cos u Z,$$

$$\frac{d\delta\pi}{dt} = \frac{1}{k\sqrt{p}} \left(-\frac{p \cos v}{\sin \varphi} R + \frac{(p+r) \sin v}{\sin \varphi} S \right) + 2 \sin^2 \frac{1}{2} i \frac{d\delta\Omega}{dt},$$

$$\frac{d\delta\varphi}{dt} = \frac{1}{kVp} (a \cos \varphi \sin vR + a \cos \varphi (\cos v + \cos E) S),$$

$$\frac{d\delta\mu}{dt} = -\frac{1}{kVp} \cdot \frac{3a\mu}{s} (\sin \varphi \sin vR + \frac{p}{r} S),$$

$$\frac{d\delta M}{dt} = \frac{1}{kVp} \left(\left(\frac{p \cos v}{\sin \varphi} - 2r \right) R - \frac{(p+r) \sin v}{\sin \varphi} S \right) \cos \varphi + \int \frac{d\delta\mu}{dt} dt;$$

and the results are the following:—

| Date. | 40 $\frac{d\delta\Omega}{dt}$ | 40 $\frac{d\delta i}{dt}$ | 40 $\frac{d\delta\pi}{dt}$ | 40 $\frac{d\delta\phi}{dt}$ | 1600 $\frac{d\delta\mu}{dt}$ | 40 $\int \frac{d\delta\mu}{dt} dt$ | 40 $\frac{d\delta M}{dt}$ |
|-----------------|-------------------------------|---------------------------|----------------------------|-----------------------------|------------------------------|------------------------------------|---------------------------|
| 1863 Dec. 12.0, | -0".004 | -0".001 | -16".730 | +0".022 | -0".0790 | +0".027 | +11".092 |
| 1864 Jan. 21.0, | 0.108 | 0.017 | 17.255 | -0.992 | +0.4524 | 0.162 | 11.864 |
| March 1.0, | 0.302 | 0.026 | 19.578 | 1.810 | 0.9396 | 0.863 | 15.381 |
| April 10.0, | 0.555 | 0.028 | 22.986 | 2.294 | 1.3321 | 2.008 | 20.746 |
| May 20.0, | 0.822 | 0.022 | 26.572 | 2.418 | 1.6169 | 3.492 | 26.898 |
| June 29.0, | 1.037 | -0.007 | 29.271 | 2.228 | 1.7750 | 5.198 | 32.617 |
| Aug. 8.0, | 1.189 | +0.012 | 30.698 | 1.829 | 1.8196 | 7.004 | 37.293 |
| Sept. 17.0, | 1.233 | 0.033 | 30.500 | 1.406 | 1.7591 | 8.801 | 40.445 |
| Oct. 27.0, | 1.169 | 0.052 | 28.953 | 1.055 | 1.6206 | 10.498 | 42.144 |
| Dec. 6.0, | 1.004 | 0.065 | 26.498 | 0.902 | 1.4074 | 12.017 | 42.741 |
| 1865 Jan. 15.0, | -0.742 | +0.066 | -23.336 | -1.004 | +1.1388 | +13.292 | +42.323 |

The values thus obtained give, by means of the formulæ for integration by mechanical quadrature, the following perturbations of the elements:—

| Berlin Mean Time. | $\delta\Omega$ | δi | $\delta\pi$ | $\delta\phi$ | $\delta\mu$ | δM |
|-------------------|----------------|------------|-------------|--------------|-------------|------------|
| 1863 Dec. 12.0, | +0".01 | -0".00 | +8".43 | +0".12 | +0".0007 | -5".48 |
| 1864 Jan. 21.0, | -0.04 | 0.01 | -8.49 | -0.38 | 0.0040 | +5.72 |
| March 1.0, | 0.24 | 0.03 | 26.78 | 1.80 | 0.0216 | 19.15 |
| April 10.0, | 0.66 | 0.06 | 48.01 | 3.88 | 0.0502 | 37.11 |
| May 20.0, | 1.35 | 0.08 | 72.82 | 6.27 | 0.0875 | 60.91 |
| June 29.0, | 2.28 | 0.10 | 100.83 | 8.61 | 0.1299 | 90.73 |
| Aug. 8.0, | 3.40 | 0.09 | 130.93 | 10.65 | 0.1751 | 125.79 |
| Sept. 17.0, | 4.63 | 0.07 | 161.66 | 12.26 | 0.2200 | 164.79 |
| Oct. 27.0, | 5.84 | -0.03 | 191.48 | 13.48 | 0.2624 | 206.19 |
| Dec. 6.0, | 6.93 | +0.03 | 219.27 | 14.44 | 0.3004 | 248.72 |
| 1865 Jan. 15.0, | -7.81 | +0.10 | -244.24 | -15.37 | +0.3323 | +291.33 |

Applying the variations of the elements thus obtained to the osculating elements for 1864 Jan. 1.0, as given in Art. 166, the osculating elements for the instant 1865 Jan. 15.0 are found to be the following:—

Epoch = 1865 Jan. 15.0 Berlin mean time.

$M = 99^\circ 34' 48''.81$

$\pi = 44 \ 13 \ 7.93$
 $\Omega = 206 \ 38 \ 57.88$
 $i = 4 \ 36 \ 52.21$

} Ecliptic and Mean
 Equinox 1860.0.

$\varphi = 11 \ 15 \ 35.65$

$\log a = 0.3880283$

$\mu = 928''.8897.$

In order to compare the results thus derived with the perturbations computed by the other methods which have been given, let us compute the heliocentric longitude and latitude, in the case of the disturbed orbit, for the date 1865 Jan. 15.0, Berlin mean time. Thus, by means of the new elements, we find

$$\begin{array}{ll} M = 99^\circ 34' 48''.81, & E = 110^\circ 5' 14''.15, \\ \log r = 0.4162182, & v = 120 19 18 .01, \\ l = 164^\circ 37' 59''.04, & b = -3 5 32 .54, \end{array}$$

agreeing completely with the results already obtained by the other methods. The heliocentric place thus found is referred to the ecliptic and mean equinox of 1860.0, to which the elements π , Ω , and i are referred; and it may be reduced to any other ecliptic and equinox by means of the usual formulæ. Throughout the calculation of the perturbations it will be convenient to adopt a fixed equinox and ecliptic, the results being subsequently reduced by the application of the corrections for precession and nutation.

In the determination of δM , if we denote by ΔM the value which is obtained when we neglect the last term of the equation for $\frac{d\delta M}{dt}$, we shall have

$$\delta M = \Delta M + \iint \frac{d\delta\mu}{dt} dt^2,$$

which form is equally convenient in the numerical calculation. Thus, for 1865 Jan. 15.0, we find

$$\Delta M = + 234''.74,$$

and from the several values of $1600 \frac{d\delta\mu}{dt}$ we obtain, for the same date, by means of the formula for double integration,

$$\iint \frac{d\delta\mu}{dt} dt^2 = + 56''.59.$$

Hence we derive

$$\delta M = + 234''.74 + 56''.59 = + 291''.33,$$

agreeing with the result already obtained.

If we compute the variation of the mean anomaly at the epoch, by means of equation (209), we find, in the case under consideration,

$$\delta M_0 = + 165''.29,$$

and since the place of the body in the case of the instantaneous orbit is to be computed precisely as if the planet had been moving constantly in that orbit, we have, for 1865 Jan. 15.0,

$$(t - t_0) \delta\mu = + 126''.27,$$

and hence

$$\delta M = \delta M_0 + (t - t_0) \delta\mu = + 291''.56.$$

The error of this result is $-0''.23$, and arises chiefly from the increase of the accidental and unavoidable errors of the numerical calculation by the factor $t - t_0$, which appears in the last term of the equation (209). Hence it is evident that it will always be preferable to compute the variation of the mean anomaly directly; and if the variation of the mean anomaly at a given epoch be required, it may easily be found from δM by means of the equation

$$\delta M_0 = \delta M - (t - t_0) \delta\mu.$$

If the osculating elements of one of the asteroid planets are thus determined for the date of the opposition of the planet, they will suffice, without further change, to compute an ephemeris for the brief period included by the observations in the vicinity of the opposition, unless the disturbed planet shall be very near to Jupiter, in which case the perturbations during the period included by the ephemeris may become sensible. The variation of the geocentric place of the disturbed body arising from the action of the disturbing forces, may be obtained by substituting the corresponding variations of the elements in the differential formulæ as derived from the equation (1)₂, whenever the terms of the second order may be neglected. It should be observed, however, that if we substitute the value of δM directly in the equations for the variations of the geocentric co-ordinates, the coefficient of $\delta\mu$ must be that which depends solely on the variation of the semi-transverse axis. But when the coefficient of $\delta\mu$ has been computed so as to involve the effect of this quantity during the interval $t - t_0$, the value of δM_0 must be found from δM and substituted in the equations.

200. It will be observed that, on account of the divisor e in the expressions for $\frac{d\chi}{dt}$, $\frac{d\omega}{dt}$, and $\frac{d\pi}{dt}$, these elements will be subject to large perturbations whenever e is very small, although the absolute effect on the heliocentric place of the disturbed body may be small; and on

account of the divisor $\sin i$ in the expression for $\frac{d\Omega}{dt}$ the variation of Ω will be large whenever i is very small. To avoid the difficulties thus encountered, new elements must be introduced. Thus, in the case of Ω , let us put

$$\alpha'' = \sin i \sin \Omega, \quad \beta'' = \sin i \cos \Omega; \quad (224)$$

then we shall have

$$\begin{aligned} \frac{d\alpha''}{dt} &= \sin \Omega \cos i \frac{di}{dt} + \sin i \cos \Omega \frac{d\Omega}{dt}, \\ \frac{d\beta''}{dt} &= \cos \Omega \cos i \frac{di}{dt} - \sin i \sin \Omega \frac{d\Omega}{dt}. \end{aligned}$$

Introducing the values of $\frac{di}{dt}$ and $\frac{d\Omega}{dt}$ given by the equations (212), and introducing further the auxiliary constants a, b, A , and B computed by means of the formulæ (94)₁ with respect to the fundamental plane to which Ω and i are referred, we obtain

$$\begin{aligned} \frac{d\alpha''}{dt} &= - \frac{1}{kVp(1+m)} rZ \sin a \cos (A + u), \\ \frac{d\beta''}{dt} &= \frac{1}{kVp(1+m)} rZ \sin b \cos (B + u), \end{aligned} \quad (225)$$

by means of which the variations of α'' and β'' may be found. If the integrals are put equal to zero at the beginning of the integration, the values of $\delta\alpha''$ and $\delta\beta''$ will be obtained, so that we shall have

$$\begin{aligned} \sin i \sin \Omega &= \sin i_0 \sin \Omega_0 + \delta\alpha'', \\ \sin i \cos \Omega &= \sin i_0 \cos \Omega_0 + \delta\beta'', \end{aligned}$$

or

$$\begin{aligned} \sin i \sin (\Omega - \Omega_0) &= \cos \Omega_0 \delta\alpha'' - \sin \Omega_0 \delta\beta'', \\ \sin i \cos (\Omega - \Omega_0) &= \sin i_0 + \sin \Omega_0 \delta\alpha'' + \cos \Omega_0 \delta\beta'', \end{aligned} \quad (226)$$

by means of which i and $\Omega - \Omega_0$ may be found.

In the case of χ , let us put

$$\eta'' = e \sin \chi, \quad \zeta'' = e \cos \chi, \quad (227)$$

and we have

$$\begin{aligned} \frac{d\eta''}{dt} &= \sin \chi \frac{de}{dt} + e \cos \chi \frac{d\chi}{dt}, \\ \frac{d\zeta''}{dt} &= \cos \chi \frac{de}{dt} - e \sin \chi \frac{d\chi}{dt}. \end{aligned}$$

Substituting for $\frac{de}{dt}$ and $\frac{d\chi}{dt}$ the values given by the equations (203) and (205), and reducing, we obtain

$$\begin{aligned}\frac{d\eta''}{dt} &= \frac{1}{kV\sqrt{p(1+m)}} \left(-p \cos(v+\chi) R + \{(p+r) \sin(v+\chi) \right. \\ &\quad \left. + er \sin \chi\} S \right), \\ \frac{d\zeta''}{dt} &= \frac{1}{kV\sqrt{p(1+m)}} \left(p \sin(v+\chi) R + \{(p+r) \cos(v+\chi) \right. \\ &\quad \left. + er \cos \chi\} S \right),\end{aligned}\tag{228}$$

by means of which the values of $\delta\eta''$ and $\delta\zeta''$ may be found. Then we shall have

$$\begin{aligned}e \sin \chi &= e_0 \sin \pi_0 + \delta\eta'', \\ e \cos \chi &= e_0 \cos \pi_0 + \delta\zeta'',\end{aligned}$$

or

$$\begin{aligned}e \sin(\chi - \pi_0) &= \cos \pi_0 \delta\eta'' - \sin \pi_0 \delta\zeta'', \\ e \cos(\chi - \pi_0) &= e_0 + \sin \pi_0 \delta\eta'' + \cos \pi_0 \delta\zeta'',\end{aligned}\tag{229}$$

from which to find e and χ . If, in order to find the variation of π , we write π instead of χ in these formulæ, the terms $+ 2e \cos \pi \sin^2 \frac{1}{2}i \frac{d\Omega}{dt}$ and $- 2e \sin \pi \sin^2 \frac{1}{2}i \frac{d\Omega}{dt}$ must be added to the second members of (228), respectively.

201. By means of the four methods which we have developed and illustrated, the special perturbations of a heavenly body may be determined with entire accuracy, and the choice of the particular method will depend on the circumstances of the case. By computing the perturbations of the elements, correcting these elements as often as may be required, the terms depending on the higher powers of the masses may be included, and no indirect calculation becomes necessary. The frequent correction of the elements will also render insensible the effect of whatever uncertainty remains in regard to their true values. But, since the perturbations of the elements are in general much greater than those of the co-ordinates, the effect of the terms of the second order will be much greater upon the values of the elements than upon those of the co-ordinates. Hence, the frequency with which a change of the elements will be required will fully compensate the labor of the indirect part of the calculation in the case of the perturbations of the co-ordinates.

The determination of the perturbations of the polar co-ordinates r , w , and z , and that of the perturbations δM , ν , and δz , are effected with almost equal facility, especially when the effect of the disturbing forces is to be determined for a long interval of time. If the perturbations are required only for a brief period, it will be preferable to determine δM , ν , and δz , rather than δw , δr , and z , since the indirect part of the calculation will thus be effected with less repetition. In both of these cases the values of the perturbations are generally smaller than in the case of the rectangular co-ordinates, and hence they are less affected by terms of the second order; but on account of the simplicity of the formulæ, even when we include the terms depending on the higher powers of the masses, so long as the magnitude of the values of δx , δy , and δz is not so large as to render troublesome the indirect part of the calculation, the method of the variation of rectangular co-ordinates may be advantageously employed when the perturbations are to be determined for a long period.

By whatever method the perturbations are determined, if the fundamental osculating elements are correct, the final elements of the instantaneous orbit will be the same. But, since the effect of the errors of the elements will differ in degree in the different methods of treating the problem, if these elements are affected with small errors, the agreement of the final osculating elements obtained by the different methods, in connection with the corrections derived by the comparison of observations, may not be complete.

When the disturbed body approaches very near to a disturbing planet, the magnitude of the perturbations will be such as to enable us by means of accurate observations to correct the adopted value of the disturbing mass. In this case the perturbations, computed by means of either of the methods applicable, must be converted into the corresponding perturbations of the geocentric spherical co-ordinates. Let the variation of either of the geocentric co-ordinates arising from the action of the disturbing planet be denoted by $\delta\theta$; then, if we suppose the correct value of the disturbing mass to be $1 + n$ times the assumed value used in computing $\delta\theta$, the corresponding variation of the geocentric spherical co-ordinate will be

$$(1 + n) \delta\theta.$$

The value $\delta\theta$ may be included in the determination of the difference between computation and observation in the formation of the equations of condition for finding the corrections to be applied to the ele-

ments; and, finally, the term $n\delta\theta$ may be added to each of the equations of condition, so that we thus introduce a new unknown quantity n . The solution of all the equations thus formed, by the method of least squares, will then furnish the most probable values of the corrections to be applied to the adopted elements, and also the value of n , by means of which a corrected value of the mass of the disturbing body will be obtained.

202. If the determination of the perturbations of a heavenly body required that all the disturbing bodies in the system should be constantly considered, the labor would be very great. But, fortunately, it so happens that the masses of many of the planets are so small in comparison with that of the sun, that the sphere of their disturbing influence is very much restricted. Thus, in the determination of the perturbations of the asteroid planets, only the action of Mars, Jupiter, and Saturn need be considered; and of these disturbing planets Jupiter exerts the principal influence. It is true, however, that, on account of the elongated form of the orbits of the periodic comets, they may at different times be sensibly disturbed by each of the planets of the system. But since in the remote parts of their orbits they are very distant from many of the disturbing planets, the determination of their perturbations will then be much facilitated by considering them as revolving around the common centre of gravity of the sun and disturbing planet. When the motion is referred to the centre of the sun, the disturbing force is the difference of the direct action of the disturbing body upon the disturbed body and upon the sun; and in the case of those disturbing planets whose periodic time is short, the term which expresses the action upon the sun will change value so rapidly that it will be necessary to adopt small intervals in the direct numerical calculation. But when we refer the motion to the centre of gravity of the system, which does not receive any motion in virtue of the mutual attractions of the bodies which compose the system, that part of the disturbing force which expresses the action of the disturbing planet upon the sun will disappear, and the magnitude of the disturbing force will be less than that of the force which disturbs the motion of the comet relative to the sun, so that the intervals for quadrature may be greatly extended. It will be observed, further, that, if the distance of the comet from the sun is far greater than the distance of the disturbing body, the direct action of the planet upon the comet becomes so small that its effect upon the motion will be quite insignificant. In this case the motion of the

comet will be sensibly the same as the pure elliptic motion around the common centre of gravity of the sun and disturbing planet.

In order to exhibit these principles more clearly, let us denote by ξ, η, ζ , the co-ordinates of the sun referred to the centre of gravity of the system; by x_0, y_0, z_0 , the co-ordinates of the comet; and by x'_0, y'_0, z'_0 , the co-ordinates of the disturbing planet referred to the same origin. Let x, y, z be the co-ordinates of the comet, and x', y', z' those of the planet referred to the centre of the sun; then we shall have

$$\begin{aligned} x_0 &= \xi + x, & y_0 &= \eta + y, & z_0 &= \zeta + z, \\ \xi &= -m'x'_0, & \eta &= -m'y'_0, & \zeta &= -m'z'_0, \end{aligned}$$

and hence

$$\begin{aligned} x &= x_0 + m'x'_0, & y &= y_0 + m'y'_0, & z &= z_0 + m'z'_0, \\ x' &= x'_0 + m'x'_0, & y' &= y'_0 + m'y'_0, & z' &= z'_0 + m'z'_0, \\ r' &= r'_0 + m'r'_0. \end{aligned}$$

From these we derive

$$\xi = -\frac{m'x'}{1+m'}, \quad \eta = -\frac{m'y'}{1+m'}, \quad \zeta = -\frac{m'z'}{1+m'}. \quad (230)$$

The equations (15)₁ are now easily transformed into the following:—

$$\begin{aligned} \frac{d^2x_0}{dt^2} + \frac{k^2(1+m')x_0}{r_0^3} &= m'k^2(x'_0 - x_0) \left(\frac{1}{\rho^3} - \frac{1}{r_0^3} \right) \\ &\quad + k^2(x_0 + m'x'_0) \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right), \\ \frac{d^2y_0}{dt^2} + \frac{k^2(1+m')y_0}{r_0^3} &= m'k^2(y'_0 - y_0) \left(\frac{1}{\rho^3} - \frac{1}{r_0^3} \right) \\ &\quad + k^2(y_0 + m'y'_0) \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right), \\ \frac{d^2z_0}{dt^2} + \frac{k^2(1+m')z_0}{r_0^3} &= m'k^2(z'_0 - z_0) \left(\frac{1}{\rho^3} - \frac{1}{r_0^3} \right) \\ &\quad + k^2(z_0 + m'z'_0) \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right), \end{aligned} \quad (231)$$

which completely determine the motion of the comet about the common centre of gravity of the sun and planet. The second members express the forces which disturb the pure elliptic motion; and it is evident, by an inspection of the terms, that when the comet is remote from both the planet and the sun these forces become extremely

small. If, therefore, we compute the perturbations of the motion relative to the sun as far as to the point at which the second members of (231) have not any appreciable influence on the results, it will suffice simply to convert the elements which refer to the centre of the sun into those relative to the common centre of gravity of the sun and disturbing planet, and then to regard the motion as undisturbed until the comet again approaches so near that the direct perturbations must be considered, at which point the motion will again be referred to the centre of the sun.

203. The reduction of the elements from the centre of gravity of the sun to the common centre of gravity of the sun and the disturbing planet, may be easily effected by means of the variations of the rectangular co-ordinates and of the corresponding velocities. To derive the co-ordinates of the comet referred to the centre of gravity of the sun and planet, it is only necessary to add to the heliocentric co-ordinates the co-ordinates of the sun referred to this origin, so that, according to (230), we shall have

$$\delta x = -\frac{m'}{1+m'}x', \quad \delta y = -\frac{m'}{1+m'}y', \quad \delta z = -\frac{m'}{1+m'}z', \quad (232)$$

and, also,

$$\begin{aligned} \delta \frac{dx}{dt} &= -\frac{m'}{1+m'} \cdot \frac{dx'}{dt}, & \delta \frac{dy}{dt} &= -\frac{m'}{1+m'} \cdot \frac{dy'}{dt}, \\ \delta \frac{dz}{dt} &= -\frac{m'}{1+m'} \cdot \frac{dz'}{dt}. \end{aligned} \quad (233)$$

If, therefore, from the elements of the orbit of the disturbing planet we compute the auxiliary constants for the adopted fundamental plane by means of the equations (94)₁ or (99)₁, and also V' and U' from

$$\begin{aligned} \frac{kV\sqrt{1+m'}}{\sqrt{p'}} (e' \sin \omega' + \sin u') &= V' \sin U', \\ \frac{kV\sqrt{1+m'}}{\sqrt{p'}} (e' \cos \omega' + \cos u') &= V' \cos U', \end{aligned}$$

the equations (100)₁ and (49), in connection with (232) and (233), give

$$\delta x = -\frac{m'}{1+m'} r' \sin \alpha' \sin (A' + u'), \quad (234)$$

$$\begin{aligned}
 \delta y &= -\frac{m'}{1+m'} r' \sin b' \sin (B' + u'), \\
 \delta z &= -\frac{m'}{1+m'} r' \sin c' \sin (C' + u'); \\
 \delta \frac{dx}{dt} &= -\frac{m'}{1+m'} V' \sin a' \cos (A' + U'), \\
 \delta \frac{dy}{dt} &= -\frac{m'}{1+m'} V' \sin b' \cos (B' + U'), \\
 \delta \frac{dz}{dt} &= -\frac{m'}{1+m'} V' \sin c' \cos (C' + U'),
 \end{aligned} \tag{234}$$

If we add the values of δx , δy , δz , $\delta \frac{dx}{dt}$, $\delta \frac{dy}{dt}$, and $\delta \frac{dz}{dt}$ to the corresponding co-ordinates and velocities of the comet in reference to the centre of gravity of the sun, the results will give the co-ordinates and velocities of the comet in reference to the common centre of gravity of the sun and disturbing planet, and from these the new elements of the orbit may be determined as explained in Art. 168.

The time at which the elements of the orbit of the comet may be referred to the common centre of gravity of the sun and planet, can be readily estimated in the actual application of the formulæ, by means of the magnitude of the disturbing force. In the case of Mercury as the disturbing planet, this transformation may generally be effected when the radius-vector of the comet has attained the value 1.5, and in the case of Venus when it has the value 2.5. It should be remarked, however, that the distance here assigned may be increased or diminished by the relative position of the bodies in their orbits. The motion relative to the common centre of gravity of the sun and planet—disregarding the perturbations produced by the other planets, which should be considered separately—may then be regarded as undisturbed until the comet has again arrived at the point at which the motion must be referred to the centre of the sun, and at which the perturbations of this motion by the planet under consideration must be determined. The reduction to the centre of the sun will be effected by means of the values obtained from (234), when the second member of each of these equations is taken with a contrary sign.

204. In the cases in which the motion of the comet will be referred to the common centre of gravity of the sun and disturbing planet, the resulting variations of the co-ordinates and velocities will be so small that their squares and products may be neglected, and, there-

fore, instead of using the complete formulæ in finding the new elements, it will suffice to employ differential formulæ. The formulæ (100)₁ give

$$\begin{aligned}\frac{dx}{dt} &= \sin a \sin (A + u) \frac{dr}{dt} + r \sin a \cos (A + u) \frac{dv}{dt}, \\ \frac{dy}{dt} &= \sin b \sin (B + u) \frac{dr}{dt} + r \sin b \cos (B + u) \frac{dv}{dt}, \\ \frac{dz}{dt} &= \sin c \sin (C + u) \frac{dr}{dt} + r \sin c \cos (C + u) \frac{dv}{dt}.\end{aligned}\quad (235)$$

If we multiply the first of these equations by δx , the second by δy , and the third by δz ; then multiply the first by $\delta \frac{dx}{dt}$, the second by $\delta \frac{dy}{dt}$, and the third by $\delta \frac{dz}{dt}$, and put

$$\begin{aligned}P &= \sin a \sin (A + u) \delta x + \sin b \sin (B + u) \delta y \\ &\quad + \sin c \sin (C + u) \delta z, \\ Q &= \sin a \cos (A + u) \delta x + \sin b \cos (B + u) \delta y \\ &\quad + \sin c \cos (C + u) \delta z; \\ P' &= \sin a \sin (A + u) \delta \frac{dx}{dt} + \sin b \sin (B + u) \delta \frac{dy}{dt} \\ &\quad + \sin c \sin (C + u) \delta \frac{dz}{dt}, \\ Q' &= \sin a \cos (A + u) \delta \frac{dx}{dt} + \sin b \cos (B + u) \delta \frac{dy}{dt} \\ &\quad + \sin c \cos (C + u) \delta \frac{dz}{dt},\end{aligned}\quad (236)$$

we shall have, observing that $\frac{dr}{dt} = \frac{k}{\sqrt{p}} e \sin v$ and that $\frac{dv}{dt} = \frac{k\sqrt{p}}{r^2}$,

$$\begin{aligned}\frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z &= \frac{k}{\sqrt{p}} e \sin v P + \frac{k\sqrt{p}}{r} Q, \\ \frac{dx}{dt} \delta \frac{dx}{dt} + \frac{dy}{dt} \delta \frac{dy}{dt} + \frac{dz}{dt} \delta \frac{dz}{dt} &= \frac{k}{\sqrt{p}} e \sin v P' + \frac{k\sqrt{p}}{r} Q'.\end{aligned}\quad (237)$$

From the equations

$$\begin{aligned}r \frac{dr}{dt} &= x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt}, \\ V^2 &= \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2},\end{aligned}$$

we get

$$\delta\left(\frac{rdr}{dt}\right) = \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z + x \delta \frac{dx}{dt} + y \delta \frac{dy}{dt} + z \delta \frac{dz}{dt},$$

$$V \delta V = \frac{dx}{dt} \delta \frac{dx}{dt} + \frac{dy}{dt} \delta \frac{dy}{dt} + \frac{dz}{dt} \delta \frac{dz}{dt},$$

which by means of (237) become

$$\delta\left(\frac{rdr}{dt}\right) = \frac{k}{\sqrt{p}} e \sin v P + \frac{k\sqrt{p}}{r} Q + P'r,$$

$$V \delta V = \frac{k}{\sqrt{p}} e \sin v P' + \frac{k\sqrt{p}}{r} Q'. \quad (238)$$

From the equation

$$k^2 p = V^2 r^2 - \frac{r^2 dr^2}{dt^2}$$

we get

$$2pk\delta k + k^2\delta p = 2r^2 V \delta V + 2V^2 r \delta r - 2 \frac{rdr}{dt} \delta\left(\frac{rdr}{dt}\right).$$

Substituting the values given by (238), observing also that $P = \delta r$, this becomes

$$\frac{\delta k}{k} + \frac{\delta p}{2p} = \frac{V^2 r}{k^2 p} P - \frac{re^2 \sin^2 v}{p^2} P - \frac{e \sin v}{p} Q + \frac{r}{k\sqrt{p}} Q';$$

and, since

$$V^2 = \frac{k^2}{p} (1 + 2e \cos v + e^2),$$

we obtain

$$\delta(\sqrt{p}) = \frac{\sqrt{p}}{r} P - \frac{e \sin v}{\sqrt{p}} Q + \frac{r}{k} Q' - \sqrt{p} \frac{\delta k}{k}, \quad (239)$$

by means of which the variation of \sqrt{p} may be found.

The equation

$$\frac{k^2}{a} = \frac{2k^2}{r} - V^2$$

gives

$$\delta \frac{1}{a} = -\frac{2}{r^2} \delta r - \frac{2}{k^2} V \delta V + 2\left(\frac{2}{r} - \frac{1}{a}\right) \frac{\delta k}{k},$$

from which we derive

$$\delta \frac{1}{a} = -\frac{2}{r^2} P - \frac{2e \sin v}{k\sqrt{p}} P' - \frac{2\sqrt{p}}{rk} Q' + 2\left(\frac{2}{r} - \frac{1}{a}\right) \frac{\delta k}{k}, \quad (240)$$

from which the new value of the semi-transverse axis a may be found. To find $\delta\mu$ we have

$$\delta\mu = \frac{3}{2}\mu a \delta \frac{1}{a} + \mu \frac{\delta k}{k}, \quad (241)$$

or

$$\delta\mu = -\frac{3\mu a}{r^2} P - \frac{3\mu a e \sin v}{k\sqrt{p}} P' - \frac{3\mu a \sqrt{p}}{rk} Q' + \left(\frac{6a}{r} - 2\right) \mu \frac{\delta k}{k}. \quad (242)$$

Next, to find δe , we have, from $p = a(1 - e^2)$,

$$\delta e = -\frac{p}{2e} \delta \frac{1}{a} - \frac{\sqrt{p}}{ae} \delta(\sqrt{p}), \quad (243)$$

or

$$\delta e = \frac{p \cos E}{r^2} P + \frac{\sin v}{a} Q + \frac{\sin v \sqrt{p}}{k} P' + \frac{\sqrt{p}}{k} (\cos v + \cos E) Q' - \frac{2p \cos E}{r} \cdot \frac{\delta k}{k}. \quad (244)$$

The equation (12)₂ gives

$$\delta M = \frac{r^2}{a^2 \cos \phi} \delta v - \frac{r^2 \sin v}{a^2 \cos^3 \phi} (2 + e \cos v) \delta e, \quad (245)$$

and from $\frac{p}{r} = 1 + e \cos v$ we get

$$\delta v = \frac{\cos v}{e \sin v} \delta e + \frac{p}{r^2 e \sin v} \delta r - \frac{2\sqrt{p}}{re \sin v} \delta(\sqrt{p}). \quad (246)$$

Substituting this value of δv in (245), and reducing, we find

$$\delta M = -\left(\frac{\cot \phi}{r} + \frac{\tan \phi}{a}\right) \sin v P + \frac{\cos v}{a \tan \phi} Q + \frac{1}{k\sqrt{p}} (p \cot \phi \cos v - 2r \cos \phi) P' - \frac{1}{k\sqrt{p}} \cdot \frac{(p+r) \sin v}{\tan \phi} Q' + \left(\frac{\cot \phi}{r} + \frac{\tan \phi}{a}\right) 2r \sin v \frac{\delta k}{k}, \quad (247)$$

from which to derive the variation of the mean anomaly.

205. Let us now denote by x'', y'', z'' the heliocentric co-ordinates of the comet referred to a system in which the plane of the orbit is the fundamental plane, and in which the positive axis of x is directed to the ascending node on the ecliptic. Let us also denote by x', y', z' the co-ordinates referred to a system in which the plane of the ecliptic is the plane of xy , and in which the positive axis of x is directed to the vernal equinox. Then we shall have

$$\begin{aligned}x'' &= x' \cos \Omega + y' \sin \Omega, \\y'' &= -x' \sin \Omega \cos i + y' \cos \Omega \cos i + z' \sin i, \\z'' &= x' \sin \Omega \sin i - y' \cos \Omega \sin i + z' \cos i.\end{aligned}$$

If we transform the co-ordinates still further, and denote by x, y, z the co-ordinates referred to the equator or to any other plane making the angle ε with the ecliptic, the positive axis of x being directed to the point from which longitudes are measured in this plane; and if we introduce also the auxiliary constants $a, A, b, B, \&c.$, we shall have

$$\begin{aligned}\delta x'' &= \sin a \sin A \delta x + \sin b \sin B \delta y + \sin c \sin C \delta z, \\\delta y'' &= \sin a \cos A \delta x + \sin b \cos B \delta y + \sin c \cos C \delta z, \\\delta z'' &= \cos a \delta x + \cos b \delta y + \cos c \delta z.\end{aligned}\tag{248}$$

Multiplying the first of these by $-\sin u$, and the second by $\cos u$, adding the results, and introducing Q as given by the second of equations (236), we get

$$\cos u \delta y'' - \sin u \delta x'' = Q.$$

Substituting for $\delta x''$ and $\delta y''$ the values given by the equations (73), the result is

$$r(\delta v + \delta \chi) = Q,$$

and, introducing the value of δv given by (246), we obtain

$$\delta \chi = \frac{Q}{r} - \frac{\cos v}{e \sin v} \delta e - \frac{p}{r^2 e \sin v} \delta r + \frac{2\sqrt{p}}{re \sin v} \delta(\sqrt{p}).$$

Substituting further for δe , δr , and $\delta(\sqrt{p})$ the values already obtained, and reducing, we find

$$\begin{aligned}\delta \chi = \frac{\sin v}{er} P - \frac{\cos E}{er} Q - \frac{\cos v \sqrt{p}}{ek} P' + \frac{(p+r) \sin v}{ek \sqrt{p}} Q' \\ - \frac{2 \sin v}{e} \cdot \frac{\delta k}{k},\end{aligned}\tag{249}$$

by means of which $\delta \chi$ may be found.

If we put

$$\begin{aligned}\cos a \delta x + \cos b \delta y + \cos c \delta z &= R, \\\cos a \delta \frac{dx}{dt} + \cos b \delta \frac{dy}{dt} + \cos c \delta \frac{dz}{dt} &= R',\end{aligned}\tag{250}$$

the last of the equations (248) gives

$$\delta z'' = R; \quad (251)$$

and if we differentiate the equation

$$\cos a \frac{dx}{dt} + \cos b \frac{dy}{dt} + \cos c \frac{dz}{dt} = 0,$$

which exists in the case of the unchanged elements, we shall have

$$0 = \cos a \delta \frac{dx}{dt} + \cos b \delta \frac{dy}{dt} + \cos c \delta \frac{dz}{dt} \\ - \frac{dx}{dt} \sin a \delta a - \frac{dy}{dt} \sin b \delta b - \frac{dz}{dt} \sin c \delta c.$$

Substituting for δa , δb , and δc the values given in Art. 60, observing that $\delta \varepsilon = 0$, we have

$$0 = R' + \left(\frac{dx}{dt} \sin a \sin A + \frac{dy}{dt} \sin b \sin B + \frac{dz}{dt} \sin c \sin C \right) \sin i \delta \Omega \\ - \left(\frac{dx}{dt} \sin a \cos A + \frac{dy}{dt} \sin b \cos B + \frac{dz}{dt} \sin c \cos C \right) \delta i. \quad (252)$$

From the equations (100)₁, observing that the relations between the auxiliary constants are not changed when the variable u is put equal to zero, or equal to 90° , we get

$$\sin^2 a \sin^2 A + \sin^2 b \sin^2 B + \sin^2 c \sin^2 C = 1, \\ \sin^2 a \cos^2 A + \sin^2 b \cos^2 B + \sin^2 c \cos^2 C = 1, \quad (253)$$

and from (235) we find

$$\sin^2 a \sin A \cos A + \sin^2 b \sin B \cos B + \sin^2 c \sin C \cos C = 0. \quad (254)$$

Substituting in (252) for $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ the values given by the equations (49), and reducing by means of (253) and (254), we get

$$0 = R' - V \sin U \sin i \delta \Omega - V \cos U \delta i. \quad (255)$$

Substituting further for $\delta z''$ in (251) the value given by the last of the equations (73)₂, there results

$$0 = R + r \cos u \sin i \delta \Omega - r \sin u \delta i. \quad (256)$$

From these equations we derive, by elimination,

$$\begin{aligned}\delta\Omega &= -\frac{e \cos \omega + \cos u}{p \sin i} R + \frac{1}{k\sqrt{p}} \cdot \frac{r \sin u}{\sin i} R', \\ \delta i &= \frac{e \sin \omega + \sin u}{p} R + \frac{r \cos u}{k\sqrt{p}} R',\end{aligned}\quad (257)$$

by means of which $\delta\Omega$ and δi may be found. To find $\delta\omega$ and $\delta\pi$ we have

$$\delta\omega = \delta\chi - \cos i \delta\Omega, \quad \delta\pi = \delta\chi + 2 \sin^2 \frac{1}{2} i \delta\Omega, \quad (258)$$

$\delta\chi$ being found from equation (249).

Neglecting the mass of the comet as inappreciable in comparison with that of the sun, the attractive force which acts upon the comet in the case of the undisturbed motion relative to the sun is k^2 , but in the case of the motion relative to the common centre of gravity of the sun and planet this force is $k^2(1+m')$. Hence it follows that the increment of this force will be $m'k^2$, and we shall have

$$\frac{\delta k}{k} = \frac{1}{2} m', \quad (259)$$

by means of which the value of this factor, which is required in the formulæ for $\delta(\sqrt{p})$, $\delta\frac{1}{a}$, &c., may be found.

206. The formulæ thus derived enable us to effect the required transformation of the elements. In the first place, we compute the values of δx , δy , δz , $\delta\frac{dx}{dt}$, $\delta\frac{dy}{dt}$, and $\delta\frac{dz}{dt}$ by means of the formulæ (234); then, by means of (236) and (250), we compute P , Q , R , P' , Q' , and R' , the auxiliary constants α , A , &c. being determined in reference to the fundamental plane to which the co-ordinates are referred. When the fundamental plane is the plane of the ecliptic, or that to which Ω and i are referred, we have

$$\sin c = \sin i, \quad C = 0.$$

The algebraic signs of $\cos \alpha$, $\cos b$, and $\cos c$, as indicated by the equations (101), must be carefully attended to. The formulæ for the variations of the elements will then give the corrections to be applied to the elements of the orbit relative to the sun in order to obtain those of the orbit relative to the common centre of gravity of the sun and planet. Whenever the elements of the orbit about the sun are again required, the corrections will be determined in the same manner, but will be applied each with a contrary sign.

Since the equations have been derived for the variations of more than the six elements usually employed, the additional formulæ, as well as those which give different relations between the elements employed, may be used to check the numerical calculation; and this proof should not be omitted. It is obvious, also, that these differential formulæ will serve to convert the perturbations of the rectangular co-ordinates into perturbations of the elements, whenever the terms of the second order may be neglected, observing that in this case $\delta k = 0$. If some of the elements considered are expressed in angular measure, and some in parts of other units, the quantity $s = 206264''.8$ should be introduced, in the numerical application, so as to preserve the homogeneity of the formulæ.

When the motion of the comet is regarded as undisturbed about the centre of gravity of the system, the variations of the elements for the instant t in order to reduce them to the centre of gravity of the system, added algebraically to those for the instant t' in order to reduce them again to the centre of the sun, will give the total perturbations of the elements of the orbit relative to the sun during the interval $t' - t$. It should be observed, however, that the value of δM for the instant t should be reduced to that for the instant t' , so that the total variation of M during the interval $t' - t$ will be

$$\delta M_t + (t' - t) \delta \mu_t + \delta M_{t'}.$$

In this manner, by considering the action of the several disturbing bodies separately, referring the motion of the comet to the common centre of gravity of the sun and planet whenever it may subsequently be regarded as undisturbed about this point, and again referring it to the centre of the sun when such an assumption is no longer admissible, the determination of the perturbations during an entire revolution of the comet is very greatly facilitated.

207. If we consider the position and dimensions of the orbits of the comets, it will at once appear that a very near approach of some of these bodies to a planet may often happen, and that when they approach very near some of the large planets their orbits may be entirely changed. It is, indeed, certainly known that the orbits of comets have been thus modified by a near approach to Jupiter, and there are periodic comets now known which will be eventually thus acted upon. It becomes an interesting problem, therefore, to consider the formulæ applicable to this special case in which the ordinary methods of calculating perturbations cannot be applied.

If we denote by x', y', z', r' , the co-ordinates and radius-vector of the planet referred to the centre of the sun, and regard its motion relative to the sun as disturbed by the comet, we shall have

$$\begin{aligned}\frac{d^2x'}{dt^2} + \frac{k^2(1+m')}{r'^3}x' &= mk^2\left(\frac{x-x'}{\rho^3} - \frac{x}{r^3}\right), \\ \frac{d^2y'}{dt^2} + \frac{k^2(1+m')}{r'^3}y' &= mk^2\left(\frac{y-y'}{\rho^3} - \frac{y}{r^3}\right), \\ \frac{d^2z'}{dt^2} + \frac{k^2(1+m')}{r'^3}z' &= mk^2\left(\frac{z-z'}{\rho^3} - \frac{z}{r^3}\right).\end{aligned}\quad (260)$$

Let us now denote by ξ, η, ζ the co-ordinates of the comet referred to the centre of gravity of the planet; then will

$$\xi = x - x', \quad \eta = y - y', \quad \zeta = z - z'.$$

Substituting the resulting values of x', y', z' in the preceding equations, and subtracting these from the corresponding equations (1) for the disturbed motion of the comet, we derive

$$\begin{aligned}\frac{d^2\xi}{dt^2} + \frac{k^2(m+m')}{\rho^3}\xi &= k^2\left(\frac{x'}{r'^3} - \frac{x'+\xi}{r^3}\right), \\ \frac{d^2\eta}{dt^2} + \frac{k^2(m+m')}{\rho^3}\eta &= k^2\left(\frac{y'}{r'^3} - \frac{y'+\eta}{r^3}\right), \\ \frac{d^2\zeta}{dt^2} + \frac{k^2(m+m')}{\rho^3}\zeta &= k^2\left(\frac{z'}{r'^3} - \frac{z'+\zeta}{r^3}\right).\end{aligned}\quad (261)$$

These equations express the motion of the comet relative to the centre of gravity of the disturbing planet; and when the comet approaches very near to the planet, so that the second member of each of these equations becomes very small in comparison with the second term of the first member, we may take, for a first approximation,

$$\begin{aligned}\frac{d^2\xi}{dt^2} + \frac{k^2(m+m')}{\rho^3}\xi &= 0, \\ \frac{d^2\eta}{dt^2} + \frac{k^2(m+m')}{\rho^3}\eta &= 0, \\ \frac{d^2\zeta}{dt^2} + \frac{k^2(m+m')}{\rho^3}\zeta &= 0;\end{aligned}\quad (262)$$

and, since $\frac{k^2(m+m')}{\rho^3}$ is the sum of the attractive force of the planet on the comet and of the reciprocal action of the comet on the planet,

these equations, being of the same form as those for the undisturbed motion of the comet relative to the sun, show that when the action of the disturbing planet on the comet exceeds that of the sun, the result of the first approximation to the motion of the comet is that it describes a conic section around the centre of gravity of the planet. Further, since $-x'$, $-y'$, $-z'$ are the co-ordinates of the sun referred to the centre of gravity of the planet, it appears that the second members of (261) express the disturbing force of the sun on the comet resolved in directions parallel to the co-ordinate axes respectively. Hence when a comet approaches so near a planet that the action of the latter upon it exceeds that of the sun, its motion will be in a conic section relatively to the planet, and will be disturbed by the action of the sun. But the disturbing action of the sun is the difference between its action on the comet and on the planet, and the masses of the larger bodies of the solar system are such that when the comet is equally attracted by the sun and by the planet, the distances of the comet and planet from the sun differ so little that the disturbing force of the sun on the comet, regarded as describing a conic section about the planet, will be extremely small. Thus, in a direction parallel to the co-ordinate ξ the disturbing force exercised by the sun is

$$k^2 \left(\frac{x'}{r'^3} - \frac{x' + \xi}{r^3} \right) = k^2 \left(\frac{x'}{r'^3} - \frac{x}{r^3} \right),$$

and when the comet approaches very near the planet this force will be extremely small. It is evident, further, that the action of the sun regarded as the disturbing body will be very small even when its direct action upon the comet considerably exceeds that of the planet, and, therefore, that we may consider the orbit of the comet to be a conic section about the planet and disturbed by the sun, when it is actually attracted more by the sun than by the planet.

208. In order to show more clearly that the disturbing force of the sun is very small even when its direct action on the comet exceeds that of the planet, let us suppose the sun, planet, and comet to be situated on the same straight line, in which case the disturbing force of the sun will be a maximum for a given distance of the comet from the planet. Then will the direct action of the sun be $\frac{k^2}{r^2}$, and that of the planet $\frac{m'k^2}{\rho^2}$. The disturbing action of the sun will be

$$\frac{k^2}{r^2} - \frac{k^2}{(r + \rho)^2} = \frac{k^2 \rho}{r^2} \cdot \frac{2r + \rho}{(r + \rho)^2},$$

which, since ρ is supposed to be small in comparison with r , may be put equal to

$$\frac{2k^2 \rho}{r^3},$$

and hence the ratio of the disturbing action of the sun to the direct action of the planet on the comet cannot exceed

$$R = \frac{2\rho^3}{m'r^3}.$$

If the comet is at a distance, such that the direct action of the sun is equal to the direct action of the planet, we have

$$\rho^2 = m'r^2,$$

and the ratio of the direct action of the sun to its disturbing action cannot in this case exceed $2\sqrt{m'}$. In the case of Jupiter this amounts to only 0.06.

So long as ρ is small, the disturbing action of the planet is very nearly $\frac{m'k^2}{\rho^2}$ in all positions of the comet relative to the planet, and hence the ratio of the disturbing action of the planet to the direct action of the sun cannot exceed

$$R' = \frac{m'r^2}{\rho^2}.$$

At the point for which the value of ρ corresponds to $R = R'$, the comet, sun, and planet being supposed to be situated in the same straight line, it will be immaterial whether we consider the sun or the planet as the disturbing body; but for values of ρ less than this R will be less than R' , and the planet must be regarded as the controlling and the sun as the disturbing body. The supposition that R is equal to R' gives

$$\frac{2\rho^3}{m'r^3} = \frac{m'r^2}{\rho^2},$$

and therefore

$$\rho = r\sqrt[5]{\frac{1}{2}m'}. \quad (263)$$

Hence we may compute the perturbations of the comet, regarding the planet as the disturbing body, until it approaches so near the

planet that ρ has the value given by this equation, after which, so long as ρ does not exceed the value here assigned, the sun must be regarded as the disturbing body.

If ψ represents the angle at the planet between the sun and comet, the disturbing force of the sun, for any position of the comet near the planet, will be very nearly

$$\frac{2k^2\rho}{r^3} \cos \psi,$$

and when this angle is considerable, the disturbing action of the sun will be small even when ρ is greater than $r\sqrt[5]{\frac{1}{2}m'^2}$. Hence we may commence to consider the sun as the disturbing body even before the comet reaches the point for which

$$\rho = r\sqrt[5]{\frac{1}{2}m'^2},$$

and, since the ratio of the disturbing action of the planet to the direct action of the sun remains nearly the same for all values of ψ , when ρ is within the limits here assigned the sun must in all cases be so considered. Corresponding to the value of ρ given by equation (263), we have

$$R' = \sqrt[5]{4m'},$$

and in the case of a near approach to Jupiter the results are

$$\rho = 0.054r, \quad R' = 0.33.$$

209. In the actual calculation of the perturbations of any particular comet when very near a large planet, it will be easy to determine the point at which it will be advantageous to commence to regard the sun as the disturbing body; and, having found the elements of the orbit of the comet relative to the planet, the perturbations of these elements or of the co-ordinates will be obtained by means of the formulæ already derived, the necessary distinctions being made in the notation. When the planet again becomes the disturbing body, the elements will be found in reference to the sun; and thus we are enabled to trace the motion of the comet before and subsequent to its being considered as subject principally to the planet. In the case of the first transformation, the co-ordinates and velocities of the comet and planet in reference to the sun being determined for the instant at which the sun is regarded as ceasing to be the controlling body, we shall have

$$\begin{aligned}\xi &= x - x', & \eta &= y - y', & \zeta &= z - z', \\ \frac{d\xi}{dt} &= \frac{dx}{dt} - \frac{dx'}{dt}, & \frac{d\eta}{dt} &= \frac{dy}{dt} - \frac{dy'}{dt}, & \frac{d\zeta}{dt} &= \frac{dz}{dt} - \frac{dz'}{dt};\end{aligned}$$

and from $\xi, \eta, \zeta, \frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$, the elements of the orbit of the comet about the planet are to be determined precisely as the elements in reference to the sun are found from $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, and as explained in Art. 168. Having computed the perturbations of the motion relative to the planet to the point at which the planet is again considered as the disturbing body, it only remains to find, for the corresponding time, the co-ordinates and velocities of the comet in reference to the centre of gravity of the planet, and from these the co-ordinates and velocities relative to the centre of the sun, and the elements of the orbit about the sun may be determined. As the interval of time during which the sun will be regarded as the disturbing body will always be small, it will be most convenient to compute the perturbations of the rectangular co-ordinates, in which case the values of $\xi, \eta, \zeta, \frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$ will be obtained directly, and then, having found the corresponding co-ordinates x', y', z' and velocities $\frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt}$ of the planet in reference to the sun, we have

$$\begin{aligned}x &= x' + \xi, & y &= y' + \eta, & z &= z' + \zeta, \\ \frac{dx}{dt} &= \frac{dx'}{dt} + \frac{d\xi}{dt}, & \frac{dy}{dt} &= \frac{dy'}{dt} + \frac{d\eta}{dt}, & \frac{dz}{dt} &= \frac{dz'}{dt} + \frac{d\zeta}{dt},\end{aligned}$$

by means of which the elements of the orbit relative to the sun will be found. If it is not considered necessary to compute rigorously the path of the comet before and after it is subject principally to the action of the planet, but simply to find the principal effect of the action of the planet in changing its elements, it will be sufficient, during the time in which the sun is regarded as the disturbing body, to suppose the comet to move in an undisturbed orbit about the planet. For the point at which we cease to regard the sun as the disturbing body, the co-ordinates and velocities of the comet relative to the centre of gravity of the planet will be determined from the elements of the orbit in reference to the planet, precisely as the corresponding quantities are determined in the case of the motion relative to the sun, the necessary distinctions being made in the notation.

210. The results obtained from the observations of the periodic comets at their successive returns to the perihelion, render it probable that there exists in space a resisting medium which opposes the motion of all the heavenly bodies in their orbits; but since the observations of the planets do not exhibit any effect of such a resistance, it is inferred that the density of the ethereal fluid is so slight that it can have an appreciable effect only in the case of rare and attenuated bodies like the comets. If, however, we adopt the hypothesis of a resisting medium in space, in considering the motion of a heavenly body we simply introduce a new disturbing force acting in the direction of the tangent to the instantaneous orbit, and in a sense contrary to that of the motion. The amount of the resistance will depend chiefly on the density of the ethereal fluid and on the velocity of the body. In accordance with what takes place within the limits of our observation, we may assume that the resistance, in a medium of constant density, is proportional to the square of the velocity. The density of the fluid may be assumed to diminish as the distance from the sun increases, and hence it may be expressed as a function of the reciprocal of this distance.

Let ds be the element of the path of the body, and r the radius-vector; then will the resistance be

$$T = -K\varphi\left(\frac{1}{r}\right)\frac{ds^2}{dt^2}, \quad (264)$$

K being a constant quantity depending on the nature of the body, and $\varphi\left(\frac{1}{r}\right)$ the density of the ethereal fluid at the distance r . Since the force acts only in the plane of the orbit, the elements which define the position of this plane will not be changed, and hence we have only to determine the variations of the elements M , e , a , and χ . If we denote by ϕ_0 the angle which the tangent makes with the prolongation of the radius-vector, the components R and S will be given by

$$R = T \cos \phi_0, \quad S = T \sin \phi_0,$$

and, since

$$V \cos \phi_0 = \frac{k}{\sqrt{p}} e \sin v, \quad V \sin \phi_0 = \frac{k\sqrt{p}}{r}, \quad V = \frac{ds}{dt},$$

we have

$$R = -K\varphi\left(\frac{1}{r}\right)\frac{k}{\sqrt{p}} e \sin v \frac{ds}{dt}, \quad S = -K\varphi\left(\frac{1}{r}\right)\frac{k\sqrt{p}}{r} \cdot \frac{ds}{dt}. \quad (265)$$

Substituting these values of R and S in the equation (205), it reduces to

$$e d\chi = -2K\varphi \left(\frac{1}{r} \right) \sin v ds.$$

Now, since

$$V = \frac{k}{\sqrt{p}} (1 + 2e \cos v + e^2)^{\frac{1}{2}},$$

we have

$$ds = V dt = \frac{r^2}{p} (1 + 2e \cos v + e^2)^{\frac{1}{2}} dv,$$

and hence

$$e d\chi = -\frac{2}{p} K\varphi \left(\frac{1}{r} \right) r^2 (1 + 2e \cos v + e^2)^{\frac{1}{2}} \sin v dv. \quad (266)$$

If we suppose the function

$$K\varphi \left(\frac{1}{r} \right) r^2 (1 + 2e \cos v + e^2)^{\frac{1}{2}},$$

the value of which is always positive, to be developed in a series arranged in reference to the cosines of v and of its multiples, so that we have

$$K\varphi \left(\frac{1}{r} \right) r^2 (1 + 2e \cos v + e^2)^{\frac{1}{2}} = A + B \cos v + C \cos 2v + \&c., \quad (267)$$

in which A , B , &c. are positive and functions of e , the equation (266) becomes

$$e d\chi = -\frac{2}{p} (A + B \cos v + \dots) \sin v dv.$$

Hence, by integrating, we derive

$$e \delta\chi = \frac{2}{p} (A \cos v + \frac{1}{4} B \cos 2v + \dots), \quad (268)$$

from which it appears that χ is subject only to periodic perturbations on account of the resisting medium.

In a similar manner it may be shown that the second term of the second member of equation (210) produces only periodic terms in the value of δM , so that if we seek only the secular perturbations due to the action of the ethereal fluid, the first and second terms of the second member of (210) will not be considered, and only the secular perturbations arising from the variation of μ will be required.

Let us next consider the elements a and e . Substituting in the

equations (198) and (202) the values of R and S given by (265), and reducing, we get

$$\begin{aligned} da &= -\frac{2a^2}{p^2} K\varphi\left(\frac{1}{r}\right) r^2 (1 + 2e \cos v + e^2)^{\frac{3}{2}} dv, \\ de &= -\frac{2}{p} K\varphi\left(\frac{1}{r}\right) r^2 (1 + 2e \cos v + e^2)^{\frac{1}{2}} (e + \cos v) dv. \end{aligned} \quad (269)$$

If we introduce into these the series (267), and integrate, it will be found that, in addition to the periodic terms, the expressions for δa and δe contain each a term multiplied by v , and hence increasing with the time. It is to be observed, further, that since A and B are positive, the secular variation of a , and also that of e , will be negative, and hence the resisting medium acts continuously to diminish both the mean distance and the eccentricity.

211. The magnitude of the disturbing force arising from the action of the resisting medium is so small that the periodic terms have no sensible influence on the place of the comet during the period in which it may be observed; and hence, since the effect of the resistance will be exhibited only by a comparison of observations made at its successive returns to the perihelion, the effect of the planetary perturbations being first completely eliminated, it is only necessary to consider the secular variations. Further, since χ is subject only to periodic changes in virtue of the action of the resistance, and since the mean longitude is subjected to a secular change only through μ , it will suffice to employ the formulæ for $\delta\mu$ and δe or $\delta\varphi$. The variations of these elements may be computed most conveniently by mechanical quadrature from given values of $\frac{d\mu}{dt}$ and $\frac{de}{dt}$ or $\frac{d\varphi}{dt}$, although their values for one complete revolution of the comet may be determined directly, the values of the coefficients A and B which appear in the series (267) being found by means of elliptic functions. The calculation of the effect of the resisting medium will be made in connection with the determination of the planetary perturbations, so that there will be no inconvenience in adding to the results the terms depending on this resistance. Since

$$\frac{d\mu}{dt} = -\frac{3}{2} \frac{\mu}{a} \cdot \frac{da}{dt}, \quad \frac{d\varphi}{dt} = \sec \varphi \frac{de}{dt},$$

the equations (269) give, putting $K = k^2 U$,

$$\begin{aligned}\frac{d\mu}{dt} &= 3a\mu U\varphi\left(\frac{1}{r}\right)V^3, \\ \frac{d\varphi}{dt} &= -\frac{2k^2p \cos E}{r \cos \varphi} U\varphi\left(\frac{1}{r}\right)V.\end{aligned}\quad (270)$$

It remains now to make an assumption in regard to the law of the density of the resisting medium. In the case of Encke's comet it has been assumed that

$$\varphi\left(\frac{1}{r}\right) = \frac{1}{r^2},$$

and this hypothesis gives results which suffice to represent the observations at its successive returns to the perihelion. Substituting for V its value in terms of r and a , the equations (270) thus become

$$\begin{aligned}\frac{d\mu}{dt} &= 3k^2 U \frac{a\mu}{r^2} \left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{3}{2}}, \\ \frac{d\varphi}{dt} &= -2k^2 U \frac{a \cos \varphi \cos E}{r^3} \left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{1}{2}},\end{aligned}\quad (271)$$

by means of which $\delta\mu$ and $\delta\varphi$ may be found; and from any given value of $\delta\mu$ we may derive the corresponding value of δa . The variation of M , neglecting the periodic terms arising from the first and second terms of the second member of equation (210), will be given by

$$\delta M = \iint \frac{d\mu}{dt} dt^2,$$

which will be integrated by mechanical quadrature so as to include the interval of an entire revolution of the comet. The quantity U has been determined, by means of observations of Encke's comet, to be

$$U = \frac{1}{894.892}$$

This value may be corrected by introducing a term in the equations of condition precisely as in the case of the determination of the correction to be applied to the mass of a disturbing planet. Introducing U into the equation (264), and adopting the hypothesis that $\varphi\left(\frac{1}{r}\right) = \frac{1}{r^2}$, the expression for the action of the ethereal fluid becomes

$$T = -\frac{k^2 U}{r^2} V^2.$$

Since the constant U depends on the nature of the comet, the value obtained in the case of Encke's comet may be very different from that in the case of another comet. Thus, in the case of Faye's comet the value has been found to be

$$U = \frac{1}{10.232};$$

and in the application of the formulæ to the motion of any particular body it will be necessary to make an independent determination of this constant.

212. The assumption that the density of the ethereal fluid varies inversely as the square of the distance from the sun, is that which appears to be the most probable, and the results obtained in accordance therewith seem to satisfy the data furnished by observation. It is true, however, that the whole subject is involved in great uncertainty as regards the nature of the resisting medium, so that the results obtained by means of any assumed law of density are not to be regarded as absolutely correct.

From the formulæ which have been given, it appears that, whatever may be the law of the density of the resisting fluid, the mean motion is constantly accelerated and the eccentricity diminished, and we may determine, by means of observations at the successive appearances of the comet, the amount of these secular changes independently of any assumption in regard to the density of the ether. Let x denote the variation of μ during the interval τ , which may be approximately the time of one revolution of the comet, and let y denote the corresponding variation of φ ; then, after the lapse of any interval $t - T_0$, we shall have

$$\mu = \mu_0 + \frac{t - T_0}{\tau} x, \quad \varphi = \varphi_0 + \frac{t - T_0}{\tau} y, \quad (272)$$

and, since the average variation of μ during the interval $t - T_0$ is $\frac{1}{2} \frac{t - T_0}{\tau} x$,

$$M = M_0 + \mu_0(t - T_0) + \frac{(t - T_0)^2}{2\tau} x. \quad (273)$$

If we introduce x and y as unknown quantities in the equations of condition for the correction of the elements by means of the differences between computation and observation, the secular variations of μ and φ may be determined in connection with the corrections to b

applied to the elements. For this purpose the partial differential coefficients of the geocentric spherical co-ordinates with respect to x and y must be determined. Thus, if we substitute the values of μ , φ , and M given by (272) and (273) in the equations (12)₂ and (14)₂, we obtain

$$\begin{aligned}\frac{dr}{dx} &= a \tan \varphi \sin v \frac{(t - T_0)^2}{2\tau} - \frac{2r}{3\mu} \cdot \frac{t - T_0}{\tau} s, \\ \frac{dv}{dx} &= \frac{a^2 \cos \varphi}{r^2} \cdot \frac{(t - T_0)^2}{2\tau}, \quad \frac{dr}{dy} = -a \cos \varphi \cos v \frac{t - T_0}{\tau}, \quad (274) \\ \frac{dv}{dy} &= \left(\frac{2}{\cos \varphi} + \tan \varphi \cos v \right) \sin v \frac{t - T_0}{\tau},\end{aligned}$$

in which $s = 206264''.8$, μ being expressed in seconds of arc. Combining the results thus obtained with the differential coefficients of the geocentric spherical co-ordinates with respect to r and v , as indicated by the equations (42)₂, we obtain the required coefficients of x and y to be introduced into the equations of condition. The solution of all the equations of condition by the method of least squares will then furnish the most probable values of y and x , or of the secular variations of the eccentricity and mean motion, without any assumption being made in reference either to the density of the ethereal fluid or to the modifications of the resistance on account of the changes in the form and dimensions of the comet, and the results thus derived may be employed in determining the values of M , μ , and φ for the subsequent returns of the comet to the perihelion.

In all the cases in which the periodic comets have been observed sufficiently, the existence of these secular changes of the elements seems to be well established; and if we grant that they arise from the resistance of an ethereal fluid, the total obliteration of our solar system is to be the final result. The fact that no such inequalities have yet been detected in the case of the motion of any of the planets, shows simply the immensity of the period which must elapse before the final catastrophe, and does not render it any the less certain. Such, indeed, appear to be the present indications of science in regard to this important question; but it is by no means impossible that, as in at least one similar case already, the operation of the simple and unique law of gravitation will alone completely explain these inequalities, and assign a limit which they can never pass, and thus afford a sublime proof of the provident care of the OMNIPOTENT CREATOR.



TABLES.

TABLE I. Angle of the Vertical and Logarithm of the Earth's Radius.

Argument ϕ = Geographical Latitude.Compression = $\frac{1}{299.15}$

| ϕ | $\phi - \phi'$ | Diff. | log ρ | Diff. | ϕ | $\phi - \phi'$ | Diff. | log ρ | Diff. |
|--------|----------------|-------|------------|-------|--------|----------------|-------|------------|-------|
| 0 0 | 0 0.00 | " | 0.000 0000 | | 35 0 | 10 48.25 | 1.38 | 9.999 5248 | |
| 1 0 | 0 24.02 | 24.02 | 9.999 9996 | 4 | 10 | 49.63 | 1.35 | 5208 | 40 |
| 2 0 | 0 48.02 | 24.00 | 9982 | 14 | 20 | 50.98 | 1.33 | 5169 | 39 |
| 3 0 | 1 11.95 | 23.93 | 9961 | 21 | 30 | 52.31 | 1.33 | 5129 | 40 |
| 4 0 | 1 35.80 | 23.85 | 9930 | 31 | 40 | 53.62 | 1.31 | 5089 | 40 |
| 5 0 | 1 59.54 | 23.74 | 9891 | 39 | 50 | 54.90 | 1.28 | 5049 | 40 |
| | | 23.58 | | 48 | | | 1.26 | | 40 |
| 6 0 | 2 23.12 | 23.42 | 9.999 9843 | 57 | 36 0 | 10 56.16 | 1.25 | 9.999 5009 | 40 |
| 7 0 | 2 46.54 | 23.22 | 9786 | 65 | 10 | 57.41 | 1.22 | 4969 | 40 |
| 8 0 | 3 9.76 | 22.98 | 9721 | 73 | 20 | 58.63 | 1.19 | 4929 | 40 |
| 9 0 | 3 32.74 | 22.73 | 9648 | 82 | 30 | 59.82 | 1.18 | 4888 | 40 |
| 10 0 | 3 55.47 | 22.45 | 9566 | 90 | 40 | 1.00 | 1.15 | 4848 | 40 |
| 11 0 | 4 17.92 | 22.14 | 9476 | 99 | 50 | 2.15 | 1.13 | 4807 | 40 |
| 12 0 | 4 40.06 | 21.79 | 9.999 9377 | 106 | 37 0 | 11 3.28 | 1.11 | 9.999 4767 | 41 |
| 13 0 | 5 1.85 | 21.43 | 9271 | 114 | 10 | 4.39 | 1.08 | 4726 | 41 |
| 14 0 | 5 23.28 | 21.05 | 9157 | 122 | 20 | 5.47 | 1.07 | 4686 | 41 |
| 15 0 | 5 44.33 | 20.62 | 9035 | 130 | 30 | 6.54 | 1.04 | 4645 | 41 |
| 16 0 | 6 4.95 | 20.19 | 8905 | 137 | 40 | 7.58 | 1.01 | 4604 | 41 |
| 17 0 | 6 25.14 | 19.72 | 8768 | 144 | 50 | 8.59 | 1.00 | 4563 | 41 |
| 18 0 | 6 44.86 | 19.23 | 9.999 8624 | 152 | 38 0 | 11 9.59 | 0.97 | 9.999 4522 | 41 |
| 19 0 | 7 4.09 | 18.71 | 8472 | 158 | 10 | 10.56 | 0.95 | 4481 | 41 |
| 20 0 | 7 22.80 | 18.19 | 8314 | 165 | 20 | 11.51 | 0.93 | 4440 | 41 |
| 21 0 | 7 40.99 | 17.62 | 8149 | 172 | 30 | 12.44 | 0.90 | 4399 | 41 |
| 22 0 | 7 58.61 | 17.05 | 7977 | 178 | 40 | 13.34 | 0.88 | 4358 | 41 |
| 23 0 | 8 15.66 | 16.44 | 7799 | 185 | 50 | 14.22 | 0.86 | 4317 | 41 |
| 24 0 | 8 32.10 | 15.83 | 9.999 7614 | 190 | 39 0 | 11 15.08 | 0.84 | 9.999 4276 | 42 |
| 25 0 | 8 47.93 | 15.19 | 7424 | 196 | 10 | 15.92 | 0.81 | 4234 | 41 |
| 26 0 | 9 3.12 | 14.53 | 7228 | 201 | 20 | 16.73 | 0.79 | 4193 | 41 |
| 27 0 | 9 17.65 | 13.85 | 7027 | 207 | 30 | 17.52 | 0.77 | 4152 | 42 |
| 28 0 | 9 31.50 | 13.16 | 6820 | 212 | 40 | 18.29 | 0.75 | 4110 | 41 |
| 29 0 | 9 44.66 | 12.46 | 6608 | 216 | 50 | 19.04 | 0.72 | 4069 | 42 |
| 30 0 | 9 57.12 | 2.00 | 9.999 6392 | 37 | 40 0 | 11 19.76 | 0.70 | 9.999 4027 | 42 |
| 10 | 9 59.12 | 1.99 | 6355 | 36 | 10 | 20.46 | 0.67 | 3985 | 41 |
| 20 | 10 1.11 | 1.96 | 6319 | 37 | 20 | 21.13 | 0.66 | 3944 | 42 |
| 30 | 3.07 | 1.95 | 6282 | 37 | 30 | 21.79 | 0.63 | 3902 | 42 |
| 40 | 5.02 | 1.92 | 6245 | 37 | 40 | 22.42 | 0.60 | 3860 | 41 |
| 50 | 6.94 | 1.91 | 6208 | 37 | 50 | 23.02 | 0.59 | 3819 | 42 |
| 31 0 | 10 8.85 | 1.88 | 9.999 6171 | 37 | 41 0 | 11 23.61 | 0.56 | 9.999 3777 | 42 |
| 10 | 10.73 | 1.86 | 6134 | 38 | 10 | 24.17 | 0.53 | 3735 | 42 |
| 20 | 12.59 | 1.85 | 6096 | 37 | 20 | 24.70 | 0.52 | 3693 | 42 |
| 30 | 14.44 | 1.82 | 6059 | 37 | 30 | 25.22 | 0.49 | 3651 | 42 |
| 40 | 16.26 | 1.80 | 6021 | 38 | 40 | 25.71 | 0.47 | 3609 | 42 |
| 50 | 18.06 | 1.78 | 5984 | 38 | 50 | 26.18 | 0.44 | 3567 | 42 |
| 32 0 | 10 19.84 | 1.76 | 9.999 5946 | 38 | 42 0 | 11 26.62 | 0.42 | 9.999 3525 | 42 |
| 10 | 21.60 | 1.74 | 5908 | 38 | 10 | 27.04 | 0.40 | 3483 | 42 |
| 20 | 23.34 | 1.71 | 5870 | 38 | 20 | 27.44 | 0.38 | 3441 | 42 |
| 30 | 25.05 | 1.70 | 5832 | 38 | 30 | 27.82 | 0.35 | 3399 | 42 |
| 40 | 26.75 | 1.68 | 5794 | 39 | 40 | 28.17 | 0.33 | 3357 | 42 |
| 50 | 28.43 | 1.65 | 5755 | 38 | 50 | 28.50 | 0.30 | 3315 | 42 |
| 33 0 | 10 30.08 | 1.63 | 9.999 5717 | 39 | 43 0 | 11 28.80 | 0.28 | 9.999 3273 | 43 |
| 10 | 31.71 | 1.61 | 5678 | 38 | 10 | 29.08 | 0.26 | 3230 | 42 |
| 20 | 33.32 | 1.59 | 5640 | 39 | 20 | 29.34 | 0.24 | 3188 | 42 |
| 30 | 34.91 | 1.57 | 5601 | 39 | 30 | 29.58 | 0.21 | 3146 | 42 |
| 40 | 36.48 | 1.55 | 5562 | 39 | 40 | 29.79 | 0.19 | 3104 | 42 |
| 50 | 38.03 | 1.52 | 5523 | 39 | 50 | 29.98 | 0.16 | 3062 | 43 |
| 34 0 | 10 39.55 | 1.51 | 9.999 5484 | 39 | 44 0 | 11 30.14 | 0.15 | 9.999 3019 | 42 |
| 10 | 41.06 | 1.48 | 5445 | 39 | 10 | 30.29 | 0.12 | 2977 | 42 |
| 20 | 42.54 | 1.46 | 5406 | 39 | 20 | 30.41 | 0.09 | 2935 | 43 |
| 30 | 44.00 | 1.44 | 5367 | 40 | 30 | 30.50 | 0.07 | 2892 | 42 |
| 40 | 45.44 | 1.42 | 5327 | 39 | 40 | 30.57 | 0.05 | 2850 | 42 |
| 50 | 46.86 | 1.39 | 5288 | 40 | 50 | 30.62 | 0.03 | 2808 | 42 |
| 35 0 | 10 48.25 | | 9.999 5248 | | 45 0 | 11 30.65 | | 9.999 2766 | |

TABLE I. Angle of the Vertical and Logarithm of the Earth's Radius.

 ϕ' = Geocentric Latitude. ρ = Earth's Radius.

| ϕ | $\phi - \phi'$ | Diff. | $\log \rho$ | Diff. | ϕ | $\phi - \phi'$ | Diff. | $\log \rho$ | Diff. |
|--------|----------------|-------|-------------|-------|--------|----------------|-------|-------------|-------|
| ° ' " | ' " | " | | | ° ' " | ' " | " | | |
| 45 0 | 11 30.65 | 0.00 | 9.999 2766 | 43 | 55 0 | 10 49.74 | 1.38 | 9.999 0275 | 40 |
| 10 | 30.65 | 0.02 | 2723 | 42 | 10 | 48.36 | 1.39 | 0235 | 40 |
| 20 | 30.63 | 0.05 | 2681 | 42 | 20 | 46.97 | 1.42 | 0195 | 40 |
| 30 | 30.58 | 0.07 | 2639 | 43 | 30 | 45.55 | 1.44 | 0155 | 39 |
| 40 | 30.51 | 0.09 | 2596 | 42 | 40 | 44.11 | 1.46 | 0116 | 40 |
| 50 | 30.42 | 0.11 | 2554 | 42 | 50 | 42.65 | 1.49 | 0076 | 39 |
| 46 0 | 11 30.31 | 0.14 | 9.999 2512 | 42 | 56 0 | 10 41.16 | 1.51 | 9.999 0037 | 39 |
| 10 | 30.17 | 0.16 | 2470 | 43 | 10 | 39.65 | 1.52 | 9.998 9998 | 40 |
| 20 | 30.01 | 0.19 | 2427 | 42 | 20 | 38.13 | 1.55 | 9958 | 39 |
| 30 | 29.82 | 0.21 | 2385 | 42 | 30 | 36.58 | 1.57 | 9919 | 39 |
| 40 | 29.61 | 0.23 | 2343 | 43 | 40 | 35.01 | 1.60 | 9880 | 39 |
| 50 | 29.38 | 0.26 | 2300 | 42 | 50 | 33.41 | 1.61 | 9841 | 39 |
| 47 0 | 11 29.12 | 0.27 | 9.999 2258 | 42 | 57 0 | 10 31.80 | 1.64 | 9.998 9802 | 38 |
| 10 | 28.85 | 0.31 | 2216 | 42 | 10 | 30.16 | 1.66 | 9764 | 39 |
| 20 | 28.54 | 0.32 | 2174 | 42 | 20 | 28.50 | 1.67 | 9725 | 39 |
| 30 | 28.22 | 0.35 | 2132 | 43 | 30 | 26.83 | 1.70 | 9686 | 38 |
| 40 | 27.87 | 0.37 | 2089 | 42 | 40 | 25.13 | 1.73 | 9648 | 38 |
| 50 | 27.50 | 0.40 | 2047 | 42 | 50 | 23.40 | 1.74 | 9610 | 39 |
| 48 0 | 11 27.10 | 0.41 | 9.999 2005 | 42 | 58 0 | 10 21.66 | 1.76 | 9.998 9571 | 38 |
| 10 | 26.69 | 0.45 | 1963 | 42 | 10 | 19.90 | 1.79 | 9533 | 38 |
| 20 | 26.24 | 0.46 | 1921 | 42 | 20 | 18.11 | 1.80 | 9495 | 38 |
| 30 | 25.78 | 0.49 | 1879 | 42 | 30 | 16.31 | 1.83 | 9457 | 38 |
| 40 | 25.29 | 0.51 | 1837 | 42 | 40 | 14.48 | 1.85 | 9419 | 38 |
| 50 | 24.78 | 0.54 | 1795 | 42 | 50 | 12.63 | 1.86 | 9382 | 37 |
| 49 0 | 11 24.24 | 0.55 | 9.999 1753 | 42 | 59 0 | 10 10.77 | 1.89 | 9.998 9344 | 37 |
| 10 | 23.69 | 0.58 | 1711 | 42 | 10 | 8.88 | 1.91 | 9307 | 38 |
| 20 | 23.11 | 0.61 | 1669 | 42 | 20 | 6.97 | 1.93 | 9269 | 37 |
| 30 | 22.50 | 0.63 | 1627 | 41 | 30 | 5.04 | 1.96 | 9232 | 37 |
| 40 | 21.87 | 0.65 | 1586 | 42 | 40 | 3.08 | 1.97 | 9195 | 37 |
| 50 | 21.22 | 0.67 | 1544 | 42 | 50 | 1.11 | 1.99 | 9158 | 37 |
| 50 0 | 11 20.55 | 0.70 | 9.999 1502 | 42 | 60 0 | 9 59.12 | 12.38 | 9.998 9121 | 219 |
| 10 | 19.85 | 0.72 | 1460 | 41 | 61 0 | 9 46.74 | 13.09 | 8902 | 214 |
| 20 | 19.13 | 0.74 | 1419 | 42 | 62 0 | 9 33.65 | 13.80 | 8688 | 209 |
| 30 | 18.39 | 0.76 | 1377 | 42 | 63 0 | 9 19.85 | 14.49 | 8479 | 204 |
| 40 | 17.63 | 0.79 | 1335 | 41 | 64 0 | 9 5.36 | 15.15 | 8275 | 198 |
| 50 | 16.84 | 0.82 | 1294 | 42 | 65 0 | 8 50.21 | 15.81 | 8077 | 193 |
| 51 0 | 11 16.02 | 0.83 | 9.999 1252 | 41 | 66 0 | 8 34.40 | 16.43 | 9.998 7884 | 187 |
| 10 | 15.19 | 0.86 | 1211 | 41 | 67 0 | 8 17.97 | 17.05 | 7697 | 180 |
| 20 | 14.33 | 0.88 | 1170 | 42 | 68 0 | 8 0.92 | 17.63 | 7517 | 175 |
| 30 | 13.45 | 0.90 | 1128 | 41 | 69 0 | 7 43.29 | 18.21 | 7342 | 168 |
| 40 | 12.55 | 0.93 | 1087 | 41 | 70 0 | 7 25.08 | 18.75 | 7174 | 161 |
| 50 | 11.62 | 0.95 | 1046 | 41 | 71 0 | 7 6.33 | 19.27 | 7013 | 154 |
| 52 0 | 11 10.67 | 0.97 | 9.999 1005 | 42 | 72 0 | 6 47.06 | 19.78 | 9.998 6859 | 146 |
| 10 | 9.70 | 0.99 | 0963 | 41 | 73 0 | 6 27.28 | 20.25 | 6713 | 140 |
| 20 | 8.71 | 1.02 | 0922 | 41 | 74 0 | 6 7.03 | 20.70 | 6573 | 132 |
| 30 | 7.69 | 1.03 | 0881 | 41 | 75 0 | 5 46.33 | 21.13 | 6441 | 124 |
| 40 | 6.66 | 1.06 | 0840 | 40 | 76 0 | 5 25.20 | 21.53 | 6317 | 116 |
| 50 | 5.60 | 1.09 | 0800 | 41 | 77 0 | 5 3.67 | 21.90 | 6201 | 108 |
| 53 0 | 11 4.51 | 1.11 | 9.999 0759 | 41 | 78 0 | 4 41.77 | 22.24 | 9.998 6093 | 100 |
| 10 | 3.40 | 1.13 | 0718 | 41 | 79 0 | 4 19.53 | 22.57 | 5993 | 92 |
| 20 | 2.27 | 1.15 | 0677 | 40 | 80 0 | 3 56.96 | 22.86 | 5901 | 83 |
| 30 | 1.12 | 1.18 | 0637 | 41 | 81 0 | 3 34.10 | 23.12 | 5818 | 75 |
| 40 | 59.94 | 1.20 | 0596 | 40 | 82 0 | 3 10.98 | 23.35 | 5743 | 67 |
| 50 | 58.74 | 1.22 | 0556 | 41 | 83 0 | 2 47.63 | 23.56 | 5676 | 57 |
| 54 0 | 10 57.52 | 1.24 | 9.999 0515 | 40 | 84 0 | 2 24.07 | 23.74 | 9.998 5619 | 49 |
| 10 | 56.28 | 1.26 | 0475 | 40 | 85 0 | 2 0.33 | 23.89 | 5570 | 40 |
| 20 | 55.02 | 1.29 | 0435 | 40 | 86 0 | 1 36.44 | 24.01 | 5530 | 32 |
| 30 | 53.73 | 1.31 | 0395 | 40 | 87 0 | 1 12.43 | 24.09 | 5498 | 22 |
| 40 | 52.42 | 1.33 | 0355 | 40 | 88 0 | 0 48.34 | 24.16 | 5476 | 13 |
| 50 | 51.09 | 1.35 | 0315 | 40 | 89 0 | 0 24.18 | 24.18 | 5463 | 5 |
| 55 0 | 10 49.74 | | 9.999 0275 | | 90 0 | 0 0.00 | | 9.998 5458 | |

TABLE II.

For converting intervals of Mean Solar Time into equivalent intervals of Sidereal Time

| Hours. | | | Minutes. | | | Seconds. | | | Decimals. | | |
|----------|----------------|------------|----------|----------------|--|----------|----------------|--|-----------|----------------|--|
| Mean T. | Sidereal Time. | | Mean T. | Sidereal Time. | | Mean T. | Sidereal Time. | | Mean T. | Sidereal Time. | |
| <i>h</i> | <i>h</i> | <i>m s</i> | <i>m</i> | <i>m s</i> | | <i>s</i> | <i>s</i> | | <i>s</i> | <i>s</i> | |
| 1 | 1 0 | 9.856 | 1 | 1 0.164 | | 1 | 1.003 | | 0.02 | 0.020 | |
| 2 | 2 0 | 19.713 | 2 | 2 0.329 | | 2 | 2.005 | | 0.04 | 0.040 | |
| 3 | 3 0 | 29.569 | 3 | 3 0.493 | | 3 | 3.008 | | 0.06 | 0.060 | |
| 4 | 4 0 | 39.426 | 4 | 4 0.657 | | 4 | 4.011 | | 0.08 | 0.080 | |
| 5 | 5 0 | 49.282 | 5 | 5 0.821 | | 5 | 5.014 | | 0.10 | 0.100 | |
| 6 | 6 0 | 59.139 | 6 | 6 0.986 | | 6 | 6.016 | | 0.12 | 0.120 | |
| 7 | 7 1 | 8.995 | 7 | 7 1.150 | | 7 | 7.019 | | 0.14 | 0.140 | |
| 8 | 8 1 | 18.852 | 8 | 8 1.314 | | 8 | 8.022 | | 0.16 | 0.160 | |
| 9 | 9 1 | 28.708 | 9 | 9 1.478 | | 9 | 9.025 | | 0.18 | 0.180 | |
| 10 | 10 1 | 38.565 | 10 | 10 1.643 | | 10 | 10.027 | | 0.20 | 0.201 | |
| 11 | 11 1 | 48.421 | 11 | 11 1.807 | | 11 | 11.030 | | 0.22 | 0.221 | |
| 12 | 12 1 | 58.278 | 12 | 12 1.971 | | 12 | 12.033 | | 0.24 | 0.241 | |
| 13 | 13 2 | 8.134 | 13 | 13 2.136 | | 13 | 13.036 | | 0.26 | 0.261 | |
| 14 | 14 2 | 17.991 | 14 | 14 2.300 | | 14 | 14.038 | | 0.28 | 0.281 | |
| 15 | 15 2 | 27.847 | 15 | 15 2.464 | | 15 | 15.041 | | 0.30 | 0.301 | |
| 16 | 16 2 | 37.704 | 16 | 16 2.628 | | 16 | 16.044 | | 0.32 | 0.321 | |
| 17 | 17 2 | 47.560 | 17 | 17 2.793 | | 17 | 17.047 | | 0.34 | 0.341 | |
| 18 | 18 2 | 57.416 | 18 | 18 2.957 | | 18 | 18.049 | | 0.36 | 0.361 | |
| 19 | 19 3 | 7.273 | 19 | 19 3.121 | | 19 | 19.052 | | 0.38 | 0.381 | |
| 20 | 20 3 | 17.129 | 20 | 20 3.285 | | 20 | 20.055 | | 0.40 | 0.401 | |
| 21 | 21 3 | 26.986 | 21 | 21 3.450 | | 21 | 21.057 | | 0.42 | 0.421 | |
| 22 | 22 3 | 36.842 | 22 | 22 3.614 | | 22 | 22.060 | | 0.44 | 0.441 | |
| 23 | 23 3 | 46.699 | 23 | 23 3.778 | | 23 | 23.063 | | 0.46 | 0.461 | |
| 24 | 24 3 | 56.555 | 24 | 24 3.943 | | 24 | 24.066 | | 0.48 | 0.481 | |
| | | | 25 | 25 4.107 | | 25 | 25.068 | | 0.50 | 0.501 | |
| | | | 26 | 26 4.271 | | 26 | 26.071 | | 0.52 | 0.521 | |
| | | | 27 | 27 4.435 | | 27 | 27.074 | | 0.54 | 0.541 | |
| | | | 28 | 28 4.600 | | 28 | 28.077 | | 0.56 | 0.562 | |
| | | | 29 | 29 4.764 | | 29 | 29.079 | | 0.58 | 0.582 | |
| | | | 30 | 30 4.928 | | 30 | 30.082 | | 0.60 | 0.602 | |
| | | | 31 | 31 5.092 | | 31 | 31.085 | | 0.62 | 0.622 | |
| | | | 32 | 32 5.257 | | 32 | 32.088 | | 0.64 | 0.642 | |
| | | | 33 | 33 5.421 | | 33 | 33.090 | | 0.66 | 0.662 | |
| | | | 34 | 34 5.585 | | 34 | 34.093 | | 0.68 | 0.682 | |
| | | | 35 | 35 5.750 | | 35 | 35.096 | | 0.70 | 0.702 | |
| | | | 36 | 36 5.914 | | 36 | 36.099 | | 0.72 | 0.722 | |
| | | | 37 | 37 6.078 | | 37 | 37.101 | | 0.74 | 0.742 | |
| | | | 38 | 38 6.242 | | 38 | 38.104 | | 0.76 | 0.762 | |
| | | | 39 | 39 6.407 | | 39 | 39.107 | | 0.78 | 0.782 | |
| | | | 40 | 40 6.571 | | 40 | 40.110 | | 0.80 | 0.802 | |
| | | | 41 | 41 6.735 | | 41 | 41.112 | | 0.82 | 0.822 | |
| | | | 42 | 42 6.899 | | 42 | 42.115 | | 0.84 | 0.842 | |
| | | | 43 | 43 7.064 | | 43 | 43.118 | | 0.86 | 0.862 | |
| | | | 44 | 44 7.228 | | 44 | 44.120 | | 0.88 | 0.882 | |
| | | | 45 | 45 7.392 | | 45 | 45.123 | | 0.90 | 0.902 | |
| | | | 46 | 46 7.557 | | 46 | 46.126 | | 0.92 | 0.923 | |
| | | | 47 | 47 7.721 | | 47 | 47.129 | | 0.94 | 0.943 | |
| | | | 48 | 48 7.885 | | 48 | 48.131 | | 0.96 | 0.963 | |
| | | | 49 | 49 8.049 | | 49 | 49.134 | | 0.98 | 0.983 | |
| | | | 50 | 50 8.214 | | 50 | 50.137 | | 1.00 | 1.003 | |
| | | | 51 | 51 8.378 | | 51 | 51.140 | | | | |
| | | | 52 | 52 8.542 | | 52 | 52.142 | | | | |
| | | | 53 | 53 8.707 | | 53 | 53.145 | | | | |
| | | | 54 | 54 8.871 | | 54 | 54.148 | | | | |
| | | | 55 | 55 9.035 | | 55 | 55.151 | | | | |
| | | | 56 | 56 9.199 | | 56 | 56.153 | | | | |
| | | | 57 | 57 9.364 | | 57 | 57.156 | | | | |
| | | | 58 | 58 9.528 | | 58 | 58.159 | | | | |
| | | | 59 | 59 9.692 | | 59 | 59.162 | | | | |
| | | | 60 | 60 9.856 | | 60 | 60.164 | | | | |

This table is useful for the conversion of Mean Solar into Sidereal Time.
 Sidereal Time required = sidereal time at the preceding mean noon
 + the equivalent to the given mean time.

TABLE III.

For converting intervals of Sidereal Time into equivalent intervals of Mean Solar Time.

| Hours. | | | Minutes. | | | Seconds. | | Decimals. | |
|----------|------------|------------|----------|------------|--------|----------|------------|-----------|------------|
| Sid. T. | Mean Time. | | Sid. T. | Mean Time. | | Sid. T. | Mean Time. | Sid. T. | Mean Time. |
| <i>h</i> | <i>h</i> | <i>m s</i> | <i>m</i> | <i>m s</i> | | <i>s</i> | <i>s</i> | <i>s</i> | <i>s</i> |
| 1 | 0 | 59 50.170 | 1 | 0 | 59.836 | 1 | 0.997 | 0.02 | 0.020 |
| 2 | 1 | 59 40.341 | 2 | 1 | 59.672 | 2 | 1.995 | 0.04 | 0.040 |
| 3 | 2 | 59 30.511 | 3 | 2 | 59.509 | 3 | 2.992 | 0.06 | 0.060 |
| 4 | 3 | 59 20.682 | 4 | 3 | 59.345 | 4 | 3.989 | 0.08 | 0.080 |
| 5 | 4 | 59 10.852 | 5 | 4 | 59.181 | 5 | 4.986 | 0.10 | 0.100 |
| 6 | 5 | 59 1.023 | 6 | 5 | 59.017 | 6 | 5.984 | 0.12 | 0.120 |
| 7 | 6 | 58 51.193 | 7 | 6 | 58.853 | 7 | 6.981 | 0.14 | 0.140 |
| 8 | 7 | 58 41.363 | 8 | 7 | 58.689 | 8 | 7.978 | 0.16 | 0.160 |
| 9 | 8 | 58 31.534 | 9 | 8 | 58.526 | 9 | 8.975 | 0.18 | 0.180 |
| 10 | 9 | 58 21.704 | 10 | 9 | 58.362 | 10 | 9.973 | 0.20 | 0.199 |
| 11 | 10 | 58 11.875 | 11 | 10 | 58.198 | 11 | 10.970 | 0.22 | 0.219 |
| 12 | 11 | 58 2.045 | 12 | 11 | 58.034 | 12 | 11.967 | 0.24 | 0.239 |
| 13 | 12 | 57 52.216 | 13 | 12 | 57.870 | 13 | 12.964 | 0.26 | 0.259 |
| 14 | 13 | 57 42.386 | 14 | 13 | 57.706 | 14 | 13.962 | 0.28 | 0.279 |
| 15 | 14 | 57 32.557 | 15 | 14 | 57.543 | 15 | 14.959 | 0.30 | 0.299 |
| 16 | 15 | 57 22.727 | 16 | 15 | 57.379 | 16 | 15.956 | 0.32 | 0.319 |
| 17 | 16 | 57 12.897 | 17 | 16 | 57.215 | 17 | 16.954 | 0.34 | 0.339 |
| 18 | 17 | 57 3.068 | 18 | 17 | 57.051 | 18 | 17.951 | 0.36 | 0.359 |
| 19 | 18 | 56 53.238 | 19 | 18 | 56.887 | 19 | 18.948 | 0.38 | 0.379 |
| 20 | 19 | 56 43.409 | 20 | 19 | 56.723 | 20 | 19.945 | 0.40 | 0.399 |
| 21 | 20 | 56 33.579 | 21 | 20 | 56.560 | 21 | 20.943 | 0.42 | 0.419 |
| 22 | 21 | 56 23.750 | 22 | 21 | 56.396 | 22 | 21.940 | 0.44 | 0.439 |
| 23 | 22 | 56 13.920 | 23 | 22 | 56.232 | 23 | 22.937 | 0.46 | 0.459 |
| 24 | 23 | 56 4.091 | 24 | 23 | 56.068 | 24 | 23.934 | 0.48 | 0.479 |
| | | | 25 | 24 | 55.904 | 25 | 24.932 | 0.50 | 0.499 |
| | | | 26 | 25 | 55.740 | 26 | 25.929 | 0.52 | 0.519 |
| | | | 27 | 26 | 55.577 | 27 | 26.926 | 0.54 | 0.539 |
| | | | 28 | 27 | 55.413 | 28 | 27.924 | 0.56 | 0.558 |
| | | | 29 | 28 | 55.249 | 29 | 28.921 | 0.58 | 0.578 |
| | | | 30 | 29 | 55.085 | 30 | 29.918 | 0.60 | 0.598 |
| | | | 31 | 30 | 54.921 | 31 | 30.915 | 0.62 | 0.618 |
| | | | 32 | 31 | 54.758 | 32 | 31.913 | 0.64 | 0.638 |
| | | | 33 | 32 | 54.594 | 33 | 32.910 | 0.66 | 0.658 |
| | | | 34 | 33 | 54.430 | 34 | 33.907 | 0.68 | 0.678 |
| | | | 35 | 34 | 54.266 | 35 | 34.904 | 0.70 | 0.698 |
| | | | 36 | 35 | 54.102 | 36 | 35.902 | 0.72 | 0.718 |
| | | | 37 | 36 | 53.938 | 37 | 36.899 | 0.74 | 0.738 |
| | | | 38 | 37 | 53.775 | 38 | 37.896 | 0.76 | 0.758 |
| | | | 39 | 38 | 53.611 | 39 | 38.894 | 0.78 | 0.778 |
| | | | 40 | 39 | 53.447 | 40 | 39.891 | 0.80 | 0.798 |
| | | | 41 | 40 | 53.283 | 41 | 40.888 | 0.82 | 0.818 |
| | | | 42 | 41 | 53.119 | 42 | 41.885 | 0.84 | 0.838 |
| | | | 43 | 42 | 52.955 | 43 | 42.883 | 0.86 | 0.858 |
| | | | 44 | 43 | 52.792 | 44 | 43.880 | 0.88 | 0.878 |
| | | | 45 | 44 | 52.628 | 45 | 44.877 | 0.90 | 0.898 |
| | | | 46 | 45 | 52.464 | 46 | 45.874 | 0.92 | 0.917 |
| | | | 47 | 46 | 52.300 | 47 | 46.872 | 0.94 | 0.937 |
| | | | 48 | 47 | 52.136 | 48 | 47.869 | 0.96 | 0.957 |
| | | | 49 | 48 | 51.972 | 49 | 48.866 | 0.98 | 0.977 |
| | | | 50 | 49 | 51.809 | 50 | 49.863 | 1.00 | 0.997 |
| | | | 51 | 50 | 51.645 | 51 | 50.861 | | |
| | | | 52 | 51 | 51.481 | 52 | 51.858 | | |
| | | | 53 | 52 | 51.317 | 53 | 52.855 | | |
| | | | 54 | 53 | 51.153 | 54 | 53.853 | | |
| | | | 55 | 54 | 50.990 | 55 | 54.850 | | |
| | | | 56 | 55 | 50.826 | 56 | 55.847 | | |
| | | | 57 | 56 | 50.662 | 57 | 56.844 | | |
| | | | 58 | 57 | 50.498 | 58 | 57.842 | | |
| | | | 59 | 58 | 50.334 | 59 | 58.839 | | |
| | | | 60 | 59 | 50.170 | 60 | 59.836 | | |

This table is useful for the conversion of Sidereal into Mean Solar Time.
Mean solar time required = mean time at the preceding sidereal noon
+ the equivalent to the given sidereal time.

TABLE IV.

For converting Hours, Minutes, and Seconds into Decimals of a Day.

| Hours. | Decimal. | Min. | Decimal. | Min. | Decimal. | Sec. | Decimal. | Sec. | Decimal. |
|--------|----------|------|-----------|------|-----------|------|----------|------|----------|
| 1 | 0.0416 + | 1 | .000694 + | 31 | .021527 + | 1 | .0000116 | 31 | .0003588 |
| 2 | .0833 + | 2 | .001388 + | 32 | .022222 + | 2 | .0000231 | 32 | .0003704 |
| 3 | .1250 + | 3 | .002083 + | 33 | .022916 + | 3 | .0000347 | 33 | .0003819 |
| 4 | .1666 + | 4 | .002777 + | 34 | .023611 + | 4 | .0000463 | 34 | .0003935 |
| 5 | .2083 + | 5 | .003472 + | 35 | .024305 + | 5 | .0000579 | 35 | .0004051 |
| 6 | .2500 + | 6 | .004166 + | 36 | .025000 + | 6 | .0000694 | 36 | .0004167 |
| 7 | 0.2916 + | 7 | .004861 + | 37 | .025694 + | 7 | .0000810 | 37 | .0004282 |
| 8 | .3333 + | 8 | .005555 + | 38 | .026388 + | 8 | .0000925 | 38 | .0004398 |
| 9 | .3750 + | 9 | .006250 + | 39 | .027083 + | 9 | .0001042 | 39 | .0004514 |
| 10 | .4166 + | 10 | .006944 + | 40 | .027777 + | 10 | .0001157 | 40 | .0004630 |
| 11 | .4583 + | 11 | .007638 + | 41 | .028472 + | 11 | .0001273 | 41 | .0004745 |
| 12 | .5000 + | 12 | .008333 + | 42 | .029166 + | 12 | .0001389 | 42 | .0004861 |
| 13 | 0.5416 + | 13 | .009027 + | 43 | .029861 + | 13 | .0001505 | 43 | .0004977 |
| 14 | .5833 + | 14 | .009722 + | 44 | .030555 + | 14 | .0001620 | 44 | .0005093 |
| 15 | .6250 + | 15 | .010416 + | 45 | .031250 + | 15 | .0001736 | 45 | .0005208 |
| 16 | .6666 + | 16 | .011111 + | 46 | .031944 + | 16 | .0001852 | 46 | .0005324 |
| 17 | .7083 + | 17 | .011805 + | 47 | .032638 + | 17 | .0001968 | 47 | .0005440 |
| 18 | .7500 + | 18 | .012500 + | 48 | .033333 + | 18 | .0002083 | 48 | .0005556 |
| 19 | 0.7916 + | 19 | .013194 + | 49 | .034027 + | 19 | .0002199 | 49 | .0005671 |
| 20 | .8333 + | 20 | .013888 + | 50 | .034722 + | 20 | .0002315 | 50 | .0005787 |
| 21 | .8750 + | 21 | .014583 + | 51 | .035416 + | 21 | .0002431 | 51 | .0005903 |
| 22 | .9166 + | 22 | .015277 + | 52 | .036111 + | 22 | .0002546 | 52 | .0006019 |
| 23 | 0.9583 + | 23 | .015972 + | 53 | .036805 + | 23 | .0002662 | 53 | .0006134 |
| 24 | 1.0000 + | 24 | .016666 + | 54 | .037500 + | 24 | .0002778 | 54 | .0006250 |
| | | 25 | .017361 + | 55 | .038194 + | 25 | .0002894 | 55 | .0006366 |
| | | 26 | .018055 + | 56 | .038888 + | 26 | .0003009 | 56 | .0006481 |
| | | 27 | .018750 + | 57 | .039583 + | 27 | .0003125 | 57 | .0006597 |
| | | 28 | .019444 + | 58 | .040277 + | 28 | .0003241 | 58 | .0006713 |
| | | 29 | .020138 + | 59 | .040972 + | 29 | .0003356 | 59 | .0006829 |
| | | 30 | .020833 + | 60 | .041666 + | 30 | .0003472 | 60 | .0006944 |

The sign +, appended to numbers in this table, signifies that the last figure repeats to infinity

TABLE V.

For finding the number of Days from the beginning of the Year.

| Date. | Com. | Bis. |
|---------------|------|------|
| January 0.0 | 0 | 0 |
| February 0.0 | 31 | 31 |
| March 0.0 | 59 | 60 |
| April 0.0 | 90 | 91 |
| May 0.0 | 120 | 121 |
| June 0.0 | 151 | 152 |
| July 0.0 | 181 | 182 |
| August 0.0 | 212 | 213 |
| September 0.0 | 243 | 244 |
| October 0.0 | 273 | 274 |
| November 0.0 | 304 | 305 |
| December 0.0 | 334 | 335 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 0° | | 1° | | 2° | | 3° | |
|----|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0' | 0.000000 | 181.81 | 0.654532 | 181.83 | 1.309263 | 181.92 | 1.964393 | 182.05 |
| 1 | 0.010908 | 181.81 | 0.665442 | 181.83 | 1.320178 | 181.92 | 1.975316 | 182.06 |
| 2 | 0.021817 | 181.81 | 0.676352 | 181.83 | 1.331093 | 181.92 | 1.986240 | 182.06 |
| 3 | 0.032725 | 181.81 | 0.687262 | 181.84 | 1.342008 | 181.92 | 1.997164 | 182.06 |
| 4 | 0.043633 | 181.81 | 0.698172 | 181.84 | 1.352923 | 181.92 | 2.008087 | 182.07 |
| 5 | 0.054542 | 181.81 | 0.709082 | 181.84 | 1.363839 | 181.93 | 2.019011 | 182.07 |
| 6 | 0.065450 | 181.81 | 0.719993 | 181.84 | 1.374755 | 181.93 | 2.029936 | 182.07 |
| 7 | 0.076358 | 181.81 | 0.730903 | 181.84 | 1.385670 | 181.93 | 2.040860 | 182.07 |
| 8 | 0.087267 | 181.81 | 0.741813 | 181.84 | 1.396586 | 181.93 | 2.051785 | 182.08 |
| 9 | 0.098175 | 181.81 | 0.752724 | 181.84 | 1.407502 | 181.93 | 2.062709 | 182.08 |
| 10 | 0.109083 | 181.81 | 0.763634 | 181.84 | 1.418418 | 181.94 | 2.073634 | 182.08 |
| 11 | 0.119992 | 181.81 | 0.774545 | 181.84 | 1.429334 | 181.94 | 2.084559 | 182.08 |
| 12 | 0.130900 | 181.81 | 0.785456 | 181.84 | 1.440251 | 181.94 | 2.095485 | 182.09 |
| 13 | 0.141808 | 181.81 | 0.796366 | 181.85 | 1.451167 | 181.94 | 2.106410 | 182.09 |
| 14 | 0.152717 | 181.81 | 0.807277 | 181.85 | 1.462083 | 181.94 | 2.117335 | 182.09 |
| 15 | 0.163625 | 181.81 | 0.818188 | 181.85 | 1.473000 | 181.95 | 2.128261 | 182.10 |
| 16 | 0.174534 | 181.81 | 0.829099 | 181.85 | 1.483917 | 181.95 | 2.139187 | 182.10 |
| 17 | 0.185442 | 181.81 | 0.840010 | 181.85 | 1.494834 | 181.95 | 2.150114 | 182.10 |
| 18 | 0.196350 | 181.81 | 0.850921 | 181.85 | 1.505751 | 181.95 | 2.161040 | 182.11 |
| 19 | 0.207259 | 181.81 | 0.861832 | 181.85 | 1.516668 | 181.95 | 2.171966 | 182.11 |
| 20 | 0.218167 | 181.81 | 0.872743 | 181.85 | 1.527585 | 181.96 | 2.182894 | 182.11 |
| 21 | 0.229076 | 181.81 | 0.883654 | 181.86 | 1.538503 | 181.96 | 2.193820 | 182.12 |
| 22 | 0.239984 | 181.81 | 0.894566 | 181.86 | 1.549420 | 181.96 | 2.204747 | 182.12 |
| 23 | 0.250893 | 181.81 | 0.905478 | 181.86 | 1.560338 | 181.96 | 2.215674 | 182.12 |
| 24 | 0.261801 | 181.81 | 0.916389 | 181.86 | 1.571256 | 181.96 | 2.226602 | 182.13 |
| 25 | 0.272710 | 181.81 | 0.927301 | 181.86 | 1.582174 | 181.97 | 2.237529 | 182.13 |
| 26 | 0.283619 | 181.81 | 0.938212 | 181.86 | 1.593092 | 181.97 | 2.248457 | 182.13 |
| 27 | 0.294527 | 181.81 | 0.949124 | 181.86 | 1.604010 | 181.97 | 2.259385 | 182.14 |
| 28 | 0.305436 | 181.81 | 0.960036 | 181.86 | 1.614928 | 181.97 | 2.270313 | 182.14 |
| 29 | 0.316345 | 181.81 | 0.970948 | 181.87 | 1.625847 | 181.97 | 2.281242 | 182.14 |
| 30 | 0.327253 | 181.81 | 0.981860 | 181.87 | 1.636766 | 181.98 | 2.292170 | 182.14 |
| 31 | 0.338162 | 181.81 | 0.992772 | 181.87 | 1.647684 | 181.98 | 2.303099 | 182.15 |
| 32 | 0.349071 | 181.81 | 1.003684 | 181.87 | 1.658603 | 181.98 | 2.314028 | 182.15 |
| 33 | 0.359980 | 181.81 | 1.014596 | 181.87 | 1.669522 | 181.98 | 2.324957 | 182.15 |
| 34 | 0.370888 | 181.81 | 1.025509 | 181.87 | 1.680441 | 181.99 | 2.335887 | 182.16 |
| 35 | 0.381797 | 181.81 | 1.036421 | 181.87 | 1.691361 | 181.99 | 2.346816 | 182.16 |
| 36 | 0.392706 | 181.81 | 1.047334 | 181.87 | 1.702280 | 181.99 | 2.357746 | 182.16 |
| 37 | 0.403615 | 181.81 | 1.058246 | 181.88 | 1.713200 | 181.99 | 2.368676 | 182.17 |
| 38 | 0.414524 | 181.82 | 1.069159 | 181.88 | 1.724120 | 182.00 | 2.379606 | 182.17 |
| 39 | 0.425433 | 181.82 | 1.080072 | 181.88 | 1.735039 | 182.00 | 2.390536 | 182.17 |
| 40 | 0.436342 | 181.82 | 1.090985 | 181.88 | 1.745960 | 182.00 | 2.401467 | 182.18 |
| 41 | 0.447251 | 181.82 | 1.101898 | 181.88 | 1.756880 | 182.00 | 2.412398 | 182.18 |
| 42 | 0.458160 | 181.82 | 1.112811 | 181.89 | 1.767800 | 182.01 | 2.423329 | 182.18 |
| 43 | 0.469069 | 181.82 | 1.123724 | 181.89 | 1.778721 | 182.01 | 2.434260 | 182.19 |
| 44 | 0.479979 | 181.82 | 1.134637 | 181.89 | 1.789641 | 182.01 | 2.445191 | 182.19 |
| 45 | 0.490888 | 181.82 | 1.145550 | 181.89 | 1.800562 | 182.01 | 2.456123 | 182.19 |
| 46 | 0.501797 | 181.82 | 1.156464 | 181.89 | 1.811483 | 182.02 | 2.467055 | 182.20 |
| 47 | 0.512706 | 181.82 | 1.167377 | 181.89 | 1.822404 | 182.02 | 2.477987 | 182.20 |
| 48 | 0.523616 | 181.82 | 1.178291 | 181.89 | 1.833325 | 182.02 | 2.488919 | 182.20 |
| 49 | 0.534525 | 181.82 | 1.189205 | 181.90 | 1.844247 | 182.02 | 2.499851 | 182.21 |
| 50 | 0.545435 | 181.82 | 1.200119 | 181.90 | 1.855168 | 182.03 | 2.510784 | 182.21 |
| 51 | 0.556344 | 181.82 | 1.211033 | 181.90 | 1.866090 | 182.03 | 2.521717 | 182.22 |
| 52 | 0.567254 | 181.82 | 1.221947 | 181.90 | 1.877012 | 182.04 | 2.532650 | 182.22 |
| 53 | 0.578163 | 181.83 | 1.232861 | 181.90 | 1.887934 | 182.04 | 2.543583 | 182.22 |
| 54 | 0.589073 | 181.83 | 1.243775 | 181.91 | 1.898856 | 182.04 | 2.554517 | 182.23 |
| 55 | 0.599983 | 181.83 | 1.254689 | 181.91 | 1.909779 | 182.04 | 2.565450 | 182.23 |
| 56 | 0.610892 | 181.83 | 1.265604 | 181.91 | 1.920701 | 182.04 | 2.576384 | 182.23 |
| 57 | 0.621802 | 181.83 | 1.276518 | 181.91 | 1.931624 | 182.05 | 2.587319 | 182.24 |
| 58 | 0.632712 | 181.83 | 1.287433 | 181.91 | 1.942547 | 182.05 | 2.598253 | 182.24 |
| 59 | 0.643622 | 181.83 | 1.298348 | 181.91 | 1.953470 | 182.05 | 2.609187 | 182.24 |
| 60 | 0.654532 | 181.83 | 1.309263 | 181.92 | 1.964393 | 182.05 | 2.620122 | 182.25 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 4° | | 5° | | 6° | | 7° | |
|----|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0' | 2.620122 | 182.25 | 3.276651 | 182.50 | 3.934182 | 182.80 | 4.592917 | 183.17 |
| 1 | 2.631057 | 182.25 | 3.287602 | 182.50 | 3.945151 | 182.81 | 4.603907 | 183.18 |
| 2 | 2.641993 | 182.26 | 3.298552 | 182.51 | 3.956119 | 182.82 | 4.614898 | 183.18 |
| 3 | 2.652928 | 182.26 | 3.309503 | 182.51 | 3.967088 | 182.82 | 4.625889 | 183.19 |
| 4 | 2.663864 | 182.26 | 3.320454 | 182.52 | 3.978058 | 182.83 | 4.636880 | 183.19 |
| 5 | 2.674800 | 182.27 | 3.331405 | 182.52 | 3.989028 | 182.83 | 4.647872 | 183.20 |
| 6 | 2.685736 | 182.27 | 3.342356 | 182.53 | 3.999998 | 182.84 | 4.658864 | 183.21 |
| 7 | 2.696672 | 182.27 | 3.353308 | 182.53 | 4.010968 | 182.84 | 4.669857 | 183.21 |
| 8 | 2.707609 | 182.28 | 3.364260 | 182.54 | 4.021939 | 182.85 | 4.680850 | 183.22 |
| 9 | 2.718546 | 182.28 | 3.375212 | 182.54 | 4.032911 | 182.86 | 4.691843 | 183.23 |
| 10 | 2.729483 | 182.29 | 3.386165 | 182.55 | 4.043882 | 182.86 | 4.702837 | 183.24 |
| 11 | 2.740420 | 182.29 | 3.397118 | 182.55 | 4.054854 | 182.87 | 4.713831 | 183.24 |
| 12 | 2.751358 | 182.29 | 3.408071 | 182.56 | 4.065826 | 182.87 | 4.724826 | 183.25 |
| 13 | 2.762295 | 182.30 | 3.419024 | 182.56 | 4.076799 | 182.88 | 4.735821 | 183.25 |
| 14 | 2.773233 | 182.30 | 3.429978 | 182.57 | 4.087772 | 182.88 | 4.746816 | 183.26 |
| 15 | 2.784172 | 182.31 | 3.440932 | 182.57 | 4.098745 | 182.89 | 4.757812 | 183.27 |
| 16 | 2.795110 | 182.31 | 3.451887 | 182.58 | 4.109718 | 182.90 | 4.768809 | 183.27 |
| 17 | 2.806049 | 182.31 | 3.462841 | 182.58 | 4.120692 | 182.90 | 4.779805 | 183.28 |
| 18 | 2.816988 | 182.32 | 3.473796 | 182.59 | 4.131667 | 182.91 | 4.790802 | 183.28 |
| 19 | 2.827927 | 182.32 | 3.484752 | 182.59 | 4.142641 | 182.91 | 4.801800 | 183.29 |
| 20 | 2.838867 | 182.33 | 3.495707 | 182.60 | 4.153616 | 182.92 | 4.812797 | 183.30 |
| 21 | 2.849806 | 182.33 | 3.506663 | 182.60 | 4.164592 | 182.93 | 4.823796 | 183.31 |
| 22 | 2.860746 | 182.33 | 3.517619 | 182.61 | 4.175568 | 182.93 | 4.834795 | 183.32 |
| 23 | 2.871686 | 182.34 | 3.528575 | 182.61 | 4.186544 | 182.94 | 4.845794 | 183.32 |
| 24 | 2.882627 | 182.34 | 3.539532 | 182.61 | 4.197520 | 182.94 | 4.856793 | 183.33 |
| 25 | 2.893567 | 182.35 | 3.550489 | 182.62 | 4.208497 | 182.95 | 4.867793 | 183.34 |
| 26 | 2.904508 | 182.35 | 3.561447 | 182.62 | 4.219474 | 182.95 | 4.878793 | 183.34 |
| 27 | 2.915449 | 182.36 | 3.572404 | 182.63 | 4.230451 | 182.96 | 4.889794 | 183.35 |
| 28 | 2.926391 | 182.36 | 3.583362 | 182.63 | 4.241429 | 182.97 | 4.900795 | 183.36 |
| 29 | 2.937332 | 182.36 | 3.594320 | 182.64 | 4.252408 | 182.97 | 4.911797 | 183.36 |
| 30 | 2.948274 | 182.37 | 3.605279 | 182.64 | 4.263386 | 182.98 | 4.922799 | 183.37 |
| 31 | 2.959217 | 182.37 | 3.616238 | 182.65 | 4.274365 | 182.99 | 4.933801 | 183.38 |
| 32 | 2.970159 | 182.37 | 3.627197 | 182.65 | 4.285344 | 182.99 | 4.944804 | 183.38 |
| 33 | 2.981102 | 182.38 | 3.638156 | 182.66 | 4.296324 | 183.00 | 4.955807 | 183.39 |
| 34 | 2.992045 | 182.38 | 3.649116 | 182.66 | 4.307304 | 183.00 | 4.966811 | 183.40 |
| 35 | 3.002988 | 182.39 | 3.660076 | 182.67 | 4.318284 | 183.01 | 4.977815 | 183.41 |
| 36 | 3.013931 | 182.39 | 3.671037 | 182.68 | 4.329265 | 183.01 | 4.988820 | 183.41 |
| 37 | 3.024875 | 182.39 | 3.681997 | 182.68 | 4.340246 | 183.02 | 4.999825 | 183.42 |
| 38 | 3.035819 | 182.40 | 3.692958 | 182.69 | 4.351228 | 183.03 | 5.010830 | 183.43 |
| 39 | 3.046763 | 182.40 | 3.703920 | 182.69 | 4.362210 | 183.03 | 5.021836 | 183.43 |
| 40 | 3.057707 | 182.41 | 3.714881 | 182.70 | 4.373192 | 183.04 | 5.032842 | 183.44 |
| 41 | 3.068652 | 182.41 | 3.725843 | 182.70 | 4.384175 | 183.05 | 5.043849 | 183.45 |
| 42 | 3.079597 | 182.42 | 3.736806 | 182.71 | 4.395158 | 183.05 | 5.054856 | 183.46 |
| 43 | 3.090542 | 182.42 | 3.747768 | 182.71 | 4.406141 | 183.06 | 5.065864 | 183.46 |
| 44 | 3.101488 | 182.43 | 3.758731 | 182.72 | 4.417125 | 183.06 | 5.076872 | 183.47 |
| 45 | 3.112433 | 182.43 | 3.769694 | 182.72 | 4.428109 | 183.07 | 5.087880 | 183.48 |
| 46 | 3.123379 | 182.44 | 3.780658 | 182.72 | 4.439093 | 183.08 | 5.098889 | 183.48 |
| 47 | 3.134325 | 182.44 | 3.791622 | 182.73 | 4.450078 | 183.08 | 5.109898 | 183.49 |
| 48 | 3.145272 | 182.44 | 3.802586 | 182.74 | 4.461064 | 183.09 | 5.120908 | 183.50 |
| 49 | 3.156219 | 182.45 | 3.813551 | 182.74 | 4.472049 | 183.10 | 5.131918 | 183.51 |
| 50 | 3.167166 | 182.45 | 3.824515 | 182.75 | 4.483035 | 183.10 | 5.142929 | 183.51 |
| 51 | 3.178113 | 182.46 | 3.835481 | 182.76 | 4.494022 | 183.11 | 5.153940 | 183.52 |
| 52 | 3.189061 | 182.46 | 3.846446 | 182.76 | 4.505008 | 183.12 | 5.164951 | 183.53 |
| 53 | 3.200009 | 182.47 | 3.857412 | 182.77 | 4.515995 | 183.12 | 5.175963 | 183.54 |
| 54 | 3.210957 | 182.47 | 3.868378 | 182.77 | 4.526983 | 183.13 | 5.186975 | 183.54 |
| 55 | 3.221905 | 182.48 | 3.879345 | 182.78 | 4.537971 | 183.14 | 5.197988 | 183.55 |
| 56 | 3.232854 | 182.48 | 3.890312 | 182.78 | 4.548959 | 183.14 | 5.209002 | 183.56 |
| 57 | 3.243803 | 182.49 | 3.901279 | 182.79 | 4.559948 | 183.15 | 5.220015 | 183.57 |
| 58 | 3.254752 | 182.49 | 3.912246 | 182.79 | 4.570937 | 183.15 | 5.231029 | 183.57 |
| 59 | 3.265702 | 182.49 | 3.923214 | 182.80 | 4.581927 | 183.16 | 5.242044 | 183.58 |
| 60 | 3.276651 | 182.50 | 3.934182 | 182.80 | 4.592917 | 183.17 | 5.253059 | 183.59 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 8° | | 9° | | 10° | | 11° | |
|----|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0' | 5.253059 | 183.59 | 5.914815 | 184.06 | 6.578391 | 184.60 | 7.243997 | 185.19 |
| 1 | 5.264075 | 183.59 | 5.925859 | 184.07 | 6.589467 | 184.61 | 7.255109 | 185.20 |
| 2 | 5.275090 | 183.60 | 5.936904 | 184.08 | 6.600544 | 184.62 | 7.266222 | 185.21 |
| 3 | 5.286107 | 183.61 | 5.947949 | 184.09 | 6.611622 | 184.63 | 7.277335 | 185.22 |
| 4 | 5.297124 | 183.62 | 5.958995 | 184.10 | 6.622700 | 184.64 | 7.288449 | 185.23 |
| 5 | 5.308141 | 183.62 | 5.970041 | 184.11 | 6.633778 | 184.65 | 7.299563 | 185.25 |
| 6 | 5.319159 | 183.63 | 5.981087 | 184.11 | 6.644857 | 184.66 | 7.310678 | 185.26 |
| 7 | 5.330177 | 183.64 | 5.992134 | 184.12 | 6.655937 | 184.67 | 7.321793 | 185.27 |
| 8 | 5.341195 | 183.65 | 6.003182 | 184.13 | 6.667017 | 184.67 | 7.332909 | 185.28 |
| 9 | 5.352214 | 183.66 | 6.014230 | 184.14 | 6.678098 | 184.68 | 7.344026 | 185.29 |
| 10 | 5.363234 | 183.66 | 6.025279 | 184.15 | 6.689179 | 184.69 | 7.355144 | 185.30 |
| 11 | 5.374254 | 183.67 | 6.036328 | 184.16 | 6.700261 | 184.70 | 7.366262 | 185.31 |
| 12 | 5.385275 | 183.68 | 6.047378 | 184.17 | 6.711343 | 184.71 | 7.377381 | 185.32 |
| 13 | 5.396296 | 183.69 | 6.058428 | 184.18 | 6.722426 | 184.72 | 7.388500 | 185.33 |
| 14 | 5.407317 | 183.69 | 6.069479 | 184.18 | 6.733510 | 184.73 | 7.399620 | 185.34 |
| 15 | 5.418339 | 183.70 | 6.080530 | 184.19 | 6.744594 | 184.74 | 7.410741 | 185.35 |
| 16 | 5.429361 | 183.71 | 6.091582 | 184.20 | 6.755679 | 184.75 | 7.421862 | 185.36 |
| 17 | 5.440384 | 183.72 | 6.102634 | 184.21 | 6.766764 | 184.76 | 7.432983 | 185.37 |
| 18 | 5.451407 | 183.73 | 6.113687 | 184.22 | 6.777850 | 184.77 | 7.444106 | 185.38 |
| 19 | 5.462431 | 183.73 | 6.124740 | 184.23 | 6.788937 | 184.78 | 7.455230 | 185.39 |
| 20 | 5.473455 | 183.74 | 6.135794 | 184.24 | 6.800024 | 184.79 | 7.466354 | 185.40 |
| 21 | 5.484480 | 183.75 | 6.146849 | 184.25 | 6.811112 | 184.80 | 7.477478 | 185.41 |
| 22 | 5.495505 | 183.75 | 6.157904 | 184.25 | 6.822200 | 184.81 | 7.488603 | 185.42 |
| 23 | 5.506530 | 183.76 | 6.168959 | 184.26 | 6.833289 | 184.82 | 7.499729 | 185.43 |
| 24 | 5.517556 | 183.77 | 6.180015 | 184.27 | 6.844378 | 184.83 | 7.510855 | 185.44 |
| 25 | 5.528583 | 183.78 | 6.191072 | 184.28 | 6.855468 | 184.84 | 7.521982 | 185.46 |
| 26 | 5.539610 | 183.79 | 6.202129 | 184.29 | 6.866559 | 184.85 | 7.533110 | 185.47 |
| 27 | 5.550637 | 183.79 | 6.213187 | 184.30 | 6.877650 | 184.86 | 7.544239 | 185.48 |
| 28 | 5.561665 | 183.80 | 6.224245 | 184.31 | 6.888742 | 184.87 | 7.555368 | 185.49 |
| 29 | 5.572693 | 183.81 | 6.235304 | 184.32 | 6.899834 | 184.88 | 7.566497 | 185.50 |
| 30 | 5.583722 | 183.82 | 6.246363 | 184.32 | 6.910927 | 184.89 | 7.577628 | 185.51 |
| 31 | 5.594752 | 183.83 | 6.257422 | 184.33 | 6.922021 | 184.90 | 7.588759 | 185.52 |
| 32 | 5.605782 | 183.83 | 6.268482 | 184.34 | 6.933115 | 184.91 | 7.599890 | 185.53 |
| 33 | 5.616812 | 183.84 | 6.279543 | 184.35 | 6.944210 | 184.92 | 7.611022 | 185.54 |
| 34 | 5.627843 | 183.85 | 6.290605 | 184.36 | 6.955305 | 184.93 | 7.622155 | 185.55 |
| 35 | 5.638874 | 183.86 | 6.301667 | 184.37 | 6.966401 | 184.94 | 7.633289 | 185.57 |
| 36 | 5.6499 6 | 183.87 | 6.312729 | 184.38 | 6.977498 | 184.95 | 7.644423 | 185.58 |
| 37 | 5.660938 | 183.87 | 6.323792 | 184.39 | 6.988595 | 184.96 | 7.655558 | 185.59 |
| 38 | 5.671971 | 183.88 | 6.334855 | 184.40 | 6.999693 | 184.97 | 7.666694 | 185.60 |
| 39 | 5.683004 | 183.89 | 6.345919 | 184.41 | 7.010791 | 184.98 | 7.677830 | 185.61 |
| 40 | 5.694038 | 183.90 | 6.356984 | 184.41 | 7.021890 | 184.99 | 7.688967 | 185.62 |
| 41 | 5.705072 | 183.91 | 6.368049 | 184.42 | 7.032990 | 185.00 | 7.700104 | 185.63 |
| 42 | 5.716106 | 183.92 | 6.379115 | 184.43 | 7.044090 | 185.01 | 7.711242 | 185.64 |
| 43 | 5.727141 | 183.92 | 6.390181 | 184.44 | 7.055191 | 185.02 | 7.722381 | 185.65 |
| 44 | 5.738177 | 183.93 | 6.401248 | 184.45 | 7.066292 | 185.03 | 7.733521 | 185.66 |
| 45 | 5.749213 | 183.94 | 6.412315 | 184.46 | 7.077394 | 185.04 | 7.744661 | 185.68 |
| 46 | 5.760250 | 183.95 | 6.423383 | 184.47 | 7.088497 | 185.05 | 7.755802 | 185.69 |
| 47 | 5.771287 | 183.96 | 6.434451 | 184.48 | 7.099600 | 185.06 | 7.766943 | 185.70 |
| 48 | 5.782325 | 183.96 | 6.445520 | 184.49 | 7.110704 | 185.07 | 7.778085 | 185.71 |
| 49 | 5.793363 | 183.97 | 6.456590 | 184.50 | 7.121808 | 185.08 | 7.789228 | 185.72 |
| 50 | 5.804401 | 183.98 | 6.467660 | 184.51 | 7.132913 | 185.09 | 7.800372 | 185.73 |
| 51 | 5.815440 | 183.99 | 6.478731 | 184.52 | 7.144019 | 185.10 | 7.811516 | 185.74 |
| 52 | 5.826480 | 184.00 | 6.489802 | 184.52 | 7.155125 | 185.11 | 7.822661 | 185.75 |
| 53 | 5.837520 | 184.01 | 6.500874 | 184.53 | 7.166232 | 185.12 | 7.833807 | 185.76 |
| 54 | 5.848561 | 184.01 | 6.511946 | 184.54 | 7.177340 | 185.13 | 7.844953 | 185.78 |
| 55 | 5.859602 | 184.02 | 6.523019 | 184.55 | 7.188448 | 185.14 | 7.856100 | 185.79 |
| 56 | 5.870644 | 184.03 | 6.534092 | 184.56 | 7.199557 | 185.15 | 7.867247 | 185.80 |
| 57 | 5.881686 | 184.04 | 6.545166 | 184.57 | 7.210666 | 185.16 | 7.878396 | 185.81 |
| 58 | 5.892728 | 184.05 | 6.556241 | 184.58 | 7.221776 | 185.17 | 7.889545 | 185.82 |
| 59 | 5.903771 | 184.06 | 6.567316 | 184.59 | 7.232886 | 185.18 | 7.900694 | 185.83 |
| 60 | 5.914815 | 184.06 | 6.578391 | 184.60 | 7.243997 | 185.19 | 7.911845 | 185.84 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 12° | | 13° | | 14° | | 15° | |
|----|----------|-----------|----------|-----------|----------|-----------|-----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0 | 7.911845 | 185.84 | 8.582146 | 186.56 | 9.255120 | 187.33 | 9.930984 | 188.16 |
| 1 | 7.922995 | 185.86 | 8.593340 | 186.57 | 9.266360 | 187.34 | 9.942274 | 188.18 |
| 2 | 7.934147 | 185.87 | 8.604535 | 186.58 | 9.277601 | 187.35 | 9.953565 | 188.19 |
| 3 | 7.945300 | 185.88 | 8.615730 | 186.59 | 9.288842 | 187.37 | 9.964857 | 188.21 |
| 4 | 7.956453 | 185.89 | 8.626926 | 186.61 | 9.300085 | 187.38 | 9.976149 | 188.22 |
| 5 | 7.967606 | 185.90 | 8.638123 | 186.62 | 9.311328 | 187.40 | 9.987443 | 188.23 |
| 6 | 7.978761 | 185.91 | 8.649320 | 186.63 | 9.322572 | 187.41 | 9.998738 | 188.25 |
| 7 | 7.989916 | 185.92 | 8.660518 | 186.64 | 9.333817 | 187.42 | 10.010033 | 188.26 |
| 8 | 8.001072 | 185.93 | 8.671717 | 186.66 | 9.345063 | 187.44 | 10.021329 | 188.28 |
| 9 | 8.012228 | 185.95 | 8.682917 | 186.67 | 9.356310 | 187.45 | 10.032626 | 188.29 |
| 10 | 8.023385 | 185.96 | 8.694117 | 186.68 | 9.367557 | 187.46 | 10.043924 | 188.31 |
| 11 | 8.034543 | 185.97 | 8.705318 | 186.69 | 9.378805 | 187.48 | 10.055223 | 188.32 |
| 12 | 8.045702 | 185.98 | 8.716520 | 186.71 | 9.390054 | 187.49 | 10.066523 | 188.34 |
| 13 | 8.056861 | 185.99 | 8.727723 | 186.72 | 9.401304 | 187.50 | 10.077823 | 188.35 |
| 14 | 8.068021 | 186.00 | 8.738927 | 186.73 | 9.412555 | 187.52 | 10.089125 | 188.37 |
| 15 | 8.079181 | 186.02 | 8.750131 | 186.74 | 9.423806 | 187.53 | 10.100427 | 188.38 |
| 16 | 8.090343 | 186.03 | 8.761336 | 186.76 | 9.435058 | 187.54 | 10.111730 | 188.39 |
| 17 | 8.101505 | 186.04 | 8.772542 | 186.77 | 9.446311 | 187.56 | 10.123035 | 188.41 |
| 18 | 8.112668 | 186.05 | 8.783748 | 186.78 | 9.457565 | 187.57 | 10.134340 | 188.42 |
| 19 | 8.123831 | 186.06 | 8.794955 | 186.79 | 9.468820 | 187.59 | 10.145646 | 188.44 |
| 20 | 8.134995 | 186.07 | 8.806163 | 186.81 | 9.480076 | 187.60 | 10.156952 | 188.45 |
| 21 | 8.146160 | 186.09 | 8.817372 | 186.82 | 9.491332 | 187.61 | 10.168260 | 188.47 |
| 22 | 8.157326 | 186.10 | 8.828582 | 186.83 | 9.502589 | 187.63 | 10.179568 | 188.48 |
| 23 | 8.168492 | 186.11 | 8.839792 | 186.84 | 9.513847 | 187.64 | 10.190878 | 188.50 |
| 24 | 8.179659 | 186.12 | 8.851003 | 186.86 | 9.525106 | 187.65 | 10.202188 | 188.51 |
| 25 | 8.190826 | 186.13 | 8.862215 | 186.87 | 9.536366 | 187.67 | 10.213499 | 188.53 |
| 26 | 8.201995 | 186.15 | 8.873427 | 186.88 | 9.547626 | 187.68 | 10.224812 | 188.54 |
| 27 | 8.213164 | 186.16 | 8.884641 | 186.90 | 9.558888 | 187.70 | 10.236125 | 188.56 |
| 28 | 8.224334 | 186.17 | 8.895855 | 186.91 | 9.570150 | 187.71 | 10.247439 | 188.57 |
| 29 | 8.235504 | 186.18 | 8.907070 | 186.92 | 9.581413 | 187.72 | 10.258753 | 188.59 |
| 30 | 8.246675 | 186.19 | 8.918286 | 186.93 | 9.592676 | 187.74 | 10.270069 | 188.60 |
| 31 | 8.257847 | 186.20 | 8.929502 | 186.95 | 9.603941 | 187.75 | 10.281386 | 188.62 |
| 32 | 8.269020 | 186.22 | 8.940719 | 186.96 | 9.615207 | 187.77 | 10.292703 | 188.63 |
| 33 | 8.280193 | 186.23 | 8.951937 | 186.97 | 9.626473 | 187.78 | 10.304021 | 188.65 |
| 34 | 8.291367 | 186.24 | 8.963156 | 186.99 | 9.637740 | 187.79 | 10.315341 | 188.66 |
| 35 | 8.302542 | 186.25 | 8.974376 | 187.00 | 9.649008 | 187.81 | 10.326661 | 188.68 |
| 36 | 8.313717 | 186.26 | 8.985596 | 187.01 | 9.660277 | 187.82 | 10.337982 | 188.69 |
| 37 | 8.324893 | 186.28 | 8.996817 | 187.02 | 9.671547 | 187.84 | 10.349304 | 188.71 |
| 38 | 8.336070 | 186.29 | 9.008039 | 187.04 | 9.682817 | 187.85 | 10.360627 | 188.72 |
| 39 | 8.347248 | 186.30 | 9.019262 | 187.05 | 9.694088 | 187.86 | 10.371951 | 188.74 |
| 40 | 8.358426 | 186.31 | 9.030485 | 187.06 | 9.705361 | 187.88 | 10.383275 | 188.75 |
| 41 | 8.369605 | 186.32 | 9.041709 | 187.08 | 9.716634 | 187.89 | 10.394601 | 188.77 |
| 42 | 8.380785 | 186.34 | 9.052934 | 187.09 | 9.727908 | 187.91 | 10.405927 | 188.78 |
| 43 | 8.391966 | 186.35 | 9.064160 | 187.10 | 9.739182 | 187.92 | 10.417255 | 188.80 |
| 44 | 8.403147 | 186.36 | 9.075387 | 187.12 | 9.750458 | 187.93 | 10.428583 | 188.81 |
| 45 | 8.414329 | 186.37 | 9.086614 | 187.13 | 9.761734 | 187.95 | 10.439912 | 188.83 |
| 46 | 8.425512 | 186.38 | 9.097842 | 187.14 | 9.773012 | 187.96 | 10.451242 | 188.84 |
| 47 | 8.436695 | 186.40 | 9.109071 | 187.16 | 9.784290 | 187.98 | 10.462573 | 188.86 |
| 48 | 8.447879 | 186.41 | 9.120301 | 187.17 | 9.795569 | 187.99 | 10.473905 | 188.87 |
| 49 | 8.459064 | 186.42 | 9.131531 | 187.18 | 9.806848 | 188.00 | 10.485238 | 188.89 |
| 50 | 8.470250 | 186.43 | 9.142763 | 187.20 | 9.818129 | 188.02 | 10.496572 | 188.90 |
| 51 | 8.481436 | 186.45 | 9.153995 | 187.21 | 9.829410 | 188.03 | 10.507907 | 188.92 |
| 52 | 8.492623 | 186.46 | 9.165228 | 187.22 | 9.840693 | 188.05 | 10.519242 | 188.93 |
| 53 | 8.503811 | 186.47 | 9.176462 | 187.23 | 9.851977 | 188.06 | 10.530579 | 188.95 |
| 54 | 8.515000 | 186.48 | 9.187696 | 187.25 | 9.863261 | 188.08 | 10.541916 | 188.97 |
| 55 | 8.526189 | 186.49 | 9.198931 | 187.26 | 9.874546 | 188.09 | 10.553255 | 188.98 |
| 56 | 8.537379 | 186.51 | 9.210167 | 187.27 | 9.885832 | 188.10 | 10.564594 | 189.00 |
| 57 | 8.548569 | 186.52 | 9.221404 | 187.29 | 9.897118 | 188.12 | 10.575934 | 189.01 |
| 58 | 8.559761 | 186.53 | 9.232642 | 187.30 | 9.908406 | 188.13 | 10.587276 | 189.03 |
| 59 | 8.570953 | 186.54 | 9.243880 | 187.31 | 9.919694 | 188.15 | 10.598618 | 189.04 |
| 60 | 8.582146 | 186.56 | 9.255120 | 187.33 | 9.930984 | 188.16 | 10.609961 | 189.06 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 16° | | 17° | | 18° | | 19° | |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0' | 10.609961 | 189.06 | 11.292277 | 190.02 | 11.978162 | 191.04 | 12.667850 | 192.13 |
| 1 | 10.621305 | 189.07 | 11.303679 | 190.03 | 11.989625 | 191.06 | 12.679379 | 192.15 |
| 2 | 10.632649 | 189.09 | 11.315082 | 190.05 | 12.001089 | 191.08 | 12.690908 | 192.17 |
| 3 | 10.643995 | 189.10 | 11.326485 | 190.07 | 12.012554 | 191.09 | 12.702439 | 192.19 |
| 4 | 10.655342 | 189.12 | 11.337889 | 190.08 | 12.024021 | 191.11 | 12.713970 | 192.21 |
| 5 | 10.666690 | 189.14 | 11.349295 | 190.10 | 12.035488 | 191.13 | 12.725503 | 192.22 |
| 6 | 10.678038 | 189.15 | 11.360701 | 190.12 | 12.046956 | 191.15 | 12.737037 | 192.24 |
| 7 | 10.689388 | 189.17 | 11.372109 | 190.13 | 12.058425 | 191.16 | 12.748573 | 192.26 |
| 8 | 10.700738 | 189.18 | 11.383517 | 190.15 | 12.069896 | 191.18 | 12.760109 | 192.28 |
| 9 | 10.712090 | 189.20 | 11.394927 | 190.17 | 12.081367 | 191.20 | 12.771646 | 192.30 |
| 10 | 10.723442 | 189.21 | 11.406337 | 190.18 | 12.092840 | 191.22 | 12.783185 | 192.32 |
| 11 | 10.734795 | 189.23 | 11.417749 | 190.20 | 12.104313 | 191.24 | 12.794724 | 192.34 |
| 12 | 10.746149 | 189.24 | 11.429161 | 190.22 | 12.115788 | 191.25 | 12.806265 | 192.36 |
| 13 | 10.757505 | 189.26 | 11.440575 | 190.23 | 12.127264 | 191.27 | 12.817807 | 192.37 |
| 14 | 10.768861 | 189.28 | 11.451989 | 190.25 | 12.138741 | 191.29 | 12.829350 | 192.39 |
| 15 | 10.780218 | 189.29 | 11.463405 | 190.27 | 12.150219 | 191.31 | 12.840894 | 192.41 |
| 16 | 10.791576 | 189.31 | 11.474821 | 190.28 | 12.161698 | 191.32 | 12.852440 | 192.43 |
| 17 | 10.802935 | 189.32 | 11.486239 | 190.30 | 12.173178 | 191.34 | 12.863986 | 192.45 |
| 18 | 10.814295 | 189.34 | 11.497657 | 190.32 | 12.184659 | 191.36 | 12.875534 | 192.47 |
| 19 | 10.825655 | 189.35 | 11.509077 | 190.33 | 12.196141 | 191.38 | 12.887082 | 192.49 |
| 20 | 10.837017 | 189.37 | 11.520497 | 190.35 | 12.207624 | 191.40 | 12.898632 | 192.51 |
| 21 | 10.848380 | 189.39 | 11.531919 | 190.37 | 12.219108 | 191.41 | 12.910183 | 192.53 |
| 22 | 10.859744 | 189.40 | 11.543342 | 190.39 | 12.230594 | 191.43 | 12.921736 | 192.55 |
| 23 | 10.871108 | 189.42 | 11.554765 | 190.40 | 12.242080 | 191.45 | 12.933289 | 192.56 |
| 24 | 10.882474 | 189.43 | 11.566190 | 190.42 | 12.253568 | 191.47 | 12.944843 | 192.58 |
| 25 | 10.893840 | 189.45 | 11.577616 | 190.44 | 12.265057 | 191.49 | 12.956399 | 192.60 |
| 26 | 10.905208 | 189.47 | 11.589042 | 190.45 | 12.276546 | 191.50 | 12.967956 | 192.62 |
| 27 | 10.916576 | 189.48 | 11.600470 | 190.47 | 12.288037 | 191.52 | 12.979514 | 192.64 |
| 28 | 10.927946 | 189.50 | 11.611899 | 190.49 | 12.299529 | 191.54 | 12.991073 | 192.66 |
| 29 | 10.939316 | 189.51 | 11.623328 | 190.50 | 12.311022 | 191.56 | 13.002633 | 192.68 |
| 30 | 10.950687 | 189.53 | 11.634759 | 190.52 | 12.322516 | 191.58 | 13.014195 | 192.70 |
| 31 | 10.962059 | 189.55 | 11.646191 | 190.54 | 12.334011 | 191.60 | 13.025757 | 192.72 |
| 32 | 10.973433 | 189.56 | 11.657624 | 190.56 | 12.345508 | 191.61 | 13.037321 | 192.74 |
| 33 | 10.984807 | 189.58 | 11.669057 | 190.57 | 12.357005 | 191.63 | 13.048886 | 192.76 |
| 34 | 10.996182 | 189.59 | 11.680492 | 190.59 | 12.368503 | 191.65 | 13.060452 | 192.78 |
| 35 | 11.007558 | 189.61 | 11.691928 | 190.61 | 12.380003 | 191.67 | 13.072019 | 192.80 |
| 36 | 11.018935 | 189.63 | 11.703365 | 190.62 | 12.391504 | 191.69 | 13.083587 | 192.82 |
| 37 | 11.030313 | 189.64 | 11.714803 | 190.64 | 12.403006 | 191.70 | 13.095157 | 192.83 |
| 38 | 11.041692 | 189.66 | 11.726242 | 190.66 | 12.414509 | 191.72 | 13.106727 | 192.85 |
| 39 | 11.053072 | 189.67 | 11.737682 | 190.68 | 12.426013 | 191.74 | 13.118299 | 192.87 |
| 40 | 11.064453 | 189.69 | 11.749123 | 190.69 | 12.437517 | 191.76 | 13.129872 | 192.89 |
| 41 | 11.075835 | 189.71 | 11.760565 | 190.71 | 12.449023 | 191.78 | 13.141446 | 192.91 |
| 42 | 11.087218 | 189.72 | 11.772008 | 190.73 | 12.460531 | 191.80 | 13.153022 | 192.93 |
| 43 | 11.098602 | 189.74 | 11.783452 | 190.74 | 12.472039 | 191.81 | 13.164598 | 192.95 |
| 44 | 11.109987 | 189.76 | 11.794897 | 190.76 | 12.483548 | 191.83 | 13.176176 | 192.97 |
| 45 | 11.121372 | 189.77 | 11.806344 | 190.78 | 12.495059 | 191.85 | 13.187755 | 192.99 |
| 46 | 11.132759 | 189.79 | 11.817791 | 190.80 | 12.506571 | 191.87 | 13.199335 | 193.01 |
| 47 | 11.144147 | 189.80 | 11.829239 | 190.81 | 12.518083 | 191.89 | 13.210916 | 193.03 |
| 48 | 11.155536 | 189.82 | 11.840689 | 190.83 | 12.529597 | 191.91 | 13.222498 | 193.05 |
| 49 | 11.166925 | 189.84 | 11.852139 | 190.85 | 12.541112 | 191.93 | 13.234082 | 193.07 |
| 50 | 11.178316 | 189.85 | 11.863590 | 190.87 | 12.552628 | 191.94 | 13.245667 | 193.09 |
| 51 | 11.189708 | 189.87 | 11.875043 | 190.88 | 12.564145 | 191.96 | 13.257253 | 193.11 |
| 52 | 11.201100 | 189.89 | 11.886496 | 190.90 | 12.575664 | 191.98 | 13.268840 | 193.13 |
| 53 | 11.212494 | 189.90 | 11.897951 | 190.92 | 12.587183 | 192.00 | 13.280428 | 193.15 |
| 54 | 11.223889 | 189.92 | 11.909407 | 190.94 | 12.598704 | 192.02 | 13.292017 | 193.17 |
| 55 | 11.235284 | 189.93 | 11.920863 | 190.95 | 12.610225 | 192.04 | 13.303608 | 193.19 |
| 56 | 11.246681 | 189.95 | 11.932321 | 190.97 | 12.621748 | 192.06 | 13.315200 | 193.21 |
| 57 | 11.258078 | 189.97 | 11.943780 | 190.99 | 12.633272 | 192.07 | 13.326793 | 193.23 |
| 58 | 11.269477 | 189.98 | 11.955239 | 191.01 | 12.644797 | 192.09 | 13.338387 | 193.25 |
| 59 | 11.280876 | 190.00 | 11.966700 | 191.02 | 12.656323 | 192.11 | 13.349982 | 193.27 |
| 60 | 11.292277 | 190.02 | 11.978162 | 191.04 | 12.667850 | 192.13 | 13.361579 | 193.29 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 20° | | 21° | | 22° | | 23° | |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". | M. | Diff. 1". |
| 0' | 13.361579 | 193.29 | 14.059591 | 194.51 | 14.762133 | 195.80 | 15.469459 | 197.17 |
| 1 | 13.373177 | 193.31 | 14.071262 | 194.53 | 14.773882 | 195.83 | 15.481290 | 197.19 |
| 2 | 13.384776 | 193.33 | 14.082935 | 194.55 | 14.785632 | 195.85 | 15.493122 | 197.21 |
| 3 | 13.396376 | 193.35 | 14.094608 | 194.57 | 14.797384 | 195.87 | 15.504956 | 197.24 |
| 4 | 13.407977 | 193.37 | 14.106283 | 194.59 | 14.809137 | 195.89 | 15.516791 | 197.26 |
| 5 | 13.419580 | 193.39 | 14.117960 | 194.61 | 14.820891 | 195.91 | 15.528627 | 197.28 |
| 6 | 13.431183 | 193.41 | 14.129637 | 194.64 | 14.832647 | 195.94 | 15.540465 | 197.31 |
| 7 | 13.442788 | 193.43 | 14.141316 | 194.66 | 14.844403 | 195.96 | 15.552304 | 197.33 |
| 8 | 13.454394 | 193.45 | 14.152996 | 194.68 | 14.856161 | 195.98 | 15.564144 | 197.35 |
| 9 | 13.466002 | 193.47 | 14.164677 | 194.70 | 14.867921 | 196.00 | 15.575986 | 197.38 |
| 10 | 13.477610 | 193.49 | 14.176360 | 194.72 | 14.879682 | 196.03 | 15.587830 | 197.40 |
| 11 | 13.489220 | 193.51 | 14.188044 | 194.74 | 14.891444 | 196.05 | 15.599675 | 197.43 |
| 12 | 13.500831 | 193.53 | 14.199729 | 194.76 | 14.903208 | 196.07 | 15.611521 | 197.45 |
| 13 | 13.512443 | 193.55 | 14.211415 | 194.78 | 14.914973 | 196.09 | 15.623369 | 197.47 |
| 14 | 13.524056 | 193.57 | 14.223103 | 194.81 | 14.926739 | 196.12 | 15.635218 | 197.50 |
| 15 | 13.535671 | 193.59 | 14.234792 | 194.83 | 14.938506 | 196.14 | 15.647068 | 197.52 |
| 16 | 13.547287 | 193.61 | 14.246482 | 194.85 | 14.950275 | 196.16 | 15.658920 | 197.54 |
| 17 | 13.558904 | 193.63 | 14.258174 | 194.87 | 14.962045 | 196.18 | 15.670773 | 197.57 |
| 18 | 13.570522 | 193.65 | 14.269867 | 194.89 | 14.973817 | 196.20 | 15.682628 | 197.59 |
| 19 | 13.582141 | 193.67 | 14.281561 | 194.91 | 14.985590 | 196.23 | 15.694484 | 197.61 |
| 20 | 13.593762 | 193.69 | 14.293256 | 194.93 | 14.997365 | 196.25 | 15.706342 | 197.64 |
| 21 | 13.605383 | 193.71 | 14.304953 | 194.95 | 15.009140 | 196.27 | 15.718201 | 197.66 |
| 22 | 13.617006 | 193.73 | 14.316651 | 194.98 | 15.020917 | 196.30 | 15.730061 | 197.69 |
| 23 | 13.628631 | 193.75 | 14.328350 | 195.00 | 15.032696 | 196.32 | 15.741923 | 197.71 |
| 24 | 13.640256 | 193.77 | 14.340050 | 195.02 | 15.044475 | 196.34 | 15.753786 | 197.73 |
| 25 | 13.651883 | 193.79 | 14.351752 | 195.04 | 15.056256 | 196.36 | 15.765651 | 197.76 |
| 26 | 13.663511 | 193.81 | 14.363455 | 195.06 | 15.068039 | 196.39 | 15.777517 | 197.78 |
| 27 | 13.675140 | 193.83 | 14.375159 | 195.08 | 15.079823 | 196.41 | 15.789385 | 197.80 |
| 28 | 13.686770 | 193.85 | 14.386865 | 195.10 | 15.091608 | 196.43 | 15.801254 | 197.83 |
| 29 | 13.698401 | 193.87 | 14.398572 | 195.13 | 15.103394 | 196.45 | 15.813124 | 197.85 |
| 30 | 13.710034 | 193.89 | 14.410280 | 195.15 | 15.115182 | 196.48 | 15.824996 | 197.88 |
| 31 | 13.721668 | 193.91 | 14.421990 | 195.17 | 15.126971 | 196.50 | 15.836870 | 197.90 |
| 32 | 13.733303 | 193.93 | 14.433700 | 195.19 | 15.138762 | 196.52 | 15.848744 | 197.92 |
| 33 | 13.744940 | 193.95 | 14.445412 | 195.21 | 15.150554 | 196.54 | 15.860620 | 197.95 |
| 34 | 13.756577 | 193.97 | 14.457126 | 195.23 | 15.162348 | 196.57 | 15.872498 | 197.97 |
| 35 | 13.768216 | 193.99 | 14.468841 | 195.26 | 15.174142 | 196.59 | 15.884377 | 198.00 |
| 36 | 13.779856 | 194.01 | 14.480557 | 195.28 | 15.185938 | 196.61 | 15.896258 | 198.02 |
| 37 | 13.791498 | 194.03 | 14.492274 | 195.30 | 15.197736 | 196.64 | 15.908140 | 198.04 |
| 38 | 13.803140 | 194.05 | 14.503992 | 195.32 | 15.209535 | 196.66 | 15.920023 | 198.07 |
| 39 | 13.814784 | 194.07 | 14.515712 | 195.34 | 15.221335 | 196.68 | 15.931908 | 198.09 |
| 40 | 13.826429 | 194.09 | 14.527434 | 195.36 | 15.233137 | 196.70 | 15.943794 | 198.12 |
| 41 | 13.838075 | 194.11 | 14.539156 | 195.39 | 15.244940 | 196.73 | 15.955682 | 198.14 |
| 42 | 13.849723 | 194.14 | 14.550880 | 195.41 | 15.256744 | 196.75 | 15.967571 | 198.17 |
| 43 | 13.861372 | 194.16 | 14.562605 | 195.43 | 15.268550 | 196.77 | 15.979462 | 198.19 |
| 44 | 13.873022 | 194.18 | 14.574331 | 195.45 | 15.280357 | 196.80 | 15.991354 | 198.21 |
| 45 | 13.884673 | 194.20 | 14.586059 | 195.47 | 15.292165 | 196.82 | 16.003248 | 198.24 |
| 46 | 13.896325 | 194.22 | 14.597788 | 195.50 | 15.303975 | 196.84 | 16.015143 | 198.26 |
| 47 | 13.907979 | 194.24 | 14.609519 | 195.52 | 15.315786 | 196.87 | 16.027039 | 198.29 |
| 48 | 13.919634 | 194.26 | 14.621250 | 195.54 | 15.327599 | 196.89 | 16.038937 | 198.31 |
| 49 | 13.931290 | 194.28 | 14.632983 | 195.56 | 15.339413 | 196.91 | 16.050836 | 198.34 |
| 50 | 13.942948 | 194.30 | 14.644718 | 195.58 | 15.351228 | 196.94 | 16.062737 | 198.36 |
| 51 | 13.954606 | 194.32 | 14.656453 | 195.60 | 15.363045 | 196.96 | 16.074639 | 198.38 |
| 52 | 13.966266 | 194.34 | 14.668190 | 195.63 | 15.374863 | 196.98 | 16.086543 | 198.41 |
| 53 | 13.977927 | 194.36 | 14.679929 | 195.65 | 15.386683 | 197.00 | 16.098449 | 198.43 |
| 54 | 13.989590 | 194.38 | 14.691668 | 195.67 | 15.398504 | 197.03 | 16.110355 | 198.46 |
| 55 | 14.001254 | 194.41 | 14.703409 | 195.69 | 15.410326 | 197.05 | 16.122263 | 198.48 |
| 56 | 14.012919 | 194.43 | 14.715151 | 195.71 | 15.422150 | 197.07 | 16.134173 | 198.51 |
| 57 | 14.024585 | 194.45 | 14.726895 | 195.74 | 15.433975 | 197.10 | 16.146084 | 198.53 |
| 58 | 14.036252 | 194.47 | 14.738640 | 195.76 | 15.445802 | 197.12 | 16.157997 | 198.56 |
| 59 | 14.047921 | 194.49 | 14.750386 | 195.78 | 15.457630 | 197.14 | 16.169911 | 198.58 |
| 60 | 14.059591 | 194.51 | 14.762133 | 195.80 | 15.469459 | 197.17 | 16.181826 | 198.60 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 24° | | | 25° | | | 26° | | | 27° | | |
|----|-----------|-----------|--|-----------|-----------|--|-----------|-----------|--|-----------|-----------|--|
| | M. | Diff. 1". | | M. | Diff. 1". | | M. | Diff. 1". | | M. | Diff. 1". | |
| 0' | 16.181826 | 198.60 | | 16.899499 | 200.12 | | 17.622747 | 201.70 | | 18.351847 | 203.37 | |
| 1 | 16.193743 | 198.63 | | 16.911507 | 200.14 | | 17.634850 | 201.73 | | 18.364050 | 203.40 | |
| 2 | 16.205662 | 198.65 | | 16.923516 | 200.17 | | 17.646954 | 201.76 | | 18.376255 | 203.42 | |
| 3 | 16.217582 | 198.68 | | 16.935527 | 200.19 | | 17.659060 | 201.78 | | 18.388461 | 203.45 | |
| 4 | 16.229503 | 198.70 | | 16.947539 | 200.22 | | 17.671168 | 201.81 | | 18.400669 | 203.48 | |
| 5 | 16.241426 | 198.73 | | 16.959553 | 200.24 | | 17.683278 | 201.84 | | 18.412879 | 203.51 | |
| 6 | 16.253350 | 198.75 | | 16.971568 | 200.27 | | 17.695389 | 201.87 | | 18.425090 | 203.54 | |
| 7 | 16.265276 | 198.78 | | 16.983585 | 200.30 | | 17.707502 | 201.89 | | 18.437303 | 203.57 | |
| 8 | 16.277204 | 198.80 | | 16.995604 | 200.32 | | 17.719616 | 201.92 | | 18.449518 | 203.59 | |
| 9 | 16.289133 | 198.83 | | 17.007624 | 200.35 | | 17.731732 | 201.95 | | 18.461735 | 203.62 | |
| 10 | 16.301063 | 198.85 | | 17.019646 | 200.37 | | 17.743850 | 201.97 | | 18.473953 | 203.65 | |
| 11 | 16.312995 | 198.88 | | 17.031669 | 200.40 | | 17.755969 | 202.00 | | 18.486173 | 203.68 | |
| 12 | 16.324928 | 198.90 | | 17.043694 | 200.43 | | 17.768090 | 202.03 | | 18.498395 | 203.71 | |
| 13 | 16.336863 | 198.93 | | 17.055720 | 200.45 | | 17.780213 | 202.06 | | 18.510618 | 203.74 | |
| 14 | 16.348799 | 198.95 | | 17.067748 | 200.48 | | 17.792337 | 202.08 | | 18.522843 | 203.77 | |
| 15 | 16.360737 | 198.97 | | 17.079777 | 200.50 | | 17.804462 | 202.11 | | 18.535070 | 203.80 | |
| 16 | 16.372676 | 199.00 | | 17.091808 | 200.53 | | 17.816590 | 202.14 | | 18.547299 | 203.82 | |
| 17 | 16.384617 | 199.02 | | 17.103841 | 200.56 | | 17.828719 | 202.17 | | 18.559529 | 203.85 | |
| 18 | 16.396559 | 199.05 | | 17.115875 | 200.58 | | 17.840850 | 202.19 | | 18.571761 | 203.88 | |
| 19 | 16.408503 | 199.07 | | 17.127911 | 200.61 | | 17.852982 | 202.22 | | 18.583995 | 203.91 | |
| 20 | 16.420448 | 199.10 | | 17.139948 | 200.64 | | 17.865116 | 202.25 | | 18.596230 | 203.94 | |
| 21 | 16.432395 | 199.12 | | 17.151987 | 200.66 | | 17.877252 | 202.28 | | 18.608467 | 203.97 | |
| 22 | 16.444343 | 199.15 | | 17.164028 | 200.69 | | 17.889389 | 202.30 | | 18.620706 | 204.00 | |
| 23 | 16.456292 | 199.17 | | 17.176070 | 200.71 | | 17.901528 | 202.33 | | 18.632947 | 204.03 | |
| 24 | 16.468243 | 199.20 | | 17.188114 | 200.74 | | 17.913669 | 202.36 | | 18.645190 | 204.05 | |
| 25 | 16.480196 | 199.22 | | 17.200159 | 200.77 | | 17.925811 | 202.39 | | 18.657434 | 204.08 | |
| 26 | 16.492151 | 199.25 | | 17.212206 | 200.79 | | 17.937955 | 202.41 | | 18.669679 | 204.11 | |
| 27 | 16.504107 | 199.27 | | 17.224254 | 200.82 | | 17.950101 | 202.44 | | 18.681927 | 204.14 | |
| 28 | 16.516064 | 199.30 | | 17.236304 | 200.85 | | 17.962248 | 202.47 | | 18.694177 | 204.17 | |
| 29 | 16.528022 | 199.33 | | 17.248356 | 200.87 | | 17.974397 | 202.50 | | 18.706428 | 204.20 | |
| 30 | 16.539983 | 199.35 | | 17.260409 | 200.90 | | 17.986548 | 202.52 | | 18.718680 | 204.23 | |
| 31 | 16.551945 | 199.38 | | 17.272464 | 200.93 | | 17.998700 | 202.55 | | 18.730935 | 204.26 | |
| 32 | 16.563908 | 199.40 | | 17.284520 | 200.95 | | 18.010854 | 202.58 | | 18.743191 | 204.29 | |
| 33 | 16.575873 | 199.43 | | 17.296578 | 200.98 | | 18.023010 | 202.61 | | 18.755449 | 204.32 | |
| 34 | 16.587839 | 199.45 | | 17.308637 | 201.00 | | 18.035167 | 202.64 | | 18.767709 | 204.35 | |
| 35 | 16.599807 | 199.48 | | 17.320698 | 201.03 | | 18.047326 | 202.66 | | 18.779971 | 204.37 | |
| 36 | 16.611776 | 199.50 | | 17.332761 | 201.06 | | 18.059487 | 202.69 | | 18.792234 | 204.40 | |
| 37 | 16.623747 | 199.53 | | 17.344825 | 201.08 | | 18.071649 | 202.72 | | 18.804499 | 204.43 | |
| 38 | 16.635719 | 199.55 | | 17.356891 | 201.11 | | 18.083813 | 202.75 | | 18.816767 | 204.46 | |
| 39 | 16.647693 | 199.58 | | 17.368959 | 201.14 | | 18.095979 | 202.78 | | 18.829036 | 204.49 | |
| 40 | 16.659669 | 199.60 | | 17.381028 | 201.16 | | 18.108146 | 202.80 | | 18.841305 | 204.52 | |
| 41 | 16.671646 | 199.63 | | 17.393098 | 201.19 | | 18.120315 | 202.83 | | 18.853577 | 204.55 | |
| 42 | 16.683624 | 199.65 | | 17.405171 | 201.22 | | 18.132486 | 202.86 | | 18.865851 | 204.58 | |
| 43 | 16.695604 | 199.68 | | 17.417245 | 201.24 | | 18.144658 | 202.89 | | 18.878127 | 204.61 | |
| 44 | 16.707586 | 199.70 | | 17.429320 | 201.27 | | 18.156832 | 202.92 | | 18.890404 | 204.64 | |
| 45 | 16.719569 | 199.73 | | 17.441397 | 201.30 | | 18.169008 | 202.94 | | 18.902684 | 204.67 | |
| 46 | 16.731553 | 199.76 | | 17.453476 | 201.32 | | 18.181186 | 202.97 | | 18.914965 | 204.70 | |
| 47 | 16.743539 | 199.78 | | 17.465556 | 201.35 | | 18.193365 | 203.00 | | 18.927247 | 204.73 | |
| 48 | 16.755527 | 199.81 | | 17.477638 | 201.38 | | 18.205546 | 203.03 | | 18.939532 | 204.76 | |
| 49 | 16.767516 | 199.83 | | 17.489722 | 201.41 | | 18.217728 | 203.06 | | 18.951818 | 204.79 | |
| 50 | 16.779507 | 199.86 | | 17.501807 | 201.43 | | 18.229912 | 203.08 | | 18.964106 | 204.81 | |
| 51 | 16.791499 | 199.88 | | 17.513894 | 201.46 | | 18.242098 | 203.11 | | 18.976396 | 204.84 | |
| 52 | 16.803493 | 199.91 | | 17.525982 | 201.49 | | 18.254286 | 203.14 | | 18.988687 | 204.87 | |
| 53 | 16.815488 | 199.94 | | 17.538072 | 201.51 | | 18.266475 | 203.17 | | 19.000981 | 204.90 | |
| 54 | 16.827485 | 199.96 | | 17.550163 | 201.54 | | 18.278666 | 203.20 | | 19.013276 | 204.93 | |
| 55 | 16.839484 | 199.99 | | 17.562257 | 201.57 | | 18.290859 | 203.23 | | 19.025573 | 204.96 | |
| 56 | 16.851484 | 200.01 | | 17.574352 | 201.59 | | 18.303053 | 203.25 | | 19.037871 | 204.99 | |
| 57 | 16.863485 | 200.04 | | 17.586448 | 201.62 | | 18.315249 | 203.28 | | 19.050172 | 205.02 | |
| 58 | 16.875488 | 200.06 | | 17.598546 | 201.65 | | 18.327447 | 203.31 | | 19.062474 | 205.05 | |
| 59 | 16.887493 | 200.09 | | 17.610646 | 201.68 | | 18.339646 | 203.34 | | 19.074778 | 205.08 | |
| 60 | 16.899499 | 200.12 | | 17.622747 | 201.70 | | 18.351847 | 203.37 | | 19.087084 | 205.11 | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 28° | | 29° | | 30° | | 31° | |
|----|-----------|-----------|-----------|-----------|------------|-----------|------------|-----------|
| | M. | Diff. 1". | M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 19.087084 | 205.11 | 19.828747 | 206.94 | 1.313 3849 | 44.08 | 1.329 0430 | 42.92 |
| 1 | 19.099391 | 205.14 | 19.841164 | 206.97 | .313 6493 | 44.06 | .329 3004 | 42.91 |
| 2 | 19.111701 | 205.17 | 19.853583 | 207.00 | .313 9136 | 44.04 | .329 5578 | 42.89 |
| 3 | 19.124012 | 205.20 | 19.866004 | 207.03 | .314 1778 | 44.02 | .329 8151 | 42.87 |
| 4 | 19.136325 | 205.23 | 19.878427 | 207.06 | .314 4419 | 44.00 | .330 0723 | 42.85 |
| 5 | 19.148639 | 205.26 | 19.890852 | 207.09 | 1.314 7058 | 43.98 | 1.330 3293 | 42.83 |
| 6 | 19.160956 | 205.29 | 19.903279 | 207.13 | .314 9696 | 43.96 | .330 5862 | 42.81 |
| 7 | 19.173274 | 205.32 | 19.915707 | 207.16 | .315 2333 | 43.94 | .330 8431 | 42.80 |
| 8 | 19.185594 | 205.35 | 19.928137 | 207.19 | .315 4969 | 43.92 | .331 0998 | 42.78 |
| 9 | 19.197916 | 205.38 | 19.940569 | 207.22 | .315 7604 | 43.90 | .331 3564 | 42.76 |
| 10 | 19.210240 | 205.41 | 19.953003 | 207.25 | 1.316 0237 | 43.88 | 1.331 6129 | 42.74 |
| 11 | 19.222566 | 205.44 | 19.965439 | 207.28 | .316 2869 | 43.86 | .331 8693 | 42.72 |
| 12 | 19.234893 | 205.47 | 19.977877 | 207.31 | .316 5500 | 43.84 | .332 1255 | 42.70 |
| 13 | 19.247222 | 205.50 | 19.990317 | 207.34 | .316 8130 | 43.82 | .332 3817 | 42.69 |
| 14 | 19.259553 | 205.53 | 20.002759 | 207.38 | .317 0759 | 43.80 | .332 6378 | 42.67 |
| 15 | 19.271885 | 205.56 | 20.015202 | 207.41 | 1.317 3386 | 43.78 | 1.332 8937 | 42.65 |
| 16 | 19.284220 | 205.59 | 20.027647 | 207.44 | .317 6013 | 43.76 | .333 1496 | 42.63 |
| 17 | 19.296556 | 205.62 | 20.040095 | 207.47 | .317 8638 | 43.74 | .333 4053 | 42.61 |
| 18 | 19.308894 | 205.65 | 20.052544 | 207.50 | .318 1262 | 43.72 | .333 6609 | 42.59 |
| 19 | 19.321234 | 205.68 | 20.064995 | 207.53 | .318 3885 | 43.70 | .333 9164 | 42.58 |
| 20 | 19.333576 | 205.71 | 20.077448 | 207.57 | 1.318 6506 | 43.68 | 1.334 1718 | 42.56 |
| 21 | 19.345920 | 205.74 | 20.089903 | 207.60 | .318 9127 | 43.67 | .334 4271 | 42.54 |
| 22 | 19.358265 | 205.77 | 20.102360 | 207.63 | .319 1746 | 43.65 | .334 6823 | 42.52 |
| 23 | 19.370612 | 205.80 | 20.114818 | 207.66 | .319 4364 | 43.63 | .334 9374 | 42.50 |
| 24 | 19.382961 | 205.83 | 20.127279 | 207.69 | .319 6981 | 43.61 | .335 1924 | 42.49 |
| 25 | 19.395312 | 205.86 | 20.139741 | 207.72 | 1.319 9597 | 43.59 | 1.335 4472 | 42.47 |
| 26 | 19.407665 | 205.89 | 20.152206 | 207.76 | .320 2212 | 43.57 | .335 7020 | 42.45 |
| 27 | 19.420019 | 205.92 | 20.164672 | 207.79 | .320 4825 | 43.55 | .335 9567 | 42.43 |
| 28 | 19.432375 | 205.95 | 20.177140 | 207.82 | .320 7438 | 43.53 | .336 2112 | 42.41 |
| 29 | 19.444734 | 205.98 | 20.189610 | 207.85 | .321 0049 | 43.51 | .336 4656 | 42.40 |
| 30 | 19.457094 | 206.01 | 20.202082 | 207.88 | 1.321 2659 | 43.49 | 1.336 7199 | 42.38 |
| 31 | 19.469455 | 206.04 | 20.214556 | 207.91 | .321 5268 | 43.47 | .336 9742 | 42.36 |
| 32 | 19.481819 | 206.08 | 20.227032 | 207.95 | .321 7875 | 43.45 | .337 2283 | 42.34 |
| 33 | 19.494184 | 206.11 | 20.239510 | 207.98 | .322 0482 | 43.43 | .337 4823 | 42.33 |
| 34 | 19.506551 | 206.14 | 20.251989 | 208.01 | .322 3087 | 43.41 | .337 7362 | 42.31 |
| 35 | 19.518921 | 206.17 | 20.264471 | 208.04 | 1.322 5692 | 43.40 | 1.337 9900 | 42.29 |
| 36 | 19.531292 | 206.20 | 20.276954 | 208.07 | .322 8295 | 43.38 | .338 2437 | 42.27 |
| 37 | 19.543664 | 206.23 | 20.289440 | 208.11 | .323 0897 | 43.36 | .338 4972 | 42.25 |
| 38 | 19.556039 | 206.26 | 20.301927 | 208.14 | .323 3498 | 43.34 | .338 7507 | 42.24 |
| 39 | 19.568415 | 206.29 | 20.314416 | 208.17 | .323 6097 | 43.32 | .339 0041 | 42.22 |
| 40 | 19.580794 | 206.32 | 20.326907 | 208.20 | 1.323 8696 | 43.30 | 1.339 2573 | 42.20 |
| 41 | 19.593174 | 206.35 | 20.339400 | 208.24 | .324 1294 | 43.28 | .339 5105 | 42.18 |
| 42 | 19.605556 | 206.38 | 20.351895 | 208.27 | .324 3890 | 43.26 | .339 7635 | 42.17 |
| 43 | 19.617939 | 206.41 | 20.364392 | 208.30 | .324 6485 | 43.24 | .340 0165 | 42.15 |
| 44 | 19.630325 | 206.44 | 20.376891 | 208.33 | .324 9079 | 43.22 | .340 2693 | 42.13 |
| 45 | 19.642713 | 206.47 | 20.389392 | 208.36 | 1.325 1672 | 43.21 | 1.340 5221 | 42.11 |
| 46 | 19.655102 | 206.50 | 20.401895 | 208.39 | .325 4263 | 43.19 | .340 7747 | 42.10 |
| 47 | 19.667493 | 206.53 | 20.414399 | 208.43 | .325 6854 | 43.17 | .341 0272 | 42.08 |
| 48 | 19.679886 | 206.57 | 20.426906 | 208.46 | .325 9443 | 43.15 | .341 2796 | 42.06 |
| 49 | 19.692281 | 206.60 | 20.439415 | 208.49 | .326 2032 | 43.13 | .341 5319 | 42.04 |
| 50 | 19.704678 | 206.63 | 20.451925 | 208.52 | 1.326 4619 | 43.11 | 1.341 7841 | 42.03 |
| 51 | 19.717076 | 206.66 | 20.464437 | 208.56 | .326 7205 | 43.09 | .342 0362 | 42.01 |
| 52 | 19.729477 | 206.69 | 20.476952 | 208.59 | .326 9790 | 43.07 | .342 2882 | 41.99 |
| 53 | 19.741879 | 206.72 | 20.489468 | 208.62 | .327 2374 | 43.05 | .342 5401 | 41.97 |
| 54 | 19.754283 | 206.75 | 20.501986 | 208.65 | .327 4957 | 43.04 | .342 7919 | 41.96 |
| 55 | 19.766689 | 206.78 | 20.514506 | 208.69 | 1.327 7538 | 43.02 | 1.343 0436 | 41.94 |
| 56 | 19.779097 | 206.81 | 20.527029 | 208.72 | .328 0119 | 43.00 | .343 2952 | 41.92 |
| 57 | 19.791507 | 206.84 | 20.539553 | 208.75 | .328 2698 | 42.98 | .343 5467 | 41.90 |
| 58 | 19.803919 | 206.88 | 20.552079 | 208.78 | .328 5276 | 42.96 | .343 7980 | 41.89 |
| 59 | 19.816332 | 206.91 | 20.564607 | 208.82 | .328 7853 | 42.94 | .344 0493 | 41.87 |
| 60 | 19.828747 | 206.94 | 20.577137 | 208.85 | 1.329 0430 | 42.92 | 1.344 3005 | 41.85 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 32° | | 33° | | 34° | | 35° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.344 3005 | 41.85 | 1.359 1859 | 40.86 | 1.373 7251 | 39.93 | 1.387 9418 | 39.06 |
| 1 | .344 5515 | 41.84 | .359 4310 | 40.84 | .373 9646 | 39.91 | .388 1761 | 39.05 |
| 2 | .344 8025 | 41.82 | .359 6760 | 40.82 | .374 2041 | 39.90 | .388 4104 | 39.04 |
| 3 | .345 0534 | 41.80 | .359 9209 | 40.81 | .374 4434 | 39.88 | .388 6446 | 39.02 |
| 4 | .345 3041 | 41.78 | .360 1657 | 40.79 | .374 6827 | 39.87 | .388 8787 | 39.01 |
| 5 | 1.345 5548 | 41.77 | 1.360 4104 | 40.78 | 1.374 9218 | 39.85 | 1.389 1127 | 38.99 |
| 6 | .345 8053 | 41.75 | .360 6550 | 40.76 | .375 1609 | 39.84 | .389 3466 | 38.98 |
| 7 | .346 0558 | 41.73 | .360 8995 | 40.74 | .375 3999 | 39.82 | .389 5804 | 38.97 |
| 8 | .346 3061 | 41.72 | .361 1439 | 40.73 | .375 6388 | 39.81 | .389 8142 | 38.95 |
| 9 | .346 5564 | 41.70 | .361 3883 | 40.71 | .375 8776 | 39.79 | .390 0479 | 38.94 |
| 10 | 1.346 8065 | 41.68 | 1.361 6325 | 40.70 | 1.376 1164 | 39.78 | 1.390 2815 | 38.93 |
| 11 | .347 0565 | 41.66 | .361 8766 | 40.68 | .376 3550 | 39.77 | .390 5150 | 38.91 |
| 12 | .347 3065 | 41.65 | .362 1207 | 40.66 | .376 5935 | 39.75 | .390 7484 | 38.90 |
| 13 | .347 5563 | 41.63 | .362 3646 | 40.65 | .376 8320 | 39.74 | .390 9817 | 38.88 |
| 14 | .347 8060 | 41.61 | .362 6084 | 40.63 | .377 0703 | 39.72 | .391 2150 | 38.87 |
| 15 | 1.348 0557 | 41.60 | 1.362 8522 | 40.62 | 1.377 3086 | 39.71 | 1.391 4482 | 38.86 |
| 16 | .348 3052 | 41.58 | .363 0959 | 40.60 | .377 5468 | 39.69 | .391 6813 | 38.84 |
| 17 | .348 5546 | 41.56 | .363 3394 | 40.59 | .377 7849 | 39.68 | .391 9143 | 38.83 |
| 18 | .348 8040 | 41.55 | .363 5829 | 40.57 | .378 0230 | 39.66 | .392 1472 | 38.82 |
| 19 | .349 0532 | 41.53 | .363 8263 | 40.56 | .378 2609 | 39.65 | .392 3801 | 38.80 |
| 20 | 1.349 3023 | 41.51 | 1.364 0696 | 40.54 | 1.378 4987 | 39.64 | 1.392 6128 | 38.79 |
| 21 | .349 5513 | 41.50 | .364 3128 | 40.52 | .378 7365 | 39.62 | .392 8455 | 38.77 |
| 22 | .349 8003 | 41.48 | .364 5559 | 40.51 | .378 9742 | 39.61 | .393 0781 | 38.76 |
| 23 | .350 0491 | 41.46 | .364 7989 | 40.49 | .379 2117 | 39.59 | .393 3107 | 38.75 |
| 24 | .350 2978 | 41.45 | .365 0418 | 40.48 | .379 4492 | 39.58 | .393 5431 | 38.73 |
| 25 | 1.350 5464 | 41.43 | 1.365 2846 | 40.46 | 1.379 6866 | 39.56 | 1.393 7755 | 38.72 |
| 26 | .350 7950 | 41.41 | .365 5273 | 40.45 | .379 9240 | 39.55 | .394 0078 | 38.71 |
| 27 | .351 0434 | 41.40 | .365 7699 | 40.43 | .380 1612 | 39.53 | .394 2400 | 38.69 |
| 28 | .351 2917 | 41.38 | .366 0125 | 40.41 | .380 3983 | 39.52 | .394 4721 | 38.68 |
| 29 | .351 5399 | 41.36 | .366 2549 | 40.40 | .380 6354 | 39.50 | .394 7041 | 38.67 |
| 30 | 1.351 7880 | 41.35 | 1.366 4973 | 40.38 | 1.380 8724 | 39.49 | 1.394 9361 | 38.65 |
| 31 | .352 0361 | 41.33 | .366 7395 | 40.37 | .381 1093 | 39.47 | .395 1680 | 38.64 |
| 32 | .352 2840 | 41.31 | .366 9817 | 40.35 | .381 3461 | 39.46 | .395 3998 | 38.63 |
| 33 | .352 5318 | 41.30 | .367 2238 | 40.34 | .381 5828 | 39.45 | .395 6315 | 38.61 |
| 34 | .352 7795 | 41.28 | .367 4657 | 40.32 | .381 8194 | 39.43 | .395 8631 | 38.60 |
| 35 | 1.353 0272 | 41.26 | 1.367 7076 | 40.31 | 1.382 0559 | 39.42 | 1.396 0947 | 38.59 |
| 36 | .353 2747 | 41.25 | .367 9494 | 40.29 | .382 2924 | 39.40 | .396 3262 | 38.57 |
| 37 | .353 5221 | 41.23 | .368 1911 | 40.28 | .382 5288 | 39.39 | .396 5576 | 38.56 |
| 38 | .353 7694 | 41.21 | .368 4327 | 40.26 | .382 7651 | 39.37 | .396 7889 | 38.55 |
| 39 | .354 0167 | 41.20 | .368 6742 | 40.25 | .383 0013 | 39.36 | .397 0201 | 38.53 |
| 40 | 1.354 2638 | 41.18 | 1.368 9157 | 40.23 | 1.383 2374 | 39.35 | 1.397 2513 | 38.52 |
| 41 | .354 5108 | 41.16 | .369 1570 | 40.21 | .383 4734 | 39.33 | .397 4823 | 38.51 |
| 42 | .354 7578 | 41.15 | .369 3983 | 40.20 | .383 7093 | 39.32 | .397 7133 | 38.49 |
| 43 | .355 0046 | 41.13 | .369 6394 | 40.18 | .383 9452 | 39.30 | .397 9442 | 38.48 |
| 44 | .355 2513 | 41.11 | .369 8805 | 40.17 | .384 1809 | 39.29 | .398 1751 | 38.47 |
| 45 | 1.355 4980 | 41.10 | 1.370 1214 | 40.15 | 1.384 4166 | 39.27 | 1.398 4058 | 38.45 |
| 46 | .355 7445 | 41.08 | .370 3623 | 40.14 | .384 6522 | 39.26 | .398 6365 | 38.44 |
| 47 | .355 9909 | 41.07 | .370 6031 | 40.12 | .384 8878 | 39.25 | .398 8671 | 38.43 |
| 48 | .356 2373 | 41.05 | .370 8438 | 40.11 | .385 1232 | 39.23 | .399 0976 | 38.41 |
| 49 | .356 4836 | 41.03 | .371 0844 | 40.09 | .385 3585 | 39.22 | .399 3281 | 38.40 |
| 50 | 1.356 7297 | 41.02 | 1.371 3249 | 40.08 | 1.385 5938 | 39.20 | 1.399 5584 | 38.39 |
| 51 | .356 9758 | 41.00 | .371 5654 | 40.06 | .385 8290 | 39.19 | .399 7887 | 38.37 |
| 52 | .357 2217 | 40.98 | .371 8057 | 40.05 | .386 0641 | 39.18 | .400 0189 | 38.36 |
| 53 | .357 4676 | 40.97 | .372 0459 | 40.03 | .386 2991 | 39.16 | .400 2491 | 38.35 |
| 54 | .357 7134 | 40.95 | .372 2861 | 40.02 | .386 5340 | 39.15 | .400 4791 | 38.33 |
| 55 | 1.357 9590 | 40.94 | 1.372 5261 | 40.00 | 1.386 7689 | 39.13 | 1.400 7091 | 38.32 |
| 56 | .358 2046 | 40.92 | .372 7661 | 39.99 | .387 0036 | 39.12 | .400 9390 | 38.31 |
| 57 | .358 4501 | 40.90 | .373 0060 | 39.97 | .387 2383 | 39.11 | .401 1688 | 38.30 |
| 58 | .358 6954 | 40.89 | .373 2458 | 39.96 | .387 4729 | 39.09 | .401 3985 | 38.28 |
| 59 | .358 9407 | 40.87 | .373 4855 | 39.94 | .387 7074 | 39.08 | .401 6282 | 38.27 |
| 60 | 1.359 1859 | 40.86 | 1.373 7251 | 39.93 | 1.387 9418 | 39.06 | 1.401 8578 | 38.26 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 36° | | 37° | | 38° | | 39° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.401 8578 | 38.26 | 1.415 4930 | 37.50 | 1.428 8662 | 36.80 | 1.441 9943 | 36.14 |
| 1 | .402 0873 | 38.24 | .415 7180 | 37.49 | .429 0869 | 36.79 | .442 2111 | 36.13 |
| 2 | .402 3167 | 38.23 | .415 9429 | 37.47 | .429 3076 | 36.78 | .442 4279 | 36.12 |
| 3 | .402 5460 | 38.22 | .416 1678 | 37.46 | .429 5283 | 36.77 | .442 6446 | 36.11 |
| 4 | .402 7753 | 38.20 | .416 3925 | 37.45 | .429 7488 | 36.75 | .442 8612 | 36.10 |
| 5 | 1.403 0045 | 38.19 | 1.416 6172 | 37.44 | 1.429 9693 | 36.74 | 1.443 0778 | 36.09 |
| 6 | .403 2336 | 38.18 | .416 8419 | 37.43 | .430 1897 | 36.73 | .443 2943 | 36.08 |
| 7 | .403 4626 | 38.17 | .417 0664 | 37.41 | .430 4101 | 36.72 | .443 5107 | 36.07 |
| 8 | .403 6916 | 38.15 | .417 2909 | 37.40 | .430 6304 | 36.71 | .443 7271 | 36.06 |
| 9 | .403 9205 | 38.14 | .417 5153 | 37.39 | .430 8506 | 36.70 | .443 9434 | 36.05 |
| 10 | 1.404 1493 | 38.13 | 1.417 7396 | 37.38 | 1.431 0708 | 36.69 | 1.444 1597 | 36.04 |
| 11 | .404 3780 | 38.12 | .417 9639 | 37.37 | .431 2909 | 36.68 | .444 3758 | 36.03 |
| 12 | .404 6067 | 38.10 | .418 1881 | 37.36 | .431 5109 | 36.66 | .444 5920 | 36.02 |
| 13 | .404 8352 | 38.09 | .418 4122 | 37.35 | .431 7308 | 36.65 | .444 8080 | 36.00 |
| 14 | .405 0637 | 38.08 | .418 6362 | 37.33 | .431 9507 | 36.64 | .445 0240 | 35.99 |
| 15 | 1.405 2921 | 38.06 | 1.418 8602 | 37.32 | 1.432 1705 | 36.63 | 1.445 2400 | 35.98 |
| 16 | .405 5205 | 38.05 | .419 0841 | 37.31 | .432 3903 | 36.62 | .445 4558 | 35.97 |
| 17 | .405 7488 | 38.03 | .419 3079 | 37.30 | .432 6100 | 36.61 | .445 6716 | 35.96 |
| 18 | .405 9769 | 38.02 | .419 5317 | 37.29 | .432 8296 | 36.60 | .445 8874 | 35.95 |
| 19 | .406 2051 | 38.01 | .419 7554 | 37.27 | .433 0491 | 36.59 | .446 1031 | 35.94 |
| 20 | 1.406 4331 | 38.00 | 1.419 9790 | 37.26 | 1.433 2686 | 36.57 | 1.446 3187 | 35.93 |
| 21 | .406 6611 | 37.99 | .420 2026 | 37.25 | .433 4881 | 36.56 | .446 5343 | 35.92 |
| 22 | .406 8889 | 37.97 | .420 4260 | 37.24 | .433 7074 | 36.55 | .446 7498 | 35.91 |
| 23 | .407 1168 | 37.96 | .420 6494 | 37.23 | .433 9267 | 36.54 | .446 9652 | 35.90 |
| 24 | .407 3445 | 37.95 | .420 8728 | 37.22 | .434 1459 | 36.53 | .447 1806 | 35.89 |
| 25 | 1.407 5721 | 37.94 | 1.421 0960 | 37.20 | 1.434 3651 | 36.52 | 1.447 3959 | 35.88 |
| 26 | .407 7997 | 37.92 | .421 3192 | 37.19 | .434 5842 | 36.51 | .447 6112 | 35.87 |
| 27 | .408 0272 | 37.91 | .421 5423 | 37.18 | .434 8032 | 36.50 | .447 8263 | 35.86 |
| 28 | .408 2547 | 37.90 | .421 7654 | 37.17 | .435 0221 | 36.49 | .448 0415 | 35.85 |
| 29 | .408 4820 | 37.89 | .421 9884 | 37.16 | .435 2410 | 36.48 | .448 2565 | 35.84 |
| 30 | 1.408 7093 | 37.87 | 1.422 2113 | 37.15 | 1.435 4598 | 36.47 | 1.448 4715 | 35.83 |
| 31 | .408 9365 | 37.86 | .422 4341 | 37.13 | .435 6786 | 36.46 | .448 6865 | 35.82 |
| 32 | .409 1636 | 37.85 | .422 6569 | 37.12 | .435 8973 | 36.44 | .448 9014 | 35.81 |
| 33 | .409 3907 | 37.84 | .422 8796 | 37.11 | .436 1159 | 36.43 | .449 1162 | 35.80 |
| 34 | .409 6177 | 37.82 | .423 1022 | 37.10 | .436 3345 | 36.42 | .449 3309 | 35.79 |
| 35 | 1.409 8446 | 37.81 | 1.423 3248 | 37.09 | 1.436 5530 | 36.41 | 1.449 5456 | 35.78 |
| 36 | .410 0714 | 37.80 | .423 5473 | 37.08 | .436 7714 | 36.40 | .449 7603 | 35.77 |
| 37 | .410 2981 | 37.78 | .423 7697 | 37.06 | .436 9898 | 36.39 | .449 9749 | 35.76 |
| 38 | .410 5248 | 37.77 | .423 9920 | 37.05 | .437 2081 | 36.38 | .450 1894 | 35.75 |
| 39 | .410 7514 | 37.76 | .424 2143 | 37.04 | .437 4263 | 36.37 | .450 4038 | 35.74 |
| 40 | 1.410 9780 | 37.75 | 1.424 4365 | 37.03 | 1.437 6445 | 36.36 | 1.450 6182 | 35.73 |
| 41 | .411 2044 | 37.74 | .424 6586 | 37.02 | .437 8626 | 36.35 | .450 8325 | 35.72 |
| 42 | .411 4308 | 37.72 | .424 8807 | 37.01 | .438 0806 | 36.34 | .451 0468 | 35.71 |
| 43 | .411 6571 | 37.71 | .425 1027 | 36.99 | .438 2986 | 36.32 | .451 2610 | 35.70 |
| 44 | .411 8833 | 37.70 | .425 3246 | 36.98 | .438 5165 | 36.31 | .451 4752 | 35.69 |
| 45 | 1.412 1095 | 37.69 | 1.425 5465 | 36.97 | 1.438 7344 | 36.30 | 1.451 6893 | 35.68 |
| 46 | .412 3356 | 37.68 | .425 7683 | 36.96 | .438 9522 | 36.29 | .451 9033 | 35.67 |
| 47 | .412 5616 | 37.66 | .425 9900 | 36.95 | .439 1699 | 36.28 | .452 1173 | 35.66 |
| 48 | .412 7875 | 37.65 | .426 2117 | 36.94 | .439 3875 | 36.27 | .452 3312 | 35.65 |
| 49 | .413 0134 | 37.64 | .426 4333 | 36.92 | .439 6051 | 36.26 | .452 5450 | 35.64 |
| 50 | 1.413 2392 | 37.63 | 1.426 6548 | 36.91 | 1.439 8226 | 36.25 | 1.452 7588 | 35.63 |
| 51 | .413 4649 | 37.61 | .426 8762 | 36.90 | .440 0401 | 36.24 | .452 9725 | 35.62 |
| 52 | .413 6905 | 37.60 | .427 0976 | 36.89 | .440 2575 | 36.23 | .453 1862 | 35.61 |
| 53 | .413 9161 | 37.59 | .427 3189 | 36.88 | .440 4748 | 36.22 | .453 3998 | 35.60 |
| 54 | .414 1416 | 37.58 | .427 5402 | 36.87 | .440 6921 | 36.20 | .453 6134 | 35.59 |
| 55 | 1.414 3670 | 37.56 | 1.427 7613 | 36.86 | 1.440 9093 | 36.19 | 1.453 8269 | 35.58 |
| 56 | .414 5924 | 37.55 | .427 9824 | 36.85 | .441 1264 | 36.18 | .454 0403 | 35.57 |
| 57 | .414 8176 | 37.54 | .428 2035 | 36.83 | .441 3436 | 36.17 | .454 2537 | 35.56 |
| 58 | .415 0429 | 37.53 | .428 4244 | 36.82 | .441 5605 | 36.16 | .454 4670 | 35.55 |
| 59 | .415 2680 | 37.51 | .428 6453 | 36.81 | .441 7774 | 36.15 | .454 6802 | 35.54 |
| 60 | 1.415 4930 | 37.50 | 1.428 8662 | 36.80 | 1.441 9943 | 36.14 | 1.454 8934 | 35.53 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 40° | | 41° | | 42° | | 43° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.454 8934 | 35.53 | 1.467 5782 | 34.95 | 1.480 0627 | 34.41 | 1.492 3597 | 33.91 |
| 1 | .455 1065 | 35.52 | .467 7879 | 34.94 | .480 2691 | 34.40 | .492 5631 | 33.90 |
| 2 | .455 3196 | 35.51 | .467 9976 | 34.93 | .480 4755 | 34.40 | .492 7665 | 33.89 |
| 3 | .455 5326 | 35.50 | .468 2071 | 34.92 | .480 6819 | 34.39 | .492 9698 | 33.88 |
| 4 | .455 7456 | 35.49 | .468 4166 | 34.91 | .480 8882 | 34.38 | .493 1731 | 33.87 |
| 5 | 1.455 9585 | 35.48 | 1.468 6261 | 34.90 | 1.481 0944 | 34.37 | 1.493 3764 | 33.87 |
| 6 | .456 1713 | 35.47 | .468 8355 | 34.90 | .481 3006 | 34.36 | .493 5796 | 33.86 |
| 7 | .456 3841 | 35.46 | .469 0448 | 34.89 | .481 5068 | 34.35 | .493 7827 | 33.85 |
| 8 | .456 5968 | 35.45 | .469 2541 | 34.88 | .481 7129 | 34.34 | .493 9858 | 33.84 |
| 9 | .456 8094 | 35.44 | .469 4634 | 34.87 | .481 9189 | 34.33 | .494 1888 | 33.83 |
| 10 | 1.457 0220 | 35.43 | 1.469 6725 | 34.86 | 1.482 1249 | 34.33 | 1.494 3918 | 33.83 |
| 11 | .457 2346 | 35.42 | .469 8817 | 34.85 | .482 3308 | 34.32 | .494 5948 | 33.82 |
| 12 | .457 4470 | 35.41 | .470 0907 | 34.84 | .482 5367 | 34.31 | .494 7977 | 33.81 |
| 13 | .457 6595 | 35.40 | .470 2998 | 34.83 | .482 7425 | 34.30 | .495 0005 | 33.80 |
| 14 | .457 8718 | 35.39 | .470 5087 | 34.82 | .482 9483 | 34.29 | .495 2033 | 33.79 |
| 15 | 1.458 0841 | 35.38 | 1.470 7176 | 34.81 | 1.483 1540 | 34.28 | 1.495 4061 | 33.79 |
| 16 | .458 2964 | 35.37 | .470 9265 | 34.80 | .483 3597 | 34.28 | .495 6088 | 33.78 |
| 17 | .458 5086 | 35.36 | .471 1353 | 34.79 | .483 5653 | 34.27 | .495 8114 | 33.77 |
| 18 | .458 7207 | 35.35 | .471 3440 | 34.79 | .483 7709 | 34.26 | .496 0140 | 33.76 |
| 19 | .458 9328 | 35.34 | .471 5527 | 34.78 | .483 9764 | 34.25 | .496 2166 | 33.75 |
| 20 | 1.459 1448 | 35.33 | 1.471 7613 | 34.77 | 1.484 1819 | 34.24 | 1.496 4191 | 33.75 |
| 21 | .459 3567 | 35.32 | .471 9699 | 34.76 | .484 3873 | 34.23 | .496 6216 | 33.74 |
| 22 | .459 5686 | 35.31 | .472 1784 | 34.75 | .484 5927 | 34.22 | .496 8240 | 33.73 |
| 23 | .459 7805 | 35.30 | .472 3869 | 34.74 | .484 7980 | 34.22 | .497 0264 | 33.72 |
| 24 | .459 9922 | 35.29 | .472 5953 | 34.73 | .485 0033 | 34.21 | .497 2287 | 33.71 |
| 25 | 1.460 2040 | 35.28 | 1.472 8037 | 34.73 | 1.485 2085 | 34.20 | 1.497 4310 | 33.71 |
| 26 | .460 4156 | 35.27 | .473 0120 | 34.72 | .485 4137 | 34.19 | .497 6332 | 33.70 |
| 27 | .460 6272 | 35.26 | .473 2203 | 34.71 | .485 6188 | 34.18 | .497 8354 | 33.69 |
| 28 | .460 8388 | 35.25 | .473 4285 | 34.70 | .485 8239 | 34.17 | .498 0376 | 33.68 |
| 29 | .461 0503 | 35.24 | .473 6366 | 34.69 | .486 0289 | 34.16 | .498 2396 | 33.68 |
| 30 | 1.461 2617 | 35.23 | 1.473 8447 | 34.68 | 1.486 2338 | 34.16 | 1.498 4417 | 33.67 |
| 31 | .461 4731 | 35.23 | .474 0527 | 34.67 | .486 4388 | 34.15 | .498 6437 | 33.66 |
| 32 | .461 6844 | 35.22 | .474 2607 | 34.66 | .486 6436 | 34.14 | .498 8456 | 33.65 |
| 33 | .461 8957 | 35.21 | .474 4686 | 34.65 | .486 8484 | 34.13 | .499 0475 | 33.65 |
| 34 | .462 1069 | 35.20 | .474 6765 | 34.64 | .487 0532 | 34.12 | .499 2494 | 33.64 |
| 35 | 1.462 3180 | 35.19 | 1.474 8843 | 34.63 | 1.487 2579 | 34.12 | 1.499 4512 | 33.63 |
| 36 | .462 5291 | 35.18 | .475 0921 | 34.62 | .487 4626 | 34.11 | .499 6530 | 33.62 |
| 37 | .462 7401 | 35.17 | .475 2998 | 34.61 | .487 6672 | 34.10 | .499 8547 | 33.62 |
| 38 | .462 9511 | 35.16 | .475 5075 | 34.61 | .487 8718 | 34.09 | .500 0563 | 33.61 |
| 39 | .463 1620 | 35.15 | .475 7151 | 34.60 | .488 0763 | 34.08 | .500 2580 | 33.60 |
| 40 | 1.463 3729 | 35.14 | 1.475 9227 | 34.59 | 1.488 2807 | 34.07 | 1.500 4595 | 33.59 |
| 41 | .463 5837 | 35.13 | .476 1302 | 34.58 | .488 4852 | 34.07 | .500 6611 | 33.58 |
| 42 | .463 7944 | 35.12 | .476 3376 | 34.57 | .488 6895 | 34.06 | .500 8625 | 33.58 |
| 43 | .464 0051 | 35.11 | .476 5450 | 34.56 | .488 8939 | 34.05 | .501 0640 | 33.57 |
| 44 | .464 2158 | 35.10 | .476 7524 | 34.55 | .489 0981 | 34.04 | .501 2654 | 33.56 |
| 45 | 1.464 4263 | 35.09 | 1.476 9596 | 34.54 | 1.489 3023 | 34.03 | 1.501 4667 | 33.55 |
| 46 | .464 6369 | 35.08 | .477 1669 | 34.54 | .489 5065 | 34.02 | .501 6680 | 33.55 |
| 47 | .464 8473 | 35.07 | .477 3741 | 34.53 | .489 7106 | 34.02 | .501 8693 | 33.54 |
| 48 | .465 0577 | 35.06 | .477 5812 | 34.52 | .489 9147 | 34.01 | .502 0705 | 33.53 |
| 49 | .465 2681 | 35.05 | .477 7883 | 34.51 | .490 1187 | 34.00 | .502 2716 | 33.52 |
| 50 | 1.465 4784 | 35.04 | 1.477 9953 | 34.50 | 1.490 3227 | 33.99 | 1.502 4727 | 33.51 |
| 51 | .465 6886 | 35.04 | .478 2023 | 34.49 | .490 5266 | 33.98 | .502 6738 | 33.51 |
| 52 | .465 8988 | 35.03 | .478 4092 | 34.48 | .490 7305 | 33.97 | .502 8748 | 33.50 |
| 53 | .466 1090 | 35.02 | .478 6161 | 34.47 | .490 9343 | 33.96 | .503 0758 | 33.49 |
| 54 | .466 3190 | 35.01 | .478 8229 | 34.46 | .491 1381 | 33.95 | .503 2767 | 33.48 |
| 55 | 1.466 5290 | 35.00 | 1.479 0297 | 34.46 | 1.491 3418 | 33.95 | 1.503 4776 | 33.48 |
| 56 | .466 7390 | 34.99 | .479 2364 | 34.45 | .491 5455 | 33.94 | .503 6784 | 33.47 |
| 57 | .466 9489 | 34.98 | .479 4430 | 34.44 | .491 7491 | 33.93 | .503 8792 | 33.46 |
| 58 | .467 1587 | 34.97 | .479 6496 | 34.43 | .491 9527 | 33.92 | .504 0800 | 33.45 |
| 59 | .467 3685 | 34.96 | .479 8562 | 34.42 | .492 1562 | 33.91 | .504 2807 | 33.44 |
| 60 | 1.467 5782 | 34.95 | 1.480 0627 | 34.41 | 1.492 3597 | 33.91 | 1.504 4813 | 33.44 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 44° | | 45° | | 46° | | 47° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.504 4813 | 33.44 | 1.516 4390 | 33.00 | 1.528 2435 | 32.59 | 1.539 9048 | 32.20 |
| 1 | .504 6819 | 33.43 | .516 6370 | 32.99 | .528 4390 | 32.58 | .540 0980 | 32.20 |
| 2 | .504 8825 | 33.42 | .516 8349 | 32.98 | .528 6344 | 32.57 | .540 2912 | 32.19 |
| 3 | .505 0830 | 33.42 | .517 0328 | 32.98 | .528 8299 | 32.57 | .540 4843 | 32.18 |
| 4 | .505 2835 | 33.41 | .517 2306 | 32.97 | .529 0252 | 32.56 | .540 6774 | 32.18 |
| 5 | 1.505 4839 | 33.40 | 1.517 4284 | 32.96 | 1.529 2206 | 32.55 | 1.540 8705 | 32.17 |
| 6 | .505 6843 | 33.39 | .517 6262 | 32.96 | .529 4159 | 32.55 | .541 0635 | 32.17 |
| 7 | .505 8846 | 33.39 | .517 8239 | 32.95 | .529 6112 | 32.54 | .541 2564 | 32.16 |
| 8 | .506 0849 | 33.38 | .518 0216 | 32.94 | .529 8064 | 32.53 | .541 4494 | 32.15 |
| 9 | .506 2852 | 33.37 | .518 2192 | 32.93 | .530 0016 | 32.53 | .541 6423 | 32.15 |
| 10 | 1.506 4854 | 33.36 | 1.518 4168 | 32.93 | 1.530 1967 | 32.52 | 1.541 8352 | 32.14 |
| 11 | .506 6855 | 33.36 | .518 6143 | 32.92 | .530 3918 | 32.51 | .542 0280 | 32.14 |
| 12 | .506 8856 | 33.35 | .518 8118 | 32.91 | .530 5869 | 32.51 | .542 2208 | 32.13 |
| 13 | .507 0857 | 33.34 | .519 0093 | 32.91 | .530 7819 | 32.50 | .542 4135 | 32.12 |
| 14 | .507 2857 | 33.33 | .519 2067 | 32.90 | .530 9769 | 32.49 | .542 6063 | 32.11 |
| 15 | 1.507 4857 | 33.33 | 1.519 4041 | 32.89 | 1.531 1719 | 32.49 | 1.542 7989 | 32.11 |
| 16 | .507 6856 | 33.32 | .519 6014 | 32.89 | .531 3668 | 32.48 | .542 9916 | 32.10 |
| 17 | .507 8855 | 33.31 | .519 7987 | 32.88 | .531 5616 | 32.48 | .543 1842 | 32.10 |
| 18 | .508 0853 | 33.30 | .519 9960 | 32.87 | .531 7565 | 32.47 | .543 3768 | 32.09 |
| 19 | .508 2851 | 33.29 | .520 1932 | 32.86 | .531 9513 | 32.46 | .543 5693 | 32.09 |
| 20 | 1.508 4849 | 33.29 | 1.520 3904 | 32.86 | 1.532 1460 | 32.46 | 1.543 7618 | 32.08 |
| 21 | .508 6846 | 33.28 | .520 5875 | 32.85 | .532 3407 | 32.45 | .543 9543 | 32.08 |
| 22 | .508 8843 | 33.27 | .520 7846 | 32.84 | .532 5354 | 32.44 | .544 1467 | 32.07 |
| 23 | .509 0839 | 33.27 | .520 9816 | 32.84 | .532 7300 | 32.44 | .544 3391 | 32.06 |
| 24 | .509 2835 | 33.26 | .521 1786 | 32.83 | .532 9246 | 32.43 | .544 5315 | 32.06 |
| 25 | 1.509 4830 | 33.25 | 1.521 3756 | 32.82 | 1.533 1192 | 32.43 | 1.544 7238 | 32.05 |
| 26 | .509 6825 | 33.24 | .521 5725 | 32.82 | .533 3137 | 32.42 | .544 9161 | 32.04 |
| 27 | .509 8819 | 33.24 | .521 7694 | 32.81 | .533 5082 | 32.42 | .545 1083 | 32.04 |
| 28 | .510 0813 | 33.23 | .521 9662 | 32.80 | .533 7027 | 32.41 | .545 3005 | 32.03 |
| 29 | .510 2807 | 33.22 | .522 1630 | 32.80 | .533 8971 | 32.40 | .545 4927 | 32.03 |
| 30 | 1.510 4800 | 33.21 | 1.522 3598 | 32.79 | 1.534 0914 | 32.39 | 1.545 6849 | 32.02 |
| 31 | .510 6792 | 33.21 | .522 5565 | 32.78 | .534 2858 | 32.39 | .545 8770 | 32.02 |
| 32 | .510 8785 | 33.20 | .522 7531 | 32.78 | .534 4801 | 32.38 | .546 0690 | 32.01 |
| 33 | .511 0776 | 33.19 | .522 9498 | 32.78 | .534 6743 | 32.37 | .546 2611 | 32.00 |
| 34 | .511 2768 | 33.18 | .523 1464 | 32.77 | .534 8685 | 32.37 | .546 4531 | 32.00 |
| 35 | 1.511 4759 | 33.18 | 1.523 3429 | 32.76 | 1.535 0627 | 32.36 | 1.546 6450 | 31.99 |
| 36 | .511 6749 | 33.17 | .523 5394 | 32.75 | .535 2568 | 32.35 | .546 8370 | 31.98 |
| 37 | .511 8739 | 33.16 | .523 7359 | 32.74 | .535 4509 | 32.35 | .547 0289 | 31.98 |
| 38 | .512 0729 | 33.15 | .523 9323 | 32.73 | .535 6450 | 32.34 | .547 2207 | 31.97 |
| 39 | .512 2718 | 33.15 | .524 1287 | 32.73 | .535 8390 | 32.33 | .547 4125 | 31.97 |
| 40 | 1.512 4707 | 33.14 | 1.524 3251 | 32.72 | 1.536 0330 | 32.33 | 1.547 6043 | 31.96 |
| 41 | .512 6695 | 33.13 | .524 5214 | 32.71 | .536 2270 | 32.32 | .547 7961 | 31.96 |
| 42 | .512 8683 | 33.13 | .524 7176 | 32.71 | .536 4209 | 32.32 | .547 9878 | 31.95 |
| 43 | .513 0670 | 33.12 | .524 9138 | 32.70 | .536 6148 | 32.31 | .548 1795 | 31.94 |
| 44 | .513 2657 | 33.11 | .525 1100 | 32.70 | .536 8086 | 32.30 | .548 3711 | 31.94 |
| 45 | 1.513 4644 | 33.11 | 1.525 3062 | 32.69 | 1.537 0024 | 32.30 | 1.548 5627 | 31.93 |
| 46 | .513 6630 | 33.10 | .525 5023 | 32.68 | .537 1962 | 32.29 | .548 7543 | 31.93 |
| 47 | .513 8615 | 33.09 | .525 6983 | 32.67 | .537 3899 | 32.28 | .548 9458 | 31.92 |
| 48 | .514 0601 | 33.08 | .525 8944 | 32.67 | .537 5836 | 32.28 | .549 1373 | 31.91 |
| 49 | .514 2586 | 33.07 | .526 0903 | 32.66 | .537 7772 | 32.27 | .549 3288 | 31.91 |
| 50 | 1.514 4570 | 33.07 | 1.526 2863 | 32.65 | 1.537 9708 | 32.26 | 1.549 5202 | 31.90 |
| 51 | .514 6554 | 33.06 | .526 4822 | 32.64 | .538 1644 | 32.26 | .549 7116 | 31.90 |
| 52 | .514 8537 | 33.05 | .526 6780 | 32.64 | .538 3579 | 32.25 | .549 9030 | 31.89 |
| 53 | .515 0520 | 33.05 | .526 8739 | 32.63 | .538 5514 | 32.25 | .550 0943 | 31.88 |
| 54 | .515 2503 | 33.04 | .527 0696 | 32.62 | .538 7449 | 32.24 | .550 2856 | 31.88 |
| 55 | 1.515 4485 | 33.04 | 1.527 2654 | 32.62 | 1.538 9383 | 32.23 | 1.550 4769 | 31.87 |
| 56 | .515 6467 | 33.03 | .527 4611 | 32.61 | .539 1317 | 32.23 | .550 6681 | 31.87 |
| 57 | .515 8449 | 33.02 | .527 6567 | 32.61 | .539 3250 | 32.22 | .550 8593 | 31.86 |
| 58 | .516 0430 | 33.01 | .527 8524 | 32.60 | .539 5183 | 32.21 | .551 0504 | 31.86 |
| 59 | .516 2410 | 33.01 | .528 0479 | 32.60 | .539 7116 | 32.21 | .551 2416 | 31.85 |
| 60 | 1.516 4390 | 33.00 | 1.528 2435 | 32.59 | 1.539 9048 | 32.20 | 1.551 4326 | 31.85 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 48° | | 49° | | 50° | | 51° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.551 4326 | 31.85 | 1.562 8360 | 31.51 | 1.574 1234 | 31.20 | 1.585 3031 | 30.91 |
| 1 | .551 6237 | 31.84 | .563 0250 | 31.51 | .574 3106 | 31.20 | .585 4886 | 30.91 |
| 2 | .551 8147 | 31.83 | .563 2140 | 31.50 | .574 4977 | 31.19 | .585 6740 | 30.90 |
| 3 | .552 0057 | 31.83 | .563 4030 | 31.50 | .574 6849 | 31.19 | .585 8594 | 30.90 |
| 4 | .552 1966 | 31.82 | .563 5920 | 31.49 | .574 8720 | 31.18 | .586 0448 | 30.89 |
| 5 | 1.552 3876 | 31.82 | 1.563 7809 | 31.48 | 1.575 0590 | 31.18 | 1.586 2302 | 30.89 |
| 6 | .552 5784 | 31.81 | .563 9698 | 31.48 | .575 2461 | 31.17 | .586 4155 | 30.89 |
| 7 | .552 7693 | 31.80 | .564 1586 | 31.47 | .575 4331 | 31.17 | .586 6008 | 30.88 |
| 8 | .552 9601 | 31.80 | .564 3475 | 31.47 | .575 6201 | 31.16 | .586 7859 | 30.87 |
| 9 | .553 1508 | 31.79 | .564 5363 | 31.46 | .575 8070 | 31.16 | .586 9713 | 30.87 |
| 10 | 1.553 3416 | 31.79 | 1.564 7250 | 31.46 | 1.575 9939 | 31.15 | 1.587 1565 | 30.87 |
| 11 | .553 5323 | 31.78 | .564 9138 | 31.45 | .576 1808 | 31.15 | .587 3417 | 30.86 |
| 12 | .553 7230 | 31.78 | .565 1025 | 31.45 | .576 3677 | 31.14 | .587 5268 | 30.86 |
| 13 | .553 9136 | 31.77 | .565 2911 | 31.44 | .576 5546 | 31.14 | .587 7120 | 30.85 |
| 14 | .554 1042 | 31.76 | .565 4798 | 31.44 | .576 7414 | 31.13 | .587 8971 | 30.85 |
| 15 | 1.554 2948 | 31.76 | 1.565 6684 | 31.43 | 1.576 9281 | 31.13 | 1.588 0821 | 30.84 |
| 16 | .554 4853 | 31.75 | .565 8569 | 31.43 | .577 1149 | 31.12 | .588 2672 | 30.84 |
| 17 | .554 6758 | 31.75 | .566 0455 | 31.42 | .577 3016 | 31.12 | .588 4522 | 30.83 |
| 18 | .554 8663 | 31.74 | .566 2340 | 31.41 | .577 4883 | 31.11 | .588 6372 | 30.83 |
| 19 | .555 0567 | 31.74 | .566 4225 | 31.41 | .577 6749 | 31.11 | .588 8222 | 30.83 |
| 20 | 1.555 2472 | 31.73 | 1.566 6109 | 31.40 | 1.577 8615 | 31.10 | 1.589 0071 | 30.82 |
| 21 | .555 4375 | 31.73 | .566 7993 | 31.40 | .578 0481 | 31.10 | .589 1920 | 30.82 |
| 22 | .555 6279 | 31.72 | .566 9877 | 31.39 | .578 2347 | 31.09 | .589 3769 | 30.81 |
| 23 | .555 8182 | 31.71 | .567 1761 | 31.39 | .578 4213 | 31.09 | .589 5618 | 30.81 |
| 24 | .556 0084 | 31.71 | .567 3644 | 31.38 | .578 6078 | 31.08 | .589 7466 | 30.80 |
| 25 | 1.556 1987 | 31.70 | 1.567 5527 | 31.38 | 1.578 7942 | 31.08 | 1.589 9314 | 30.80 |
| 26 | .556 3888 | 31.70 | .567 7409 | 31.37 | .578 9807 | 31.07 | .590 1162 | 30.79 |
| 27 | .556 5790 | 31.69 | .567 9291 | 31.37 | .579 1671 | 31.07 | .590 3009 | 30.79 |
| 28 | .556 7691 | 31.68 | .568 1173 | 31.36 | .579 3535 | 31.06 | .590 4857 | 30.78 |
| 29 | .556 9592 | 31.68 | .568 3055 | 31.36 | .579 5399 | 31.06 | .590 6704 | 30.78 |
| 30 | 1.557 1493 | 31.67 | 1.568 4936 | 31.35 | 1.579 7262 | 31.06 | 1.590 8550 | 30.78 |
| 31 | .557 3393 | 31.67 | .568 6817 | 31.35 | .579 9125 | 31.05 | .591 0397 | 30.77 |
| 32 | .557 5293 | 31.66 | .568 8698 | 31.34 | .580 0988 | 31.04 | .591 2243 | 30.77 |
| 33 | .557 7193 | 31.66 | .569 0579 | 31.34 | .580 2851 | 31.04 | .591 4089 | 30.76 |
| 34 | .557 9092 | 31.65 | .569 2459 | 31.33 | .580 4713 | 31.03 | .591 5935 | 30.76 |
| 35 | 1.558 0991 | 31.65 | 1.569 4338 | 31.33 | 1.580 6575 | 31.03 | 1.591 7780 | 30.75 |
| 36 | .558 2890 | 31.64 | .569 6218 | 31.32 | .580 8436 | 31.03 | .591 9625 | 30.75 |
| 37 | .558 4788 | 31.64 | .569 8097 | 31.32 | .581 0298 | 31.02 | .592 1470 | 30.75 |
| 38 | .558 6686 | 31.63 | .569 9976 | 31.31 | .581 2159 | 31.02 | .592 3315 | 30.74 |
| 39 | .558 8584 | 31.62 | .570 1854 | 31.30 | .581 4020 | 31.01 | .592 5159 | 30.74 |
| 40 | 1.559 0482 | 31.62 | 1.570 3733 | 31.30 | 1.581 5880 | 31.01 | 1.592 7003 | 30.73 |
| 41 | .559 2379 | 31.61 | .570 5611 | 31.29 | .581 7740 | 31.00 | .592 8847 | 30.73 |
| 42 | .559 4275 | 31.61 | .570 7488 | 31.29 | .581 9600 | 31.00 | .593 0690 | 30.72 |
| 43 | .559 6172 | 31.60 | .570 9366 | 31.28 | .582 1460 | 30.99 | .593 2534 | 30.72 |
| 44 | .559 8068 | 31.60 | .571 1243 | 31.28 | .582 3319 | 30.99 | .593 4377 | 30.72 |
| 45 | 1.559 9963 | 31.59 | 1.571 3119 | 31.28 | 1.582 5179 | 30.98 | 1.593 6219 | 30.71 |
| 46 | .560 1859 | 31.59 | .571 4996 | 31.27 | .582 7037 | 30.98 | .593 8062 | 30.71 |
| 47 | .560 3754 | 31.58 | .571 6872 | 31.27 | .582 8896 | 30.97 | .593 9904 | 30.70 |
| 48 | .560 5648 | 31.57 | .571 8748 | 31.26 | .583 0754 | 30.97 | .594 1746 | 30.70 |
| 49 | .560 7543 | 31.57 | .572 0623 | 31.26 | .583 2612 | 30.96 | .594 3588 | 30.69 |
| 50 | 1.560 9437 | 31.56 | 1.572 2499 | 31.25 | 1.583 4470 | 30.96 | 1.594 5429 | 30.69 |
| 51 | .561 1331 | 31.56 | .572 4373 | 31.25 | .583 6327 | 30.95 | .594 7270 | 30.68 |
| 52 | .561 3224 | 31.55 | .572 6248 | 31.24 | .583 8184 | 30.95 | .594 9111 | 30.68 |
| 53 | .561 5117 | 31.55 | .572 8123 | 31.24 | .584 0041 | 30.94 | .595 0952 | 30.68 |
| 54 | .561 7010 | 31.54 | .572 9997 | 31.23 | .584 1898 | 30.94 | .595 2792 | 30.67 |
| 55 | 1.561 8902 | 31.54 | 1.573 1870 | 31.23 | 1.584 3754 | 30.94 | 1.595 4633 | 30.67 |
| 56 | .562 0794 | 31.53 | .573 3743 | 31.22 | .584 5610 | 30.93 | .595 6473 | 30.66 |
| 57 | .562 2686 | 31.53 | .573 5616 | 31.22 | .584 7466 | 30.93 | .595 8312 | 30.66 |
| 58 | .562 4578 | 31.52 | .573 7489 | 31.21 | .584 9321 | 30.92 | .596 0151 | 30.65 |
| 59 | .562 6469 | 31.52 | .573 9362 | 31.21 | .585 1176 | 30.92 | .596 1990 | 30.65 |
| 60 | 1.562 8360 | 31.51 | 1.574 1234 | 31.20 | 1.585 3031 | 30.91 | 1.596 3829 | 30.65 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 52° | | 53° | | 54° | | 55° | |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | .596 3829 | 30.65 | .607 3703 | 30.40 | .618 2724 | 30.17 | .629 0959 | 29.96 |
| 1 | .596 5668 | 30.64 | .607 5527 | 30.39 | .618 4534 | 30.17 | .629 2757 | 29.96 |
| 2 | .596 7506 | 30.64 | .607 7350 | 30.39 | .618 6344 | 30.16 | .629 4554 | 29.96 |
| 3 | .596 9344 | 30.63 | .607 9174 | 30.39 | .618 8153 | 30.16 | .629 6351 | 29.95 |
| 4 | .597 1182 | 30.63 | .608 0997 | 30.38 | .618 9963 | 30.16 | .629 8148 | 29.95 |
| 5 | .597 3020 | 30.62 | .608 2820 | 30.38 | .619 1772 | 30.15 | .629 9945 | 29.95 |
| 6 | .597 4857 | 30.62 | .608 4642 | 30.38 | .619 3581 | 30.15 | .630 1742 | 29.94 |
| 7 | .597 6694 | 30.62 | .608 6465 | 30.37 | .619 5390 | 30.15 | .630 3538 | 29.94 |
| 8 | .597 8531 | 30.61 | .608 8287 | 30.37 | .619 7199 | 30.14 | .630 5335 | 29.94 |
| 9 | .598 0368 | 30.61 | .609 0109 | 30.36 | .619 9007 | 30.14 | .630 7131 | 29.93 |
| 10 | .598 2204 | 30.60 | .609 1931 | 30.36 | .620 0816 | 30.14 | .630 8927 | 29.93 |
| 11 | .598 4040 | 30.60 | .609 3752 | 30.36 | .620 2623 | 30.13 | .631 0722 | 29.93 |
| 12 | .598 5876 | 30.59 | .609 5573 | 30.35 | .620 4431 | 30.13 | .631 2518 | 29.92 |
| 13 | .598 7711 | 30.59 | .609 7394 | 30.35 | .620 6239 | 30.12 | .631 4313 | 29.92 |
| 14 | .598 9547 | 30.59 | .609 9215 | 30.34 | .620 8046 | 30.12 | .631 6108 | 29.92 |
| 15 | .599 1382 | 30.58 | .610 1036 | 30.34 | .620 9853 | 30.12 | .631 7903 | 29.91 |
| 16 | .599 3217 | 30.58 | .610 2856 | 30.34 | .621 1660 | 30.11 | .631 9698 | 29.91 |
| 17 | .599 5051 | 30.57 | .610 4676 | 30.33 | .621 3467 | 30.11 | .632 1492 | 29.91 |
| 18 | .599 6885 | 30.57 | .610 6496 | 30.33 | .621 5274 | 30.11 | .632 3286 | 29.90 |
| 19 | .599 8719 | 30.57 | .610 8315 | 30.32 | .621 7080 | 30.10 | .632 5081 | 29.90 |
| 20 | .600 0553 | 30.56 | .611 0135 | 30.32 | .621 8886 | 30.10 | .632 6875 | 29.90 |
| 21 | .600 2387 | 30.56 | .611 1954 | 30.32 | .622 0692 | 30.10 | .632 8668 | 29.89 |
| 22 | .600 4220 | 30.55 | .611 3773 | 30.31 | .622 2497 | 30.09 | .633 0462 | 29.89 |
| 23 | .600 6053 | 30.55 | .611 5591 | 30.31 | .622 4303 | 30.09 | .633 2255 | 29.89 |
| 24 | .600 7886 | 30.55 | .611 7410 | 30.31 | .622 6108 | 30.09 | .633 4048 | 29.88 |
| 25 | .600 9718 | 30.54 | .611 9228 | 30.30 | .622 7913 | 30.08 | .633 5841 | 29.88 |
| 26 | .601 1551 | 30.54 | .612 1046 | 30.30 | .622 9718 | 30.08 | .633 7634 | 29.88 |
| 27 | .601 3383 | 30.53 | .612 2864 | 30.29 | .623 1523 | 30.08 | .633 9427 | 29.87 |
| 28 | .601 5214 | 30.53 | .612 4681 | 30.29 | .623 3327 | 30.07 | .634 1219 | 29.87 |
| 29 | .601 7046 | 30.52 | .612 6499 | 30.29 | .623 5131 | 30.07 | .634 3011 | 29.87 |
| 30 | .601 8877 | 30.52 | .612 8316 | 30.28 | .623 6935 | 30.06 | .634 4803 | 29.86 |
| 31 | .602 0708 | 30.52 | .613 0132 | 30.28 | .623 8739 | 30.06 | .634 6595 | 29.86 |
| 32 | .602 2539 | 30.51 | .613 1949 | 30.28 | .624 0543 | 30.06 | .634 8387 | 29.86 |
| 33 | .602 4370 | 30.51 | .613 3765 | 30.27 | .624 2346 | 30.05 | .635 0178 | 29.86 |
| 34 | .602 6200 | 30.50 | .613 5582 | 30.27 | .624 4149 | 30.05 | .635 1969 | 29.85 |
| 35 | .602 8030 | 30.50 | .613 7398 | 30.26 | .624 5952 | 30.05 | .635 3760 | 29.85 |
| 36 | .602 9860 | 30.50 | .613 9213 | 30.26 | .624 7755 | 30.04 | .635 5551 | 29.85 |
| 37 | .603 1690 | 30.49 | .614 1029 | 30.26 | .624 9557 | 30.04 | .635 7342 | 29.84 |
| 38 | .603 3519 | 30.49 | .614 2844 | 30.25 | .625 1360 | 30.04 | .635 9132 | 29.84 |
| 39 | .603 5348 | 30.48 | .614 4659 | 30.25 | .625 3162 | 30.03 | .636 0922 | 29.84 |
| 40 | .603 7177 | 30.48 | .614 6474 | 30.25 | .625 4964 | 30.03 | .636 2713 | 29.83 |
| 41 | .603 9005 | 30.47 | .614 8288 | 30.24 | .625 6765 | 30.03 | .636 4502 | 29.83 |
| 42 | .604 0834 | 30.47 | .615 0103 | 30.24 | .625 8567 | 30.02 | .636 6292 | 29.83 |
| 43 | .604 2662 | 30.47 | .615 1917 | 30.23 | .626 0368 | 30.02 | .636 8082 | 29.82 |
| 44 | .604 4490 | 30.46 | .615 3731 | 30.23 | .626 2169 | 30.02 | .636 9871 | 29.82 |
| 45 | .604 6317 | 30.46 | .615 5545 | 30.23 | .626 3970 | 30.01 | .637 1660 | 29.82 |
| 46 | .604 8145 | 30.45 | .615 7358 | 30.22 | .626 5771 | 30.01 | .637 3449 | 29.82 |
| 47 | .604 9972 | 30.45 | .615 9171 | 30.22 | .626 7571 | 30.01 | .637 5238 | 29.81 |
| 48 | .605 1799 | 30.45 | .616 0984 | 30.22 | .626 9372 | 30.00 | .637 7027 | 29.81 |
| 49 | .605 3626 | 30.44 | .616 2797 | 30.21 | .627 1172 | 30.00 | .637 8815 | 29.81 |
| 50 | .605 5452 | 30.44 | .616 4610 | 30.21 | .627 2972 | 30.00 | .638 0603 | 29.80 |
| 51 | .605 7278 | 30.43 | .616 6422 | 30.20 | .627 4771 | 29.99 | .638 2391 | 29.80 |
| 52 | .605 9104 | 30.43 | .616 8234 | 30.20 | .627 6571 | 29.99 | .638 4179 | 29.80 |
| 53 | .606 0930 | 30.43 | .617 0046 | 30.20 | .627 8370 | 29.99 | .638 5967 | 29.79 |
| 54 | .606 2755 | 30.42 | .617 1858 | 30.19 | .628 0169 | 29.98 | .638 7754 | 29.79 |
| 55 | .606 4581 | 30.42 | .617 3669 | 30.19 | .628 1968 | 29.98 | .638 9542 | 29.79 |
| 56 | .606 6406 | 30.42 | .617 5481 | 30.19 | .628 3766 | 29.98 | .639 1329 | 29.78 |
| 57 | .606 8230 | 30.41 | .617 7292 | 30.18 | .628 5565 | 29.97 | .639 3116 | 29.78 |
| 58 | .607 0055 | 30.41 | .617 9102 | 30.18 | .628 7363 | 29.97 | .639 4902 | 29.78 |
| 59 | .607 1879 | 30.40 | .618 0913 | 30.17 | .628 9161 | 29.97 | .639 6689 | 29.77 |
| 60 | .607 3703 | 30.40 | .618 2724 | 30.17 | .629 0959 | 29.96 | .639 8475 | 29.77 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 56° | | 57° | | 58° | | 59° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.639 8475 | 29.77 | 1.650 5336 | 29.60 | 1.661 1601 | 29.44 | 1.671 7331 | 29.30 |
| 1 | .640 0262 | 29.77 | .650 7112 | 29.60 | .661 3368 | 29.44 | .671 9089 | 29.30 |
| 2 | .640 2048 | 29.77 | .650 8887 | 29.59 | .661 5134 | 29.44 | .672 0846 | 29.30 |
| 3 | .640 3833 | 29.76 | .651 0663 | 29.59 | .661 6900 | 29.43 | .672 2604 | 29.29 |
| 4 | .640 5619 | 29.76 | .651 2438 | 29.59 | .661 8666 | 29.43 | .672 4362 | 29.29 |
| 5 | 1.640 7405 | 29.76 | 1.651 4213 | 29.58 | 1.662 0432 | 29.43 | 1.672 6119 | 29.29 |
| 6 | .640 9190 | 29.75 | .651 5988 | 29.58 | .662 2197 | 29.43 | .672 7876 | 29.29 |
| 7 | .641 0975 | 29.75 | .651 7763 | 29.58 | .662 3963 | 29.42 | .672 9634 | 29.28 |
| 8 | .641 2760 | 29.75 | .651 9538 | 29.58 | .662 5728 | 29.42 | .673 1391 | 29.28 |
| 9 | .641 4545 | 29.74 | .652 1312 | 29.57 | .662 7493 | 29.42 | .673 3147 | 29.28 |
| 10 | 1.641 6329 | 29.74 | 1.652 3086 | 29.57 | 1.662 9258 | 29.42 | 1.673 4904 | 29.28 |
| 11 | .641 8114 | 29.74 | .652 4861 | 29.57 | .663 1023 | 29.41 | .673 6661 | 29.28 |
| 12 | .641 9898 | 29.74 | .652 6635 | 29.57 | .663 2788 | 29.41 | .673 8417 | 29.27 |
| 13 | .642 1682 | 29.73 | .652 8408 | 29.56 | .663 4553 | 29.41 | .674 0174 | 29.27 |
| 14 | .642 3466 | 29.73 | .653 0182 | 29.56 | .663 6317 | 29.41 | .674 1930 | 29.27 |
| 15 | 1.642 5250 | 29.73 | 1.653 1956 | 29.56 | 1.663 8082 | 29.40 | 1.674 3686 | 29.27 |
| 16 | .642 7033 | 29.72 | .653 3729 | 29.55 | .663 9846 | 29.40 | .674 5442 | 29.27 |
| 17 | .642 8816 | 29.72 | .653 5502 | 29.55 | .664 1610 | 29.40 | .674 7198 | 29.26 |
| 18 | .643 0599 | 29.72 | .653 7275 | 29.55 | .664 3374 | 29.40 | .674 8954 | 29.26 |
| 19 | .643 2382 | 29.71 | .653 9048 | 29.55 | .664 5137 | 29.39 | .675 0709 | 29.26 |
| 20 | 1.643 4165 | 29.71 | 1.654 0821 | 29.54 | 1.664 6901 | 29.39 | 1.675 2465 | 29.26 |
| 21 | .643 5948 | 29.71 | .654 2593 | 29.54 | .664 8664 | 29.39 | .675 4220 | 29.25 |
| 22 | .643 7730 | 29.71 | .654 4366 | 29.54 | .665 0428 | 29.39 | .675 5975 | 29.25 |
| 23 | .643 9513 | 29.70 | .654 6138 | 29.54 | .665 2191 | 29.39 | .675 7730 | 29.25 |
| 24 | .644 1295 | 29.70 | .654 7910 | 29.53 | .665 3954 | 29.38 | .675 9485 | 29.25 |
| 25 | 1.644 3077 | 29.70 | 1.654 9682 | 29.53 | 1.665 5717 | 29.38 | 1.676 1240 | 29.25 |
| 26 | .644 4858 | 29.69 | .655 1454 | 29.53 | .665 7480 | 29.38 | .676 2995 | 29.24 |
| 27 | .644 6640 | 29.69 | .655 3225 | 29.53 | .665 9242 | 29.38 | .676 4749 | 29.24 |
| 28 | .644 8421 | 29.69 | .655 4997 | 29.52 | .666 1005 | 29.37 | .676 6504 | 29.24 |
| 29 | .645 0203 | 29.69 | .655 6768 | 29.52 | .666 2767 | 29.37 | .676 8258 | 29.24 |
| 30 | 1.645 1984 | 29.68 | 1.655 8539 | 29.52 | 1.666 4529 | 29.37 | 1.677 0012 | 29.24 |
| 31 | .645 3765 | 29.68 | .656 0310 | 29.51 | .666 6291 | 29.37 | .677 1766 | 29.23 |
| 32 | .645 5545 | 29.68 | .656 2081 | 29.51 | .666 8053 | 29.36 | .677 3520 | 29.23 |
| 33 | .645 7326 | 29.67 | .656 3852 | 29.51 | .666 9815 | 29.36 | .677 5274 | 29.23 |
| 34 | .645 9106 | 29.67 | .656 5622 | 29.51 | .667 1577 | 29.36 | .677 7028 | 29.23 |
| 35 | 1.646 0886 | 29.67 | 1.656 7392 | 29.50 | 1.667 3338 | 29.36 | 1.677 8781 | 29.23 |
| 36 | .646 2666 | 29.67 | .656 9163 | 29.50 | .667 5100 | 29.35 | .678 0535 | 29.22 |
| 37 | .646 4446 | 29.66 | .657 0933 | 29.50 | .667 6861 | 29.35 | .678 2288 | 29.22 |
| 38 | .646 6226 | 29.66 | .657 2703 | 29.50 | .667 8622 | 29.35 | .678 4041 | 29.22 |
| 39 | .646 8005 | 29.66 | .657 4472 | 29.49 | .668 0383 | 29.35 | .678 5794 | 29.22 |
| 40 | 1.646 9785 | 29.65 | 1.657 6242 | 29.49 | 1.668 2144 | 29.35 | 1.678 7547 | 29.22 |
| 41 | .647 1564 | 29.65 | .657 8011 | 29.49 | .668 3904 | 29.34 | .678 9300 | 29.21 |
| 42 | .647 3343 | 29.65 | .657 9781 | 29.49 | .668 5665 | 29.34 | .679 1053 | 29.21 |
| 43 | .647 5122 | 29.65 | .658 1550 | 29.48 | .668 7425 | 29.34 | .679 2806 | 29.21 |
| 44 | .647 6900 | 29.64 | .658 3318 | 29.48 | .668 9185 | 29.34 | .679 4558 | 29.21 |
| 45 | 1.647 8679 | 29.64 | 1.658 5087 | 29.48 | 1.669 0945 | 29.33 | 1.679 6310 | 29.20 |
| 46 | .648 0457 | 29.64 | .658 6855 | 29.48 | .669 2705 | 29.33 | .679 8063 | 29.20 |
| 47 | .648 2235 | 29.63 | .658 8624 | 29.47 | .669 4465 | 29.33 | .679 9815 | 29.20 |
| 48 | .648 4013 | 29.63 | .659 0393 | 29.47 | .669 6225 | 29.33 | .680 1567 | 29.20 |
| 49 | .648 5791 | 29.63 | .659 2161 | 29.47 | .669 7984 | 29.32 | .680 3319 | 29.20 |
| 50 | 1.648 7569 | 29.63 | 1.659 3929 | 29.47 | 1.669 9744 | 29.32 | 1.680 5070 | 29.19 |
| 51 | .648 9346 | 29.62 | .659 5697 | 29.46 | .670 1503 | 29.32 | .680 6822 | 29.19 |
| 52 | .649 1123 | 29.62 | .659 7465 | 29.46 | .670 3262 | 29.32 | .680 8574 | 29.19 |
| 53 | .649 2901 | 29.62 | .659 9232 | 29.46 | .670 5021 | 29.32 | .681 0325 | 29.19 |
| 54 | .649 4677 | 29.61 | .660 1000 | 29.46 | .670 6780 | 29.31 | .681 2076 | 29.19 |
| 55 | 1.649 6454 | 29.61 | 1.660 2767 | 29.45 | 1.670 8539 | 29.31 | 1.681 3827 | 29.18 |
| 56 | .649 8231 | 29.61 | .660 4534 | 29.45 | .671 0298 | 29.31 | .681 5578 | 29.18 |
| 57 | .650 0007 | 29.61 | .660 6301 | 29.45 | .671 2056 | 29.31 | .681 7329 | 29.18 |
| 58 | .650 1784 | 29.60 | .660 8068 | 29.45 | .671 3814 | 29.30 | .681 9080 | 29.18 |
| 59 | .650 3560 | 29.60 | .660 9835 | 29.44 | .671 5573 | 29.30 | .682 0831 | 29.18 |
| 60 | 1.650 5336 | 29.60 | 1.661 1601 | 29.44 | 1.671 7331 | 29.30 | 1.682 2581 | 29.17 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 60° | | 61° | | 62° | | 63° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.682 2581 | 29.17 | 1.692 7408 | 29.07 | 1.703 1866 | 28.97 | 1.713 6006 | 28.89 |
| 1 | .682 4332 | 29.17 | .692 9152 | 29.06 | .703 3604 | 28.97 | .713 7739 | 28.89 |
| 2 | .682 6082 | 29.17 | .693 0896 | 29.06 | .703 5342 | 28.97 | .713 9473 | 28.89 |
| 3 | .682 7832 | 29.17 | .693 2640 | 29.06 | .703 7080 | 28.97 | .714 1206 | 28.88 |
| 4 | .682 9582 | 29.17 | .693 4383 | 29.06 | .703 8818 | 28.96 | .714 2939 | 28.88 |
| 5 | 1.683 1332 | 29.16 | 1.693 6127 | 29.06 | 1.704 0556 | 28.96 | 1.714 4672 | 28.88 |
| 6 | .683 3082 | 29.16 | .693 7870 | 29.05 | .704 2293 | 28.96 | .714 6405 | 28.88 |
| 7 | .683 4832 | 29.16 | .693 9613 | 29.05 | .704 4031 | 28.96 | .714 8138 | 28.88 |
| 8 | .683 6581 | 29.16 | .694 1356 | 29.05 | .704 5768 | 28.96 | .714 9870 | 28.88 |
| 9 | .683 8331 | 29.16 | .694 3099 | 29.05 | .704 7506 | 28.96 | .715 1603 | 28.88 |
| 10 | 1.684 0080 | 29.16 | 1.694 4842 | 29.05 | 1.704 9243 | 28.96 | 1.715 3336 | 28.88 |
| 11 | .684 1830 | 29.15 | .694 6585 | 29.04 | .705 0981 | 28.95 | .715 5068 | 28.88 |
| 12 | .684 3579 | 29.15 | .694 8328 | 29.04 | .705 2718 | 28.95 | .715 6801 | 28.87 |
| 13 | .684 5328 | 29.15 | .695 0070 | 29.04 | .705 4455 | 28.95 | .715 8533 | 28.87 |
| 14 | .684 7077 | 29.15 | .695 1813 | 29.04 | .705 6192 | 28.95 | .716 0266 | 28.87 |
| 15 | 1.684 8826 | 29.14 | 1.695 3555 | 29.04 | 1.705 7929 | 28.95 | 1.716 1998 | 28.87 |
| 16 | .685 0574 | 29.14 | .695 5298 | 29.04 | .705 9666 | 28.95 | .716 3730 | 28.87 |
| 17 | .685 2323 | 29.14 | .695 7040 | 29.04 | .706 1402 | 28.95 | .716 5462 | 28.87 |
| 18 | .685 4071 | 29.14 | .695 8782 | 29.03 | .706 3139 | 28.94 | .716 7194 | 28.87 |
| 19 | .685 5820 | 29.14 | .696 0524 | 29.03 | .706 4875 | 28.94 | .716 8926 | 28.87 |
| 20 | 1.685 7568 | 29.14 | 1.696 2266 | 29.03 | 1.706 6612 | 28.94 | 1.717 0658 | 28.86 |
| 21 | .685 9316 | 29.13 | .696 4008 | 29.03 | .706 8348 | 28.94 | .717 2390 | 28.86 |
| 22 | .686 1064 | 29.13 | .696 5750 | 29.03 | .707 0085 | 28.94 | .717 4122 | 28.86 |
| 23 | .686 2812 | 29.13 | .696 7491 | 29.03 | .707 1821 | 28.94 | .717 5853 | 28.86 |
| 24 | .686 4560 | 29.13 | .696 9233 | 29.02 | .707 3557 | 28.94 | .717 7585 | 28.86 |
| 25 | 1.686 6308 | 29.13 | 1.697 0974 | 29.02 | 1.707 5293 | 28.93 | 1.717 9317 | 28.86 |
| 26 | .686 8055 | 29.13 | .697 2716 | 29.02 | .707 7029 | 28.93 | .718 1048 | 28.86 |
| 27 | .686 9803 | 29.12 | .697 4457 | 29.02 | .707 8765 | 28.93 | .718 2780 | 28.86 |
| 28 | .687 1550 | 29.12 | .697 6198 | 29.02 | .708 0501 | 28.93 | .718 4511 | 28.86 |
| 29 | .687 3297 | 29.12 | .697 7939 | 29.02 | .708 2237 | 28.93 | .718 6242 | 28.85 |
| 30 | 1.687 5044 | 29.12 | 1.697 9680 | 29.02 | 1.708 3972 | 28.93 | 1.718 7974 | 28.85 |
| 31 | .687 6791 | 29.12 | .698 1421 | 29.01 | .708 5708 | 28.93 | .718 9705 | 28.85 |
| 32 | .687 8538 | 29.11 | .698 3162 | 29.01 | .708 7444 | 28.92 | .719 1436 | 28.85 |
| 33 | .688 0285 | 29.11 | .698 4902 | 29.01 | .708 9179 | 28.92 | .719 3167 | 28.85 |
| 34 | .688 2032 | 29.11 | .698 6643 | 29.01 | .709 0914 | 28.92 | .719 4898 | 28.85 |
| 35 | 1.688 3778 | 29.11 | 1.698 8383 | 29.01 | 1.709 2650 | 28.92 | 1.719 6629 | 28.85 |
| 36 | .688 5525 | 29.11 | .699 0124 | 29.01 | .709 4385 | 28.92 | .719 8360 | 28.85 |
| 37 | .688 7271 | 29.10 | .699 1864 | 29.00 | .709 6120 | 28.92 | .720 0090 | 28.85 |
| 38 | .688 9017 | 29.10 | .699 3604 | 29.00 | .709 7855 | 28.92 | .720 1821 | 28.84 |
| 39 | .689 0764 | 29.10 | .699 5345 | 29.00 | .709 9590 | 28.92 | .720 3552 | 28.84 |
| 40 | 1.689 2510 | 29.10 | 1.699 7085 | 29.00 | 1.710 1325 | 28.91 | 1.720 5282 | 28.84 |
| 41 | .689 4256 | 29.10 | .699 8824 | 29.00 | .710 3060 | 28.91 | .720 7013 | 28.84 |
| 42 | .689 6001 | 29.09 | .700 0564 | 29.00 | .710 4794 | 28.91 | .720 8743 | 28.84 |
| 43 | .689 7747 | 29.09 | .700 2304 | 29.00 | .710 6529 | 28.91 | .721 0474 | 28.84 |
| 44 | .689 9493 | 29.09 | .700 4044 | 28.99 | .710 8263 | 28.91 | .721 2204 | 28.84 |
| 45 | 1.690 1238 | 29.09 | 1.700 5783 | 28.99 | 1.710 9998 | 28.91 | 1.721 3934 | 28.84 |
| 46 | .690 2984 | 29.09 | .700 7523 | 28.99 | .711 1732 | 28.91 | .721 5665 | 28.84 |
| 47 | .690 4729 | 29.09 | .700 9262 | 28.99 | .711 3467 | 28.90 | .721 7395 | 28.84 |
| 48 | .690 6474 | 29.09 | .701 1001 | 28.99 | .711 5201 | 28.90 | .721 9125 | 28.83 |
| 49 | .690 8219 | 29.08 | .701 2741 | 28.99 | .711 6935 | 28.90 | .722 0855 | 28.83 |
| 50 | 1.690 9964 | 29.08 | 1.701 4480 | 28.98 | 1.711 8669 | 28.90 | 1.722 2585 | 28.83 |
| 51 | .691 1709 | 29.08 | .701 6219 | 28.98 | .712 0403 | 28.90 | .722 4315 | 28.83 |
| 52 | .691 3454 | 29.08 | .701 7958 | 28.98 | .712 2137 | 28.90 | .722 6044 | 28.83 |
| 53 | .691 5199 | 29.08 | .701 9697 | 28.98 | .712 3871 | 28.90 | .722 7774 | 28.83 |
| 54 | .691 6943 | 29.08 | .702 1435 | 28.98 | .712 5605 | 28.90 | .722 9504 | 28.83 |
| 55 | 1.691 8688 | 29.07 | 1.702 3174 | 28.98 | 1.712 7339 | 28.90 | 1.723 1233 | 28.83 |
| 56 | .692 0432 | 29.07 | .702 4913 | 28.98 | .712 9072 | 28.89 | .723 2963 | 28.83 |
| 57 | .692 2176 | 29.07 | .702 6651 | 28.97 | .713 0806 | 28.89 | .723 4693 | 28.82 |
| 58 | .692 3920 | 29.07 | .702 8389 | 28.97 | .713 2539 | 28.89 | .723 6422 | 28.82 |
| 59 | .692 5664 | 29.07 | .703 0128 | 28.97 | .713 4273 | 28.89 | .723 8151 | 28.82 |
| 60 | 1.692 7408 | 29.07 | 1.703 1866 | 28.97 | 1.713 6006 | 28.89 | 1.723 9881 | 28.82 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| n. | 64° | | 65° | | 66° | | 67° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.723 9881 | 28.82 | 1.734 3539 | 28.77 | 1.744 7031 | 28.73 | 1.755 0405 | 28.70 |
| 1 | .724 1610 | 28.82 | .734 5265 | 28.77 | .744 8755 | 28.73 | .755 2127 | 28.70 |
| 2 | .724 3339 | 28.82 | .734 6991 | 28.77 | .745 0479 | 28.73 | .755 3849 | 28.70 |
| 3 | .724 5068 | 28.82 | .734 8718 | 28.77 | .745 2202 | 28.73 | .755 5571 | 28.70 |
| 4 | .724 6798 | 28.82 | .735 0444 | 28.77 | .745 3926 | 28.73 | .755 7293 | 28.70 |
| 5 | 1.724 8527 | 28.82 | 1.735 2169 | 28.76 | 1.745 5650 | 28.73 | 1.755 9015 | 28.70 |
| 6 | .725 0256 | 28.82 | .735 3895 | 28.76 | .745 7373 | 28.73 | .756 0737 | 28.70 |
| 7 | .725 1984 | 28.81 | .735 5621 | 28.76 | .745 9097 | 28.73 | .756 2459 | 28.70 |
| 8 | .725 3713 | 28.81 | .735 7347 | 28.76 | .746 0820 | 28.72 | .756 4181 | 28.70 |
| 9 | .725 5442 | 28.81 | .735 9073 | 28.76 | .746 2544 | 28.72 | .756 5903 | 28.70 |
| 10 | 1.725 7171 | 28.81 | 1.736 0798 | 28.76 | 1.746 4267 | 28.72 | 1.756 7625 | 28.70 |
| 11 | .725 8900 | 28.81 | .736 2524 | 28.76 | .746 5991 | 28.72 | .756 9347 | 28.70 |
| 12 | .726 0628 | 28.81 | .736 4250 | 28.76 | .746 7714 | 28.72 | .757 1069 | 28.70 |
| 13 | .726 2357 | 28.81 | .736 5975 | 28.76 | .746 9437 | 28.72 | .757 2791 | 28.70 |
| 14 | .726 4085 | 28.81 | .736 7701 | 28.76 | .747 1161 | 28.72 | .757 4513 | 28.70 |
| 15 | 1.726 5814 | 28.81 | 1.736 9426 | 28.76 | 1.747 2884 | 28.72 | 1.757 6235 | 28.70 |
| 16 | .726 7542 | 28.81 | .737 1152 | 28.76 | .747 4607 | 28.72 | .757 7957 | 28.70 |
| 17 | .726 9270 | 28.81 | .737 2877 | 28.76 | .747 6330 | 28.72 | .757 9679 | 28.70 |
| 18 | .727 0999 | 28.80 | .737 4602 | 28.76 | .747 8054 | 28.72 | .758 1401 | 28.70 |
| 19 | .727 2727 | 28.80 | .737 6328 | 28.75 | .747 9777 | 28.72 | .758 3123 | 28.70 |
| 20 | 1.727 4455 | 28.80 | 1.737 8053 | 28.75 | 1.748 1500 | 28.72 | 1.758 4844 | 28.70 |
| 21 | .727 6183 | 28.80 | .737 9778 | 28.75 | .748 3223 | 28.72 | .758 6566 | 28.70 |
| 22 | .727 7911 | 28.80 | .738 1503 | 28.75 | .748 4946 | 28.72 | .758 8288 | 28.70 |
| 23 | .727 9639 | 28.80 | .738 3228 | 28.75 | .748 6669 | 28.72 | .759 0010 | 28.70 |
| 24 | .728 1367 | 28.80 | .738 4953 | 28.75 | .748 8392 | 28.72 | .759 1731 | 28.70 |
| 25 | 1.728 3095 | 28.80 | 1.738 6679 | 28.75 | 1.749 0115 | 28.72 | 1.759 3453 | 28.70 |
| 26 | .728 4823 | 28.80 | .738 8404 | 28.75 | .749 1838 | 28.72 | .759 5175 | 28.70 |
| 27 | .728 6551 | 28.80 | .739 0129 | 28.75 | .749 3561 | 28.72 | .759 6897 | 28.70 |
| 28 | .728 8279 | 28.80 | .739 1853 | 28.75 | .749 5284 | 28.72 | .759 8618 | 28.69 |
| 29 | .729 0006 | 28.79 | .739 3578 | 28.75 | .749 7007 | 28.71 | .760 0340 | 28.69 |
| 30 | 1.729 1734 | 28.79 | 1.739 5303 | 28.75 | 1.749 8730 | 28.71 | 1.760 2062 | 28.69 |
| 31 | .729 3461 | 28.79 | .739 7028 | 28.75 | .750 0453 | 28.71 | .760 3783 | 28.69 |
| 32 | .729 5189 | 28.79 | .739 8753 | 28.75 | .750 2176 | 28.71 | .760 5505 | 28.69 |
| 33 | .729 6916 | 28.79 | .740 0477 | 28.75 | .750 3898 | 28.71 | .760 7227 | 28.69 |
| 34 | .729 8644 | 28.79 | .740 2202 | 28.74 | .750 5621 | 28.71 | .760 8948 | 28.69 |
| 35 | 1.730 0371 | 28.79 | 1.740 3927 | 28.74 | 1.750 7344 | 28.71 | 1.761 0670 | 28.69 |
| 36 | .730 2099 | 28.79 | .740 5651 | 28.74 | .750 9067 | 28.71 | .761 2392 | 28.69 |
| 37 | .730 3826 | 28.79 | .740 7376 | 28.74 | .751 0789 | 28.71 | .761 4113 | 28.69 |
| 38 | .730 5553 | 28.79 | .740 9101 | 28.74 | .751 2512 | 28.71 | .761 5835 | 28.69 |
| 39 | .730 7280 | 28.79 | .741 0825 | 28.74 | .751 4234 | 28.71 | .761 7556 | 28.69 |
| 40 | 1.730 9007 | 28.78 | 1.741 2550 | 28.74 | 1.751 5957 | 28.71 | 1.761 9278 | 28.69 |
| 41 | .731 0735 | 28.78 | .741 4274 | 28.74 | .751 7680 | 28.71 | .762 0999 | 28.69 |
| 42 | .731 2462 | 28.78 | .741 5998 | 28.74 | .751 9402 | 28.71 | .762 2721 | 28.69 |
| 43 | .731 4189 | 28.78 | .741 7723 | 28.74 | .752 1125 | 28.71 | .762 4442 | 28.69 |
| 44 | .731 5915 | 28.78 | .741 9447 | 28.74 | .752 2847 | 28.71 | .762 6164 | 28.69 |
| 45 | 1.731 7642 | 28.78 | 1.742 1171 | 28.74 | 1.752 4570 | 28.71 | 1.762 7885 | 28.69 |
| 46 | .731 9369 | 28.78 | .742 2896 | 28.74 | .752 6292 | 28.71 | .762 9607 | 28.69 |
| 47 | .732 1096 | 28.78 | .742 4620 | 28.74 | .752 8015 | 28.71 | .763 1328 | 28.69 |
| 48 | .732 2823 | 28.78 | .742 6344 | 28.74 | .752 9737 | 28.71 | .763 3050 | 28.69 |
| 49 | .732 4549 | 28.78 | .742 8068 | 28.74 | .753 1460 | 28.71 | .763 4771 | 28.69 |
| 50 | 1.732 6276 | 28.78 | 1.742 9792 | 28.74 | 1.753 3182 | 28.71 | 1.763 6493 | 28.69 |
| 51 | .732 8002 | 28.78 | .743 1516 | 28.73 | .753 4904 | 28.71 | .763 8214 | 28.69 |
| 52 | .732 9729 | 28.77 | .743 3240 | 28.73 | .753 6627 | 28.71 | .763 9936 | 28.69 |
| 53 | .733 1455 | 28.77 | .743 4964 | 28.73 | .753 8349 | 28.71 | .764 1657 | 28.69 |
| 54 | .733 3182 | 28.77 | .743 6688 | 28.73 | .754 0071 | 28.70 | .764 3379 | 28.69 |
| 55 | 1.733 4908 | 28.77 | 1.743 8412 | 28.73 | 1.754 1794 | 28.70 | 1.764 5100 | 28.69 |
| 56 | .733 6635 | 28.77 | .744 0136 | 28.73 | .754 3516 | 28.70 | .764 6821 | 28.69 |
| 57 | .733 8361 | 28.77 | .744 1860 | 28.73 | .754 5238 | 28.70 | .764 8543 | 28.69 |
| 58 | .734 0087 | 28.77 | .744 3584 | 28.73 | .754 6960 | 28.70 | .765 0264 | 28.69 |
| 59 | .734 1813 | 28.77 | .744 5308 | 28.73 | .754 8682 | 28.70 | .765 1985 | 28.69 |
| 60 | 1.734 3539 | 28.77 | 1.744 7031 | 28.73 | 1.755 0405 | 28.70 | 1.765 3707 | 28.69 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 68° | | 69° | | 70° | | 71° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.765 3707 | 28.69 | 1.775 6985 | 28.69 | 1.786 0284 | 28.70 | 1.796 3650 | 28.73 |
| 1 | .765 5428 | 28.69 | .775 8706 | 28.69 | .786 2006 | 28.70 | .796 5374 | 28.73 |
| 2 | .765 7150 | 28.69 | .776 0427 | 28.69 | .786 3728 | 28.70 | .796 7097 | 28.73 |
| 3 | .765 8871 | 28.69 | .776 2149 | 28.69 | .786 5450 | 28.70 | .796 8821 | 28.73 |
| 4 | .766 0592 | 28.69 | .776 3870 | 28.69 | .786 7172 | 28.70 | .797 0545 | 28.73 |
| 5 | 1.766 2314 | 28.69 | 1.776 5591 | 28.69 | 1.786 8894 | 28.70 | 1.797 2268 | 28.73 |
| 6 | .766 4035 | 28.69 | .776 7313 | 28.69 | .787 0617 | 28.70 | .797 3992 | 28.73 |
| 7 | .766 5756 | 28.69 | .776 9034 | 28.69 | .787 2339 | 28.70 | .797 5716 | 28.73 |
| 8 | .766 7478 | 28.69 | .777 0755 | 28.69 | .787 4061 | 28.70 | .797 7440 | 28.73 |
| 9 | .766 9199 | 28.69 | .777 2477 | 28.69 | .787 5783 | 28.70 | .797 9164 | 28.73 |
| 10 | 1.767 0920 | 28.69 | 1.777 4198 | 28.69 | 1.787 7506 | 28.70 | 1.798 0888 | 28.73 |
| 11 | .767 2642 | 28.69 | .777 5920 | 28.69 | .787 9228 | 28.71 | .798 2611 | 28.73 |
| 12 | .767 4363 | 28.69 | .777 7641 | 28.69 | .788 0950 | 28.71 | .798 4335 | 28.73 |
| 13 | .767 6084 | 28.69 | .777 9363 | 28.69 | .788 2673 | 28.71 | .798 6060 | 28.73 |
| 14 | .767 7805 | 28.69 | .778 1084 | 28.69 | .788 4395 | 28.71 | .798 7784 | 28.73 |
| 15 | 1.767 9527 | 28.69 | 1.778 2806 | 28.69 | 1.788 6117 | 28.71 | 1.798 9508 | 28.73 |
| 16 | .768 1248 | 28.69 | .778 4527 | 28.69 | .788 7840 | 28.71 | .799 1232 | 28.74 |
| 17 | .768 2969 | 28.69 | .778 6248 | 28.69 | .788 9562 | 28.71 | .799 2956 | 28.74 |
| 18 | .768 4691 | 28.69 | .778 7970 | 28.69 | .789 1284 | 28.71 | .799 4680 | 28.74 |
| 19 | .768 6412 | 28.69 | .778 9691 | 28.69 | .789 3007 | 28.71 | .799 6404 | 28.74 |
| 20 | 1.768 8133 | 28.69 | 1.779 1413 | 28.69 | 1.789 4730 | 28.71 | 1.799 8128 | 28.74 |
| 21 | .768 9854 | 28.69 | .779 3140 | 28.69 | .789 6452 | 28.71 | .799 9853 | 28.74 |
| 22 | .769 1576 | 28.69 | .779 4862 | 28.69 | .789 8175 | 28.71 | .800 1577 | 28.74 |
| 23 | .769 3297 | 28.69 | .779 6578 | 28.69 | .789 9897 | 28.71 | .800 3301 | 28.74 |
| 24 | .769 5018 | 28.69 | .779 8299 | 28.69 | .790 1620 | 28.71 | .800 5026 | 28.74 |
| 25 | 1.769 6740 | 28.69 | 1.780 0021 | 28.69 | 1.790 3342 | 28.71 | 1.800 6750 | 28.74 |
| 26 | .769 8461 | 28.69 | .780 1742 | 28.69 | .790 5065 | 28.71 | .800 8475 | 28.74 |
| 27 | .770 0182 | 28.69 | .780 3464 | 28.69 | .790 6788 | 28.71 | .801 0199 | 28.74 |
| 28 | .770 1903 | 28.69 | .780 5185 | 28.69 | .790 8510 | 28.71 | .801 1924 | 28.74 |
| 29 | .770 3625 | 28.69 | .780 6907 | 28.69 | .791 0233 | 28.71 | .801 3648 | 28.74 |
| 30 | 1.770 5346 | 28.69 | 1.780 8629 | 28.69 | 1.791 1956 | 28.71 | 1.801 5373 | 28.74 |
| 31 | .770 7067 | 28.69 | .781 0350 | 28.69 | .791 3678 | 28.71 | .801 7107 | 28.74 |
| 32 | .770 8788 | 28.69 | .781 2072 | 28.69 | .791 5401 | 28.71 | .801 8822 | 28.74 |
| 33 | .771 0510 | 28.69 | .781 3793 | 28.69 | .791 7124 | 28.71 | .802 0547 | 28.75 |
| 34 | .771 2231 | 28.69 | .781 5515 | 28.69 | .791 8847 | 28.71 | .802 2271 | 28.75 |
| 35 | 1.771 3952 | 28.69 | 1.781 7237 | 28.69 | 1.792 0570 | 28.71 | 1.802 3996 | 28.75 |
| 36 | .771 5673 | 28.69 | .781 8959 | 28.69 | .792 2293 | 28.71 | .802 5721 | 28.75 |
| 37 | .771 7395 | 28.69 | .782 0680 | 28.70 | .792 4016 | 28.72 | .802 7446 | 28.75 |
| 38 | .771 9116 | 28.69 | .782 2402 | 28.70 | .792 5738 | 28.72 | .802 9171 | 28.75 |
| 39 | .772 0837 | 28.69 | .782 4124 | 28.70 | .792 7461 | 28.72 | .803 0896 | 28.75 |
| 40 | 1.772 2559 | 28.69 | 1.782 5845 | 28.70 | 1.792 9184 | 28.72 | 1.803 2621 | 28.75 |
| 41 | .772 4280 | 28.69 | .782 7567 | 28.70 | .793 0907 | 28.72 | .803 4346 | 28.75 |
| 42 | .772 6001 | 28.69 | .782 9289 | 28.70 | .793 2630 | 28.72 | .803 6071 | 28.75 |
| 43 | .772 7722 | 28.69 | .783 1011 | 28.70 | .793 4354 | 28.72 | .803 7796 | 28.75 |
| 44 | .772 9444 | 28.69 | .783 2732 | 28.70 | .793 6077 | 28.72 | .803 9521 | 28.75 |
| 45 | 1.773 1165 | 28.69 | 1.783 4454 | 28.70 | 1.793 7800 | 28.72 | 1.804 1246 | 28.75 |
| 46 | .773 2886 | 28.69 | .783 6176 | 28.70 | .793 9523 | 28.72 | .804 2971 | 28.75 |
| 47 | .773 4607 | 28.69 | .783 7898 | 28.70 | .794 1246 | 28.72 | .804 4697 | 28.75 |
| 48 | .773 6329 | 28.69 | .783 9620 | 28.70 | .794 2969 | 28.72 | .804 6422 | 28.76 |
| 49 | .773 8050 | 28.69 | .784 1342 | 28.70 | .794 4693 | 28.72 | .804 8147 | 28.76 |
| 50 | 1.773 9771 | 28.69 | 1.784 3064 | 28.70 | 1.794 6416 | 28.72 | 1.804 9873 | 28.76 |
| 51 | .774 1493 | 28.69 | .784 4786 | 28.70 | .794 8139 | 28.72 | .805 1598 | 28.76 |
| 52 | .774 3214 | 28.69 | .784 6508 | 28.70 | .794 9862 | 28.72 | .805 3324 | 28.76 |
| 53 | .774 4935 | 28.69 | .784 8230 | 28.70 | .795 1586 | 28.72 | .805 5049 | 28.76 |
| 54 | .774 6657 | 28.69 | .784 9952 | 28.70 | .795 3309 | 28.72 | .805 6775 | 28.76 |
| 55 | 1.774 8378 | 28.69 | 1.785 1674 | 28.70 | 1.795 5033 | 28.72 | 1.805 8500 | 28.76 |
| 56 | .775 0099 | 28.69 | .785 3396 | 28.70 | .795 6756 | 28.72 | .806 0226 | 28.76 |
| 57 | .775 1821 | 28.69 | .785 5118 | 28.70 | .795 8480 | 28.72 | .806 1952 | 28.76 |
| 58 | .775 3542 | 28.69 | .785 6840 | 28.70 | .796 0203 | 28.73 | .806 3677 | 28.76 |
| 59 | .775 5263 | 28.69 | .785 8562 | 28.70 | .796 1927 | 28.73 | .806 5403 | 28.76 |
| 60 | 1.775 6985 | 28.69 | 1.786 0284 | 28.70 | 1.796 3650 | 28.73 | 1.806 7129 | 28.76 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 72° | | 73° | | 74° | | 75° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.806 7129 | 28.76 | 1.817 0765 | 28.81 | 1.827 4602 | 28.88 | 1.837 8686 | 28.95 |
| 1 | .806 8855 | 28.76 | .817 2494 | 28.81 | .827 6335 | 28.88 | .838 0423 | 28.95 |
| 2 | .807 0581 | 28.77 | .817 4222 | 28.82 | .827 8068 | 28.88 | .838 2160 | 28.95 |
| 3 | .807 2307 | 28.77 | .817 5951 | 28.82 | .827 9800 | 28.88 | .838 3898 | 28.95 |
| 4 | .807 4033 | 28.77 | .817 7680 | 28.82 | .828 1533 | 28.88 | .838 5635 | 28.96 |
| 5 | 1.807 5759 | 28.77 | 1.817 9410 | 28.82 | 1.828 3266 | 28.88 | 1.838 7372 | 28.96 |
| 6 | .807 7485 | 28.77 | .818 1139 | 28.82 | .828 4999 | 28.88 | .838 9110 | 28.96 |
| 7 | .807 9211 | 28.77 | .818 2868 | 28.82 | .828 6732 | 28.88 | .839 0847 | 28.96 |
| 8 | .808 0937 | 28.77 | .818 4597 | 28.82 | .828 8465 | 28.88 | .839 2585 | 28.96 |
| 9 | .808 2663 | 28.77 | .818 6326 | 28.82 | .829 0198 | 28.89 | .839 4323 | 28.96 |
| 10 | 1.808 4389 | 28.77 | 1.818 8056 | 28.82 | 1.829 1931 | 28.89 | 1.839 6060 | 28.96 |
| 11 | .808 6116 | 28.77 | .818 9785 | 28.82 | .829 3665 | 28.89 | .839 7798 | 28.97 |
| 12 | .808 7842 | 28.77 | .819 1515 | 28.83 | .829 5398 | 28.89 | .839 9536 | 28.97 |
| 13 | .808 9568 | 28.77 | .819 3244 | 28.83 | .829 7131 | 28.89 | .840 1274 | 28.97 |
| 14 | .809 1295 | 28.77 | .819 4974 | 28.83 | .829 8865 | 28.89 | .840 3012 | 28.97 |
| 15 | 1.809 3021 | 28.78 | 1.819 6704 | 28.83 | 1.830 0599 | 28.89 | 1.840 4751 | 28.97 |
| 16 | .809 4748 | 28.78 | .819 8433 | 28.83 | .830 2332 | 28.89 | .840 6489 | 28.97 |
| 17 | .809 6474 | 28.78 | .820 0163 | 28.83 | .830 4066 | 28.90 | .840 8227 | 28.97 |
| 18 | .809 8201 | 28.78 | .820 1893 | 28.83 | .830 5800 | 28.90 | .840 9966 | 28.97 |
| 19 | .809 9928 | 28.78 | .820 3623 | 28.83 | .830 7533 | 28.90 | .841 1704 | 28.98 |
| 20 | 1.810 1655 | 28.78 | 1.820 5353 | 28.83 | 1.830 9267 | 28.90 | 1.841 3443 | 28.98 |
| 21 | .810 3381 | 28.78 | .820 7083 | 28.83 | .831 1001 | 28.90 | .841 5182 | 28.98 |
| 22 | .810 5108 | 28.78 | .820 8813 | 28.84 | .831 2735 | 28.90 | .841 6921 | 28.98 |
| 23 | .810 6835 | 28.78 | .821 0543 | 28.84 | .831 4470 | 28.90 | .841 8659 | 28.98 |
| 24 | .810 8562 | 28.78 | .821 2273 | 28.84 | .831 6204 | 28.90 | .842 0398 | 28.98 |
| 25 | 1.811 0289 | 28.78 | 1.821 4003 | 28.84 | 1.831 7938 | 28.91 | 1.842 2138 | 28.98 |
| 26 | .811 2016 | 28.78 | .821 5734 | 28.84 | .831 9672 | 28.91 | .842 3877 | 28.99 |
| 27 | .811 3743 | 28.78 | .821 7464 | 28.84 | .832 1407 | 28.91 | .842 5616 | 28.99 |
| 28 | .811 5470 | 28.79 | .821 9194 | 28.84 | .832 3141 | 28.91 | .842 7355 | 28.99 |
| 29 | .811 7197 | 28.79 | .822 0925 | 28.84 | .832 4876 | 28.91 | .842 9095 | 28.99 |
| 30 | 1.811 8924 | 28.79 | 1.822 2656 | 28.84 | 1.832 6611 | 28.91 | 1.843 0834 | 28.99 |
| 31 | .812 0652 | 28.79 | .822 4386 | 28.84 | .832 8345 | 28.91 | .843 2574 | 28.99 |
| 32 | .812 2379 | 28.79 | .822 6117 | 28.85 | .833 0080 | 28.92 | .843 4313 | 29.00 |
| 33 | .812 4106 | 28.79 | .822 7848 | 28.85 | .833 1815 | 28.92 | .843 6053 | 29.00 |
| 34 | .812 5834 | 28.79 | .822 9578 | 28.85 | .833 3550 | 28.92 | .843 7793 | 29.00 |
| 35 | 1.812 7561 | 28.79 | 1.823 1309 | 28.85 | 1.833 5285 | 28.92 | 1.843 9533 | 29.00 |
| 36 | .812 9289 | 28.79 | .823 3040 | 28.85 | .833 7020 | 28.92 | .844 1273 | 29.00 |
| 37 | .813 1016 | 28.79 | .823 4771 | 28.85 | .833 8755 | 28.92 | .844 3013 | 29.00 |
| 38 | .813 2744 | 28.79 | .823 6502 | 28.85 | .834 0491 | 28.92 | .844 4753 | 29.00 |
| 39 | .813 4472 | 28.79 | .823 8233 | 28.85 | .834 2226 | 28.92 | .844 6494 | 29.01 |
| 40 | 1.813 6199 | 28.80 | 1.823 9965 | 28.85 | 1.834 3961 | 28.92 | 1.844 8234 | 29.01 |
| 41 | .813 7927 | 28.80 | .824 1696 | 28.85 | .834 5697 | 28.93 | .844 9974 | 29.01 |
| 42 | .813 9655 | 28.80 | .824 3427 | 28.86 | .834 7432 | 28.93 | .845 1715 | 29.01 |
| 43 | .814 1383 | 28.80 | .824 5159 | 28.86 | .834 9168 | 28.93 | .845 3456 | 29.01 |
| 44 | .814 3111 | 28.80 | .824 6890 | 28.86 | .835 0904 | 28.93 | .845 5196 | 29.01 |
| 45 | 1.814 4839 | 28.80 | 1.824 8622 | 28.86 | 1.835 2640 | 28.93 | 1.845 6937 | 29.01 |
| 46 | .814 6567 | 28.80 | .825 0353 | 28.86 | .835 4376 | 28.93 | .845 8678 | 29.02 |
| 47 | .814 8295 | 28.80 | .825 2085 | 28.86 | .835 6112 | 28.93 | .846 0419 | 29.02 |
| 48 | .815 0023 | 28.80 | .825 3816 | 28.86 | .835 7848 | 28.93 | .846 2160 | 29.02 |
| 49 | .815 1751 | 28.80 | .825 5548 | 28.86 | .835 9584 | 28.94 | .846 3901 | 29.02 |
| 50 | 1.815 3479 | 28.80 | 1.825 7280 | 28.86 | 1.836 1320 | 28.94 | 1.846 5643 | 29.02 |
| 51 | .815 5208 | 28.81 | .825 9012 | 28.87 | .836 3056 | 28.94 | .846 7384 | 29.02 |
| 52 | .815 6936 | 28.81 | .826 0744 | 28.87 | .836 4792 | 28.94 | .846 9125 | 29.03 |
| 53 | .815 8664 | 28.81 | .826 2476 | 28.87 | .836 6529 | 28.94 | .847 0867 | 29.03 |
| 54 | .816 0393 | 28.81 | .826 4208 | 28.87 | .836 8265 | 28.94 | .847 2609 | 29.03 |
| 55 | 1.816 2121 | 28.81 | 1.826 5940 | 28.87 | 1.837 0002 | 28.94 | 1.847 4350 | 29.03 |
| 56 | .816 3850 | 28.81 | .826 7673 | 28.87 | .837 1739 | 28.95 | .847 6092 | 29.03 |
| 57 | .816 5578 | 28.81 | .826 9405 | 28.87 | .837 3475 | 28.95 | .847 7834 | 29.03 |
| 58 | .816 7307 | 28.81 | .827 1137 | 28.87 | .837 5212 | 28.95 | .847 9576 | 29.03 |
| 59 | .816 9036 | 28.81 | .827 2870 | 28.87 | .837 6949 | 28.95 | .848 1318 | 29.04 |
| 60 | 1.817 0765 | 28.81 | 1.827 4602 | 28.88 | 1.837 8686 | 28.95 | 1.848 3060 | 29.04 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 76° | | 77° | | 78° | | 79° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.848 3060 | 29.04 | 1.858 7769 | 29.14 | 1.869 2857 | 29.25 | 1.879 8369 | 29.37 |
| 1 | .848 4803 | 29.04 | .858 9517 | 29.14 | .869 4612 | 29.25 | .880 0131 | 29.37 |
| 2 | .848 6545 | 29.04 | .859 1266 | 29.14 | .869 6367 | 29.25 | .880 1894 | 29.38 |
| 3 | .848 8287 | 29.04 | .859 3014 | 29.14 | .869 8122 | 29.25 | .880 3656 | 29.38 |
| 4 | .849 0030 | 29.04 | .859 4763 | 29.15 | .869 9878 | 29.26 | .880 5419 | 29.38 |
| 5 | 1.849 1773 | 29.04 | 1.859 6512 | 29.15 | 1.870 1633 | 29.26 | 1.880 7182 | 29.38 |
| 6 | .849 3515 | 29.05 | .859 8260 | 29.15 | .870 3389 | 29.26 | .880 8945 | 29.38 |
| 7 | .849 5258 | 29.05 | .860 0009 | 29.15 | .870 5144 | 29.26 | .881 0708 | 29.39 |
| 8 | .849 7001 | 29.05 | .860 1758 | 29.15 | .870 6900 | 29.26 | .881 2471 | 29.39 |
| 9 | .849 8744 | 29.05 | .860 3507 | 29.15 | .870 8656 | 29.26 | .881 4235 | 29.39 |
| 10 | 1.850 0487 | 29.05 | 1.860 5256 | 29.15 | 1.871 0412 | 29.27 | 1.881 5998 | 29.39 |
| 11 | .850 2231 | 29.05 | .860 7006 | 29.16 | .871 2168 | 29.27 | .881 7762 | 29.39 |
| 12 | .850 3974 | 29.06 | .860 8755 | 29.16 | .871 3924 | 29.27 | .881 9526 | 29.40 |
| 13 | .850 5717 | 29.06 | .861 0505 | 29.16 | .871 5681 | 29.27 | .882 1290 | 29.40 |
| 14 | .850 7461 | 29.06 | .861 2254 | 29.16 | .871 7437 | 29.28 | .882 3054 | 29.40 |
| 15 | 1.850 9204 | 29.06 | 1.861 4004 | 29.16 | 1.871 9194 | 29.28 | 1.882 4818 | 29.40 |
| 16 | .851 0948 | 29.06 | .861 5754 | 29.16 | .872 0950 | 29.28 | .882 6582 | 29.41 |
| 17 | .851 2692 | 29.06 | .861 7504 | 29.17 | .872 2707 | 29.28 | .882 8347 | 29.41 |
| 18 | .851 4436 | 29.07 | .861 9254 | 29.17 | .872 4464 | 29.28 | .883 0112 | 29.41 |
| 19 | .851 6180 | 29.07 | .862 1004 | 29.17 | .872 6221 | 29.29 | .883 1876 | 29.41 |
| 20 | 1.851 7924 | 29.07 | 1.862 2754 | 29.17 | 1.872 7979 | 29.29 | 1.883 3641 | 29.42 |
| 21 | .851 9668 | 29.07 | .862 4505 | 29.17 | .872 9736 | 29.29 | .883 5406 | 29.42 |
| 22 | .852 1412 | 29.07 | .862 6255 | 29.18 | .873 1493 | 29.29 | .883 7171 | 29.42 |
| 23 | .852 3157 | 29.07 | .862 8006 | 29.18 | .873 3251 | 29.29 | .883 8937 | 29.42 |
| 24 | .852 4901 | 29.07 | .862 9756 | 29.18 | .873 5008 | 29.30 | .884 0702 | 29.42 |
| 25 | 1.852 6646 | 29.08 | 1.863 1507 | 29.18 | 1.873 6766 | 29.30 | 1.884 2468 | 29.43 |
| 26 | .852 8391 | 29.08 | .863 3258 | 29.18 | .873 8524 | 29.30 | .884 4233 | 29.43 |
| 27 | .853 0135 | 29.08 | .863 5009 | 29.18 | .874 0282 | 29.30 | .884 5999 | 29.43 |
| 28 | .853 1880 | 29.08 | .863 6760 | 29.19 | .874 2041 | 29.30 | .884 7765 | 29.43 |
| 29 | .853 3625 | 29.08 | .863 8512 | 29.19 | .874 3799 | 29.31 | .884 9531 | 29.44 |
| 30 | 1.853 5370 | 29.09 | 1.864 0263 | 29.19 | 1.874 5557 | 29.31 | 1.885 1297 | 29.44 |
| 31 | .853 7115 | 29.09 | .864 2015 | 29.19 | .874 7316 | 29.31 | .885 3064 | 29.44 |
| 32 | .853 8861 | 29.09 | .864 3766 | 29.19 | .874 9074 | 29.31 | .885 4830 | 29.44 |
| 33 | .854 0606 | 29.09 | .864 5518 | 29.20 | .875 0833 | 29.31 | .885 6597 | 29.45 |
| 34 | .854 2351 | 29.09 | .864 7270 | 29.20 | .875 2592 | 29.32 | .885 8364 | 29.45 |
| 35 | 1.854 4097 | 29.09 | 1.864 9022 | 29.20 | 1.875 4351 | 29.32 | 1.886 0131 | 29.45 |
| 36 | .854 5843 | 29.10 | .865 0774 | 29.20 | .875 6111 | 29.32 | .886 1898 | 29.45 |
| 37 | .854 7588 | 29.10 | .865 2526 | 29.20 | .875 7870 | 29.32 | .886 3665 | 29.45 |
| 38 | .854 9334 | 29.10 | .865 4278 | 29.20 | .875 9629 | 29.32 | .886 5432 | 29.46 |
| 39 | .855 1080 | 29.10 | .865 6030 | 29.21 | .876 1389 | 29.33 | .886 7200 | 29.46 |
| 40 | 1.855 2826 | 29.10 | 1.865 7783 | 29.21 | 1.876 3148 | 29.33 | 1.886 8967 | 29.46 |
| 41 | .855 4572 | 29.10 | .865 9536 | 29.21 | .876 4908 | 29.33 | .887 0735 | 29.46 |
| 42 | .855 6319 | 29.11 | .866 1288 | 29.21 | .876 6668 | 29.33 | .887 2503 | 29.47 |
| 43 | .855 8065 | 29.11 | .866 3041 | 29.21 | .876 8428 | 29.33 | .887 4271 | 29.47 |
| 44 | .855 9811 | 29.11 | .866 4794 | 29.22 | .877 0188 | 29.34 | .887 6039 | 29.47 |
| 45 | 1.856 1558 | 29.11 | 1.866 6547 | 29.22 | 1.877 1949 | 29.34 | 1.887 7807 | 29.47 |
| 46 | .856 3305 | 29.11 | .866 8301 | 29.22 | .877 3709 | 29.34 | .887 9576 | 29.48 |
| 47 | .856 5052 | 29.11 | .867 0054 | 29.22 | .877 5470 | 29.34 | .888 1344 | 29.48 |
| 48 | .856 6799 | 29.12 | .867 1807 | 29.22 | .877 7230 | 29.34 | .888 3113 | 29.48 |
| 49 | .856 8546 | 29.12 | .867 3561 | 29.23 | .877 8991 | 29.35 | .888 4882 | 29.48 |
| 50 | 1.857 0293 | 29.12 | 1.867 5314 | 29.23 | 1.878 0752 | 29.35 | 1.888 6651 | 29.48 |
| 51 | .857 2040 | 29.12 | .867 7068 | 29.23 | .878 2513 | 29.35 | .888 8420 | 29.49 |
| 52 | .857 3787 | 29.12 | .867 8822 | 29.23 | .878 4275 | 29.35 | .889 0189 | 29.49 |
| 53 | .857 5534 | 29.12 | .868 0576 | 29.23 | .878 6036 | 29.35 | .889 1959 | 29.49 |
| 54 | .857 7282 | 29.13 | .868 2330 | 29.24 | .878 7797 | 29.36 | .889 3728 | 29.49 |
| 55 | 1.857 9030 | 29.13 | 1.868 4084 | 29.24 | 1.878 9559 | 29.36 | 1.889 5498 | 29.49 |
| 56 | .858 0777 | 29.13 | .868 5839 | 29.24 | .879 1321 | 29.36 | .889 7268 | 29.50 |
| 57 | .858 2525 | 29.13 | .868 7593 | 29.24 | .879 3082 | 29.36 | .889 9038 | 29.50 |
| 58 | .858 4273 | 29.13 | .868 9348 | 29.24 | .879 4844 | 29.36 | .890 0808 | 29.50 |
| 59 | .858 6021 | 29.13 | .869 1102 | 29.25 | .879 6606 | 29.37 | .890 2578 | 29.51 |
| 60 | 1.858 7769 | 29.14 | 1.869 2857 | 29.25 | 1.879 8369 | 29.37 | 1.890 4349 | 29.51 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 80° | | 81° | | 82° | | 83° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.890 4349 | 29.51 | 1.901 0841 | 29.66 | 1.911 7893 | 29.82 | 1.922 5548 | 29.99 |
| 1 | .890 6119 | 29.51 | .901 2621 | 29.66 | .911 9682 | 29.82 | .922 7347 | 29.99 |
| 2 | .890 7890 | 29.51 | .901 4400 | 29.66 | .912 1471 | 29.82 | .922 9147 | 30.00 |
| 3 | .890 9661 | 29.51 | .901 6180 | 29.66 | .912 3261 | 29.83 | .923 0947 | 30.00 |
| 4 | .891 1432 | 29.52 | .901 7960 | 29.67 | .912 5050 | 29.83 | .923 2747 | 30.00 |
| 5 | 1.891 3203 | 29.52 | 1.901 9740 | 29.67 | 1.912 6840 | 29.83 | 1.923 4548 | 30.01 |
| 6 | .891 4974 | 29.52 | .902 1521 | 29.67 | .912 8630 | 29.84 | .923 6348 | 30.01 |
| 7 | .891 6745 | 29.52 | .902 3301 | 29.67 | .913 0420 | 29.84 | .923 8149 | 30.01 |
| 8 | .891 8517 | 29.53 | .902 5082 | 29.68 | .913 2211 | 29.84 | .923 9950 | 30.02 |
| 9 | .892 0289 | 29.53 | .902 6862 | 29.68 | .913 4001 | 29.84 | .924 1751 | 30.02 |
| 10 | 1.892 2061 | 29.53 | 1.902 8643 | 29.68 | 1.913 5792 | 29.85 | 1.924 3552 | 30.02 |
| 11 | .892 3833 | 29.53 | .903 0424 | 29.69 | .913 7583 | 29.85 | .924 5354 | 30.03 |
| 12 | .892 5605 | 29.54 | .903 2105 | 29.69 | .913 9374 | 29.85 | .924 7155 | 30.03 |
| 13 | .892 7377 | 29.54 | .903 3987 | 29.69 | .914 1165 | 29.85 | .924 8957 | 30.03 |
| 14 | .892 9149 | 29.54 | .903 5768 | 29.69 | .914 2956 | 29.86 | .925 0759 | 30.03 |
| 15 | 1.893 0922 | 29.54 | 1.903 7550 | 29.70 | 1.914 4748 | 29.86 | 1.925 2561 | 30.04 |
| 16 | .893 2695 | 29.55 | .903 9332 | 29.70 | .914 6540 | 29.86 | .925 4364 | 30.04 |
| 17 | .893 4467 | 29.55 | .904 1114 | 29.70 | .914 8331 | 29.87 | .925 6166 | 30.04 |
| 18 | .893 6240 | 29.55 | .904 2896 | 29.70 | .915 0124 | 29.87 | .925 7969 | 30.05 |
| 19 | .893 8013 | 29.55 | .904 4678 | 29.71 | .915 1916 | 29.87 | .925 9772 | 30.05 |
| 20 | 1.893 9787 | 29.56 | 1.904 6461 | 29.71 | 1.915 3708 | 29.87 | 1.926 1575 | 30.05 |
| 21 | .894 1560 | 29.56 | .904 8243 | 29.71 | .915 5501 | 29.88 | .926 3378 | 30.06 |
| 22 | .894 3334 | 29.56 | .905 0026 | 29.71 | .915 7294 | 29.88 | .926 5182 | 30.06 |
| 23 | .894 5108 | 29.56 | .905 1809 | 29.72 | .915 9087 | 29.88 | .926 6986 | 30.06 |
| 24 | .894 6882 | 29.57 | .905 3592 | 29.72 | .916 0880 | 29.89 | .926 8789 | 30.07 |
| 25 | 1.894 8656 | 29.57 | 1.905 5376 | 29.72 | 1.916 2673 | 29.89 | 1.927 0593 | 30.07 |
| 26 | .895 0430 | 29.57 | .905 7159 | 29.73 | .916 4466 | 29.89 | .927 2398 | 30.07 |
| 27 | .895 2204 | 29.57 | .905 8943 | 29.73 | .916 6260 | 29.90 | .927 4202 | 30.08 |
| 28 | .895 3979 | 29.58 | .906 0726 | 29.73 | .916 8054 | 29.90 | .927 6007 | 30.08 |
| 29 | .895 5753 | 29.58 | .906 2510 | 29.73 | .916 9848 | 29.90 | .927 7811 | 30.08 |
| 30 | 1.895 7528 | 29.58 | 1.906 4294 | 29.74 | 1.917 1642 | 29.90 | 1.927 9616 | 30.08 |
| 31 | .895 9303 | 29.58 | .906 6079 | 29.74 | .917 3436 | 29.91 | .928 1422 | 30.09 |
| 32 | .896 1078 | 29.59 | .906 7863 | 29.74 | .917 5231 | 29.91 | .928 3227 | 30.09 |
| 33 | .896 2854 | 29.59 | .906 9648 | 29.74 | .917 7025 | 29.91 | .928 5032 | 30.09 |
| 34 | .896 4628 | 29.59 | .907 1432 | 29.75 | .917 8820 | 29.92 | .928 6838 | 30.10 |
| 35 | 1.896 6404 | 29.59 | 1.907 3217 | 29.75 | 1.918 0615 | 29.92 | 1.928 8644 | 30.10 |
| 36 | .896 8180 | 29.60 | .907 5002 | 29.75 | .918 2410 | 29.92 | .929 0450 | 30.10 |
| 37 | .896 9955 | 29.60 | .907 6787 | 29.75 | .918 4206 | 29.92 | .929 2256 | 30.11 |
| 38 | .897 1732 | 29.60 | .907 8573 | 29.76 | .918 6001 | 29.93 | .929 4063 | 30.11 |
| 39 | .897 3508 | 29.60 | .908 0358 | 29.76 | .918 7797 | 29.93 | .929 5869 | 30.11 |
| 40 | 1.897 5284 | 29.61 | 1.908 2144 | 29.76 | 1.918 9593 | 29.93 | 1.929 7676 | 30.12 |
| 41 | .897 7060 | 29.61 | .908 3930 | 29.77 | .919 1389 | 29.94 | .929 9483 | 30.12 |
| 42 | .897 8837 | 29.61 | .908 5716 | 29.77 | .919 3185 | 29.94 | .930 1291 | 30.12 |
| 43 | .898 0614 | 29.61 | .908 7502 | 29.77 | .919 4982 | 29.94 | .930 3098 | 30.13 |
| 44 | .898 2390 | 29.62 | .908 9288 | 29.77 | .919 6778 | 29.94 | .930 4906 | 30.13 |
| 45 | 1.898 4168 | 29.62 | 1.909 1075 | 29.78 | 1.919 8575 | 29.95 | 1.930 6713 | 30.13 |
| 46 | .898 5945 | 29.62 | .909 2862 | 29.78 | .920 0372 | 29.95 | .930 8521 | 30.13 |
| 47 | .898 7722 | 29.62 | .909 4648 | 29.78 | .920 2169 | 29.95 | .931 0330 | 30.14 |
| 48 | .898 9500 | 29.63 | .909 6436 | 29.78 | .920 3966 | 29.96 | .931 2138 | 30.14 |
| 49 | .899 1277 | 29.63 | .909 8223 | 29.79 | .920 5764 | 29.96 | .931 3946 | 30.14 |
| 50 | 1.899 3055 | 29.63 | 1.910 0010 | 29.79 | 1.920 7561 | 29.96 | 1.931 5755 | 30.15 |
| 51 | .899 4833 | 29.63 | .910 1798 | 29.79 | .920 9359 | 29.97 | .931 7564 | 30.15 |
| 52 | .899 6611 | 29.64 | .910 3585 | 29.80 | .921 1157 | 29.97 | .931 9373 | 30.15 |
| 53 | .899 8389 | 29.64 | .910 5373 | 29.80 | .921 2956 | 29.97 | .932 1183 | 30.16 |
| 54 | .900 0168 | 29.64 | .910 7161 | 29.80 | .921 4754 | 29.98 | .932 2992 | 30.16 |
| 55 | 1.900 1946 | 29.64 | 1.910 8949 | 29.80 | 1.921 6552 | 29.98 | 1.932 4802 | 30.16 |
| 56 | .900 3725 | 29.65 | .911 0738 | 29.81 | .921 8351 | 29.98 | .932 6612 | 30.17 |
| 57 | .900 5504 | 29.65 | .911 2526 | 29.81 | .922 0150 | 29.98 | .932 8422 | 30.17 |
| 58 | .900 7283 | 29.65 | .911 4315 | 29.81 | .922 1949 | 29.99 | .933 0232 | 30.17 |
| 59 | .900 9062 | 29.66 | .911 6104 | 29.82 | .922 3748 | 29.99 | .933 2043 | 30.18 |
| 60 | 1.901 0841 | 29.66 | 1.911 7893 | 29.82 | 1.922 5548 | 29.99 | 1.933 3853 | 30.18 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 84° | | 85° | | 86° | | 87° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 1.933 3853 | 30.18 | 1.944 2856 | 30.38 | 1.955 2602 | 30.59 | 1.966 3140 | 30.82 |
| 1 | .933 5664 | 30.18 | .944 4678 | 30.38 | .955 4438 | 30.60 | .966 4990 | 30.82 |
| 2 | .933 7475 | 30.19 | .944 6502 | 30.39 | .955 6274 | 30.60 | .966 6839 | 30.83 |
| 3 | .933 9287 | 30.19 | .944 8325 | 30.39 | .955 8110 | 30.60 | .966 8689 | 30.83 |
| 4 | .934 1098 | 30.19 | .945 0148 | 30.39 | .955 9946 | 30.61 | .967 0539 | 30.84 |
| 5 | 1.934 2910 | 30.20 | 1.945 1972 | 30.40 | 1.956 1783 | 30.61 | 1.967 2389 | 30.84 |
| 6 | .934 4722 | 30.20 | .945 3796 | 30.40 | .956 3619 | 30.61 | .967 4240 | 30.84 |
| 7 | .934 6533 | 30.20 | .945 5620 | 30.40 | .956 5456 | 30.62 | .967 6090 | 30.85 |
| 8 | .934 8346 | 30.21 | .945 7444 | 30.41 | .956 7294 | 30.62 | .967 7941 | 30.85 |
| 9 | .935 0158 | 30.21 | .945 9269 | 30.41 | .956 9131 | 30.63 | .967 9792 | 30.85 |
| 10 | 1.935 1971 | 30.21 | 1.946 1094 | 30.41 | 1.957 0969 | 30.63 | 1.968 1644 | 30.86 |
| 11 | .935 3784 | 30.22 | .946 2919 | 30.42 | .957 2807 | 30.63 | .968 3496 | 30.86 |
| 12 | .935 5597 | 30.22 | .946 4744 | 30.42 | .957 4645 | 30.64 | .968 5347 | 30.87 |
| 13 | .935 7410 | 30.22 | .946 6569 | 30.42 | .957 6483 | 30.64 | .968 7200 | 30.87 |
| 14 | .935 9223 | 30.22 | .946 8395 | 30.43 | .957 8322 | 30.64 | .968 9052 | 30.87 |
| 15 | 1.936 1037 | 30.23 | 1.947 0221 | 30.43 | 1.958 0160 | 30.65 | 1.969 0905 | 30.88 |
| 16 | .936 2851 | 30.23 | .947 2047 | 30.44 | .958 1999 | 30.65 | .969 2757 | 30.88 |
| 17 | .936 4665 | 30.23 | .947 3873 | 30.44 | .958 3839 | 30.66 | .969 4610 | 30.89 |
| 18 | .936 6479 | 30.24 | .947 5699 | 30.44 | .958 5678 | 30.66 | .969 6464 | 30.89 |
| 19 | .936 8293 | 30.24 | .947 7526 | 30.45 | .958 7518 | 30.66 | .969 8317 | 30.89 |
| 20 | 1.937 0108 | 30.24 | 1.947 9353 | 30.45 | 1.958 9358 | 30.67 | 1.970 0171 | 30.90 |
| 21 | .937 1922 | 30.25 | .948 1180 | 30.45 | .959 1198 | 30.67 | .970 2025 | 30.90 |
| 22 | .937 3737 | 30.25 | .948 3007 | 30.46 | .959 3038 | 30.67 | .970 3879 | 30.91 |
| 23 | .937 5553 | 30.25 | .948 4834 | 30.46 | .959 4879 | 30.68 | .970 5734 | 30.91 |
| 24 | .937 7368 | 30.26 | .948 6662 | 30.46 | .959 6720 | 30.68 | .970 7589 | 30.91 |
| 25 | 1.937 9184 | 30.26 | 1.948 8490 | 30.47 | 1.959 8561 | 30.69 | 1.970 9443 | 30.92 |
| 26 | .938 0999 | 30.26 | .949 0318 | 30.47 | .960 0402 | 30.69 | .971 1299 | 30.92 |
| 27 | .938 2815 | 30.27 | .949 2146 | 30.47 | .960 2243 | 30.69 | .971 3154 | 30.93 |
| 28 | .938 4632 | 30.27 | .949 3975 | 30.48 | .960 4085 | 30.70 | .971 5010 | 30.93 |
| 29 | .938 6448 | 30.27 | .949 5804 | 30.48 | .960 5927 | 30.70 | .971 6866 | 30.93 |
| 30 | 1.938 8264 | 30.28 | 1.949 7633 | 30.48 | 1.960 7769 | 30.70 | 1.971 8722 | 30.94 |
| 31 | .939 0081 | 30.28 | .949 9462 | 30.49 | .960 9612 | 30.71 | .972 0578 | 30.94 |
| 32 | .939 1898 | 30.28 | .950 1291 | 30.49 | .961 1454 | 30.71 | .972 2435 | 30.95 |
| 33 | .939 3715 | 30.29 | .950 3121 | 30.50 | .961 3297 | 30.71 | .972 4292 | 30.95 |
| 34 | .939 5533 | 30.29 | .950 4951 | 30.50 | .961 5140 | 30.72 | .972 6149 | 30.95 |
| 35 | 1.939 7350 | 30.29 | 1.950 6781 | 30.50 | 1.961 6983 | 30.72 | 1.972 8006 | 30.96 |
| 36 | .939 9168 | 30.30 | .950 8611 | 30.51 | .961 8827 | 30.73 | .972 9864 | 30.96 |
| 37 | .940 0986 | 30.30 | .951 0441 | 30.51 | .962 0671 | 30.73 | .973 1722 | 30.97 |
| 38 | .940 2804 | 30.30 | .951 2272 | 30.51 | .962 2515 | 30.73 | .973 3580 | 30.97 |
| 39 | .940 4623 | 30.31 | .951 4103 | 30.52 | .962 4359 | 30.74 | .973 5438 | 30.97 |
| 40 | 1.940 6441 | 30.31 | 1.951 5934 | 30.52 | 1.962 6203 | 30.74 | 1.973 7297 | 30.98 |
| 41 | .940 8260 | 30.31 | .951 7766 | 30.52 | .962 8048 | 30.75 | .973 9156 | 30.98 |
| 42 | .941 0079 | 30.32 | .951 9597 | 30.53 | .962 9893 | 30.75 | .974 1015 | 30.99 |
| 43 | .941 1898 | 30.32 | .952 1429 | 30.53 | .963 1738 | 30.75 | .974 2874 | 30.99 |
| 44 | .941 3717 | 30.32 | .952 3261 | 30.53 | .963 3583 | 30.76 | .974 4734 | 30.99 |
| 45 | 1.941 5537 | 30.33 | 1.952 5093 | 30.54 | 1.963 5429 | 30.76 | 1.974 6593 | 31.00 |
| 46 | .941 7357 | 30.33 | .952 6925 | 30.54 | .963 7275 | 30.77 | .974 8454 | 31.00 |
| 47 | .941 9177 | 30.34 | .952 8758 | 30.55 | .963 9121 | 30.77 | .975 0314 | 31.01 |
| 48 | .942 0997 | 30.34 | .953 0591 | 30.55 | .964 0967 | 30.77 | .975 2174 | 31.01 |
| 49 | .942 2817 | 30.34 | .953 2424 | 30.55 | .964 2814 | 30.78 | .975 4035 | 31.01 |
| 50 | 1.942 4638 | 30.35 | 1.953 4257 | 30.56 | 1.964 4660 | 30.78 | 1.975 5896 | 31.02 |
| 51 | .942 6459 | 30.35 | .953 6091 | 30.56 | .964 6507 | 30.78 | .975 7757 | 31.02 |
| 52 | .942 8280 | 30.35 | .953 7924 | 30.56 | .964 8354 | 30.79 | .975 9619 | 31.03 |
| 53 | .943 0101 | 30.36 | .953 9758 | 30.57 | .965 0202 | 30.79 | .976 1481 | 31.03 |
| 54 | .943 1923 | 30.36 | .954 1592 | 30.57 | .965 2050 | 30.80 | .976 3343 | 31.04 |
| 55 | 1.943 3744 | 30.36 | 1.954 3427 | 30.57 | 1.965 3897 | 30.80 | 1.976 5205 | 31.04 |
| 56 | .943 5566 | 30.37 | .954 5262 | 30.58 | .965 5746 | 30.81 | .976 7067 | 31.04 |
| 57 | .943 7388 | 30.37 | .954 7090 | 30.58 | .965 7594 | 30.81 | .976 8930 | 31.05 |
| 58 | .943 9211 | 30.37 | .954 8931 | 30.59 | .965 9442 | 30.81 | .977 0793 | 31.05 |
| 59 | .944 1033 | 30.38 | .955 0766 | 30.59 | .966 1291 | 30.81 | .977 2656 | 31.06 |
| 60 | 1.944 2856 | 30.38 | 1.955 2602 | 30.59 | 1.966 3140 | 30.82 | 1.977 4520 | 31.06 |

TABLE VI

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 88° | | 89° | | 90° | | 91° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 1.977 4520 | 31.06 | 1.988 6789 | 31.31 | 2.000 0000 | 31.58 | 2.011 4203 | 31.87 |
| 1 | .977 6383 | 31.06 | .988 8668 | 31.32 | .000 1895 | 31.59 | .011 6115 | 31.87 |
| 2 | .977 8247 | 31.07 | .989 0548 | 31.32 | .000 3790 | 31.59 | .011 8027 | 31.88 |
| 3 | .978 0112 | 31.07 | .989 2427 | 31.33 | .000 5686 | 31.60 | .011 9940 | 31.88 |
| 4 | .978 1976 | 31.08 | .989 4307 | 31.33 | .000 7582 | 31.60 | .012 1853 | 31.89 |
| 5 | 1.978 3841 | 31.08 | 1.989 6187 | 31.34 | 2.000 9478 | 31.60 | 2.012 3766 | 31.89 |
| 6 | .978 5706 | 31.08 | .989 8067 | 31.34 | .001 1375 | 31.61 | .012 5680 | 31.89 |
| 7 | .978 7571 | 31.09 | .989 9948 | 31.34 | .001 3272 | 31.61 | .012 7594 | 31.90 |
| 8 | .978 9436 | 31.09 | .990 1829 | 31.35 | .001 5169 | 31.62 | .012 9508 | 31.90 |
| 9 | .979 1302 | 31.10 | .990 3710 | 31.35 | .001 7066 | 31.62 | .013 1422 | 31.91 |
| 10 | 1.979 3168 | 31.10 | 1.990 5591 | 31.36 | 2.001 8963 | 31.63 | 2.013 3337 | 31.91 |
| 11 | .979 5034 | 31.11 | .990 7473 | 31.36 | .002 0861 | 31.63 | .013 5252 | 31.92 |
| 12 | .979 6901 | 31.11 | .990 9355 | 31.37 | .002 2759 | 31.64 | .013 7167 | 31.92 |
| 13 | .979 8768 | 31.11 | .991 1237 | 31.37 | .002 4658 | 31.64 | .013 9083 | 31.93 |
| 14 | .980 0635 | 31.12 | .991 3119 | 31.38 | .002 6557 | 31.65 | .014 0999 | 31.93 |
| 15 | 1.980 2502 | 31.12 | 1.991 5002 | 31.38 | 2.002 8456 | 31.65 | 2.014 2915 | 31.94 |
| 16 | .980 4369 | 31.13 | .991 6885 | 31.38 | .003 0355 | 31.66 | .014 4831 | 31.94 |
| 17 | .980 6237 | 31.13 | .991 8768 | 31.39 | .003 2254 | 31.66 | .014 6748 | 31.95 |
| 18 | .980 8105 | 31.13 | .992 0651 | 31.39 | .003 4154 | 31.67 | .014 8665 | 31.95 |
| 19 | .980 9973 | 31.14 | .992 2535 | 31.40 | .003 6054 | 31.67 | .015 0582 | 31.96 |
| 20 | 1.981 1842 | 31.14 | 1.992 4419 | 31.40 | 2.003 7955 | 31.68 | 2.015 2500 | 31.96 |
| 21 | .981 3710 | 31.15 | .992 6304 | 31.41 | .003 9855 | 31.68 | .015 4418 | 31.97 |
| 22 | .981 5579 | 31.15 | .992 8188 | 31.41 | .004 1756 | 31.68 | .015 6336 | 31.97 |
| 23 | .981 7449 | 31.16 | .993 0073 | 31.42 | .004 3658 | 31.69 | .015 8255 | 31.98 |
| 24 | .981 9318 | 31.16 | .993 1958 | 31.42 | .004 5559 | 31.69 | .016 0174 | 31.98 |
| 25 | 1.982 1188 | 31.16 | 1.993 3843 | 31.42 | 2.004 7461 | 31.70 | 2.016 2093 | 31.99 |
| 26 | .982 3058 | 31.17 | .993 5729 | 31.43 | .004 9363 | 31.70 | .016 4012 | 31.99 |
| 27 | .982 4928 | 31.17 | .993 7615 | 31.43 | .005 1265 | 31.71 | .016 5932 | 32.00 |
| 28 | .982 6798 | 31.18 | .993 9501 | 31.44 | .005 3168 | 31.71 | .016 7852 | 32.00 |
| 29 | .982 8669 | 31.18 | .994 1387 | 31.44 | .005 5071 | 31.72 | .016 9772 | 32.01 |
| 30 | 1.983 0540 | 31.18 | 1.994 3274 | 31.45 | 2.005 6974 | 31.72 | 2.017 1693 | 32.01 |
| 31 | .983 2411 | 31.19 | .994 5161 | 31.45 | .005 8878 | 31.73 | .017 3614 | 32.02 |
| 32 | .983 4283 | 31.19 | .994 7048 | 31.46 | .006 0781 | 31.73 | .017 5535 | 32.02 |
| 33 | .983 6155 | 31.20 | .994 8936 | 31.46 | .006 2685 | 31.74 | .017 7456 | 32.03 |
| 34 | .983 8027 | 31.20 | .995 0823 | 31.46 | .006 4590 | 31.74 | .017 9378 | 32.03 |
| 35 | 1.983 9899 | 31.21 | 1.995 2711 | 31.47 | 2.006 6494 | 31.75 | 2.018 1300 | 32.04 |
| 36 | .984 1772 | 31.21 | .995 4600 | 31.47 | .006 8399 | 31.75 | .018 3223 | 32.04 |
| 37 | .984 3644 | 31.22 | .995 6488 | 31.48 | .007 0304 | 31.76 | .018 5145 | 32.05 |
| 38 | .984 5517 | 31.22 | .995 8377 | 31.48 | .007 2210 | 31.76 | .018 7068 | 32.05 |
| 39 | .984 7391 | 31.22 | .996 0266 | 31.49 | .007 4116 | 31.77 | .018 8992 | 32.06 |
| 40 | 1.984 9264 | 31.23 | 1.996 2155 | 31.49 | 2.007 6022 | 31.77 | 2.019 0915 | 32.06 |
| 41 | .985 1138 | 31.23 | .996 4045 | 31.50 | .007 7928 | 31.77 | .019 2839 | 32.07 |
| 42 | .985 3012 | 31.24 | .996 5935 | 31.50 | .007 9835 | 31.78 | .019 4763 | 32.07 |
| 43 | .985 4886 | 31.24 | .996 7825 | 31.51 | .008 1742 | 31.78 | .019 6688 | 32.08 |
| 44 | .985 6761 | 31.24 | .996 9716 | 31.51 | .008 3649 | 31.79 | .019 8613 | 32.08 |
| 45 | 1.985 8636 | 31.25 | 1.997 1606 | 31.51 | 2.008 5556 | 31.79 | 2.020 0538 | 32.09 |
| 46 | .986 0511 | 31.25 | .997 3497 | 31.52 | .008 7464 | 31.80 | .020 2463 | 32.09 |
| 47 | .986 2386 | 31.26 | .997 5389 | 31.52 | .008 9372 | 31.80 | .020 4389 | 32.10 |
| 48 | .986 4262 | 31.26 | .997 7280 | 31.53 | .009 1280 | 31.81 | .020 6315 | 32.10 |
| 49 | .986 6138 | 31.27 | .997 9172 | 31.53 | .009 3189 | 31.81 | .020 8241 | 32.11 |
| 50 | 1.986 8014 | 31.27 | 1.998 1064 | 31.54 | 2.009 5098 | 31.82 | 2.021 0168 | 32.11 |
| 51 | .986 9890 | 31.28 | .998 2956 | 31.54 | .009 7007 | 31.82 | .021 2095 | 32.12 |
| 52 | .987 1767 | 31.28 | .998 4849 | 31.55 | .009 8917 | 31.83 | .021 4022 | 32.12 |
| 53 | .987 3644 | 31.28 | .998 6742 | 31.55 | .010 0826 | 31.83 | .021 5949 | 32.13 |
| 54 | .987 5521 | 31.29 | .998 8635 | 31.56 | .010 2736 | 31.84 | .021 7877 | 32.13 |
| 55 | 1.987 7398 | 31.29 | 1.999 0529 | 31.56 | 2.010 4647 | 31.84 | 2.021 9805 | 32.14 |
| 56 | .987 9276 | 31.30 | .999 2422 | 31.56 | .010 6557 | 31.85 | .022 1734 | 32.14 |
| 57 | .988 1154 | 31.30 | .999 4316 | 31.57 | .010 8468 | 31.85 | .022 3662 | 32.15 |
| 58 | .988 3032 | 31.31 | .999 6211 | 31.57 | .011 0380 | 31.86 | .022 5591 | 32.15 |
| 59 | .988 4911 | 31.31 | .999 8105 | 31.58 | .011 2291 | 31.86 | .022 7521 | 32.16 |
| 60 | 1.988 6789 | 31.31 | 2.000 0000 | 31.58 | 2.011 4203 | 31.87 | 2.022 9450 | 32.16 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 92° | | 93° | | 94° | | 95° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.022 9450 | 32.16 | 2.034 5797 | 32.48 | 2.046 3296 | 32.80 | 2.058 2005 | 33.15 |
| 1 | .023 1380 | 32.17 | .034 7745 | 32.48 | .046 5264 | 32.81 | .058 3994 | 33.15 |
| 2 | .023 3311 | 32.17 | .034 9694 | 32.49 | .046 7233 | 32.82 | .058 5983 | 33.16 |
| 3 | .023 5241 | 32.18 | .035 1644 | 32.49 | .046 9202 | 32.82 | .058 7973 | 33.16 |
| 4 | .023 7172 | 32.18 | .035 3593 | 32.50 | .047 1172 | 32.83 | .058 9963 | 33.17 |
| 5 | 2.023 9103 | 32.19 | 2.035 5543 | 32.50 | 2.047 3141 | 32.83 | 2.059 1953 | 33.18 |
| 6 | .024 1035 | 32.19 | .035 7494 | 32.51 | .047 5111 | 32.84 | .059 3944 | 33.18 |
| 7 | .024 2967 | 32.20 | .035 9444 | 32.51 | .047 7082 | 32.84 | .059 5935 | 33.19 |
| 8 | .024 4899 | 32.20 | .036 1395 | 32.52 | .047 9053 | 32.85 | .059 7927 | 33.19 |
| 9 | .024 6831 | 32.21 | .036 3347 | 32.52 | .048 1024 | 32.85 | .059 9919 | 33.20 |
| 10 | 2.024 8764 | 32.21 | 2.036 5298 | 32.53 | 2.048 2995 | 32.86 | 2.060 1911 | 33.21 |
| 11 | .025 0697 | 32.22 | .036 7250 | 32.53 | .048 4967 | 32.87 | .060 3904 | 33.21 |
| 12 | .025 2630 | 32.22 | .036 9202 | 32.54 | .048 6939 | 32.87 | .060 5897 | 33.22 |
| 13 | .025 4564 | 32.23 | .037 1155 | 32.54 | .048 8912 | 32.88 | .060 7890 | 33.22 |
| 14 | .025 6498 | 32.23 | .037 3108 | 32.55 | .049 0884 | 32.88 | .060 9884 | 33.23 |
| 15 | 2.025 8432 | 32.24 | 2.037 5061 | 32.55 | 2.049 2857 | 32.89 | 2.061 1878 | 33.24 |
| 16 | .026 0367 | 32.24 | .037 7015 | 32.56 | .049 4831 | 32.89 | .061 3872 | 33.24 |
| 17 | .026 2301 | 32.25 | .037 8969 | 32.57 | .049 6805 | 32.90 | .061 5867 | 33.25 |
| 18 | .026 4236 | 32.26 | .038 0923 | 32.57 | .049 8779 | 32.90 | .061 7862 | 33.25 |
| 19 | .026 6172 | 32.26 | .038 2877 | 32.58 | .050 0753 | 32.91 | .061 9857 | 33.26 |
| 20 | 2.026 8108 | 32.27 | 2.038 4832 | 32.58 | 2.050 2728 | 32.92 | 2.062 1853 | 33.27 |
| 21 | .027 0044 | 32.27 | .038 6787 | 32.59 | .050 4703 | 32.92 | .062 3849 | 33.27 |
| 22 | .027 1980 | 32.28 | .038 8743 | 32.59 | .050 6679 | 32.93 | .062 5846 | 33.28 |
| 23 | .027 3917 | 32.28 | .039 0699 | 32.60 | .050 8655 | 32.93 | .062 7842 | 33.28 |
| 24 | .027 5854 | 32.29 | .039 2655 | 32.61 | .051 0631 | 32.94 | .062 9840 | 33.29 |
| 25 | 2.027 7791 | 32.29 | 2.039 4611 | 32.61 | 2.051 2608 | 32.95 | 2.063 1837 | 33.30 |
| 26 | .027 9729 | 32.30 | .039 6568 | 32.62 | .051 4585 | 32.95 | .063 3835 | 33.30 |
| 27 | .028 1667 | 32.30 | .039 8525 | 32.62 | .051 6562 | 32.96 | .063 5833 | 33.31 |
| 28 | .028 3605 | 32.31 | .040 0482 | 32.63 | .051 8539 | 32.96 | .063 7832 | 33.31 |
| 29 | .028 5544 | 32.31 | .040 2440 | 32.63 | .052 0517 | 32.97 | .063 9831 | 33.32 |
| 30 | 2.028 7483 | 32.32 | 2.040 4399 | 32.64 | 2.052 2496 | 32.97 | 2.064 1831 | 33.33 |
| 31 | .028 9422 | 32.32 | .040 6357 | 32.64 | .052 4474 | 32.98 | .064 3830 | 33.33 |
| 32 | .029 1361 | 32.33 | .040 8316 | 32.65 | .052 6453 | 32.98 | .064 5830 | 33.34 |
| 33 | .029 3301 | 32.33 | .041 0275 | 32.65 | .052 8432 | 32.99 | .064 7831 | 33.34 |
| 34 | .029 5241 | 32.34 | .041 2234 | 32.66 | .053 0412 | 33.00 | .064 9832 | 33.35 |
| 35 | 2.029 7182 | 32.34 | 2.041 4194 | 32.67 | 2.053 2392 | 33.00 | 2.065 1833 | 33.36 |
| 36 | .029 9123 | 32.35 | .041 6154 | 32.67 | .053 4372 | 33.01 | .065 3834 | 33.36 |
| 37 | .030 1064 | 32.35 | .041 8114 | 32.68 | .053 6353 | 33.01 | .065 5836 | 33.37 |
| 38 | .030 3005 | 32.36 | .042 0075 | 32.68 | .053 8334 | 33.02 | .065 7839 | 33.37 |
| 39 | .030 4947 | 32.36 | .042 2036 | 32.69 | .054 0315 | 33.03 | .065 9841 | 33.38 |
| 40 | 2.030 6889 | 32.37 | 2.042 3998 | 32.69 | 2.054 2297 | 33.03 | 2.066 1844 | 33.39 |
| 41 | .030 8831 | 32.37 | .042 5960 | 32.70 | .054 4279 | 33.04 | .066 3847 | 33.39 |
| 42 | .031 0774 | 32.38 | .042 7922 | 32.70 | .054 6262 | 33.04 | .066 5851 | 33.40 |
| 43 | .031 2717 | 32.39 | .042 9884 | 32.71 | .054 8244 | 33.05 | .066 7855 | 33.40 |
| 44 | .031 4660 | 32.39 | .043 1847 | 32.71 | .055 0227 | 33.05 | .066 9860 | 33.41 |
| 45 | 2.031 6604 | 32.40 | 2.043 3810 | 32.72 | 2.055 2211 | 33.06 | 2.067 1865 | 33.42 |
| 46 | .031 8548 | 32.40 | .043 5773 | 32.73 | .055 4195 | 33.07 | .067 3870 | 33.42 |
| 47 | .032 0492 | 32.41 | .043 7737 | 32.73 | .055 6179 | 33.07 | .067 5875 | 33.43 |
| 48 | .032 2437 | 32.41 | .043 9701 | 32.74 | .055 8163 | 33.08 | .067 7881 | 33.43 |
| 49 | .032 4382 | 32.42 | .044 1665 | 32.74 | .056 0148 | 33.08 | .067 9887 | 33.44 |
| 50 | 2.032 6327 | 32.42 | 2.044 3630 | 32.75 | 2.056 2133 | 33.09 | 2.068 1894 | 33.45 |
| 51 | .032 8272 | 32.43 | .044 5595 | 32.75 | .056 4119 | 33.10 | .068 3901 | 33.45 |
| 52 | .033 0218 | 32.43 | .044 7561 | 32.76 | .056 6105 | 33.10 | .068 5908 | 33.46 |
| 53 | .033 2164 | 32.44 | .044 9526 | 32.76 | .056 8091 | 33.11 | .068 7916 | 33.47 |
| 54 | .033 4111 | 32.44 | .045 1492 | 32.77 | .057 0078 | 33.11 | .068 9924 | 33.47 |
| 55 | 2.033 6058 | 32.45 | 2.045 3459 | 32.78 | 2.057 2065 | 33.12 | 2.069 1933 | 33.48 |
| 56 | .033 8005 | 32.45 | .045 5426 | 32.78 | .057 4052 | 33.12 | .069 3942 | 33.48 |
| 57 | .033 9952 | 32.46 | .045 7393 | 32.79 | .057 6040 | 33.13 | .069 5951 | 33.49 |
| 58 | .034 1900 | 32.47 | .045 9360 | 32.79 | .057 8028 | 33.14 | .069 7960 | 33.50 |
| 59 | .034 3848 | 32.47 | .046 1328 | 32.80 | .058 0016 | 33.14 | .069 9970 | 33.50 |
| 60 | 2.034 5797 | 32.48 | 2.046 3296 | 32.80 | 2.058 2005 | 33.15 | 2.070 1980 | 33.51 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 96° | | 97° | | 98° | | 99° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.070 1980 | 33.51 | 2.082 3282 | 33.88 | 2.094 5971 | 34.28 | 2.107 0109 | 34.69 |
| 1 | .070 3991 | 33.51 | .082 5316 | 33.89 | .094 8028 | 34.29 | .107 2190 | 34.70 |
| 2 | .070 6002 | 33.52 | .082 7349 | 33.90 | .095 0085 | 34.29 | .107 4272 | 34.70 |
| 3 | .070 8014 | 33.53 | .082 9383 | 33.90 | .095 2143 | 34.30 | .107 6355 | 34.71 |
| 4 | .071 0025 | 33.53 | .083 1418 | 33.91 | .095 4201 | 34.31 | .107 8437 | 34.72 |
| 5 | 2.071 2037 | 33.54 | 2.083 3453 | 33.92 | 2.095 6260 | 34.31 | 2.108 0521 | 34.72 |
| 6 | .071 4050 | 33.54 | .083 5488 | 33.92 | .095 8318 | 34.32 | .108 2604 | 34.73 |
| 7 | .071 6063 | 33.55 | .083 7523 | 33.93 | .096 0378 | 34.33 | .108 4689 | 34.74 |
| 8 | .071 8076 | 33.56 | .083 9559 | 33.94 | .096 2438 | 34.33 | .108 6773 | 34.75 |
| 9 | .072 0090 | 33.56 | .084 1596 | 33.94 | .096 4498 | 34.34 | .108 8858 | 34.75 |
| 10 | 2.072 2104 | 33.57 | 2.084 3633 | 33.95 | 2.096 6558 | 34.35 | 2.109 0944 | 34.76 |
| 11 | .072 4118 | 33.58 | .084 5670 | 33.96 | .096 8619 | 34.35 | .109 3029 | 34.77 |
| 12 | .072 6133 | 33.58 | .084 7707 | 33.96 | .097 0681 | 34.36 | .109 5116 | 34.77 |
| 13 | .072 8148 | 33.59 | .084 9745 | 33.97 | .097 2742 | 34.37 | .109 7202 | 34.78 |
| 14 | .073 0163 | 33.59 | .085 1783 | 33.98 | .097 4804 | 34.37 | .109 9289 | 34.79 |
| 15 | 2.073 2179 | 33.60 | 2.085 3822 | 33.98 | 2.097 6867 | 34.38 | 2.110 1377 | 34.80 |
| 16 | .073 4195 | 33.61 | .085 5861 | 33.99 | .097 8930 | 34.39 | .110 3465 | 34.80 |
| 17 | .073 6212 | 33.61 | .085 7901 | 33.99 | .098 0993 | 34.39 | .110 5553 | 34.81 |
| 18 | .073 8229 | 33.62 | .085 9941 | 34.00 | .098 3057 | 34.40 | .110 7642 | 34.82 |
| 19 | .074 0246 | 33.63 | .086 1981 | 34.01 | .098 5121 | 34.41 | .110 9731 | 34.82 |
| 20 | 2.074 2264 | 33.63 | 2.086 4021 | 34.01 | 2.098 7186 | 34.41 | 2.111 1821 | 34.83 |
| 21 | .074 4282 | 33.64 | .086 6062 | 34.02 | .098 9251 | 34.42 | .111 3911 | 34.84 |
| 22 | .074 6301 | 33.64 | .086 8104 | 34.03 | .099 1316 | 34.43 | .111 6001 | 34.85 |
| 23 | .074 8320 | 33.65 | .087 0146 | 34.03 | .099 3382 | 34.43 | .111 8092 | 34.85 |
| 24 | .075 0339 | 33.66 | .087 2188 | 34.04 | .099 5449 | 34.44 | .112 0184 | 34.86 |
| 25 | 2.075 2358 | 33.66 | 2.087 4231 | 34.05 | 2.099 7515 | 34.45 | 2.112 2275 | 34.87 |
| 26 | .075 4378 | 33.67 | .087 6274 | 34.05 | .099 9582 | 34.45 | .112 4368 | 34.87 |
| 27 | .075 6399 | 33.67 | .087 8317 | 34.06 | .100 1650 | 34.46 | .112 6460 | 34.88 |
| 28 | .075 8419 | 33.68 | .088 0361 | 34.07 | .100 3718 | 34.47 | .112 8553 | 34.89 |
| 29 | .076 0440 | 33.69 | .088 2405 | 34.07 | .100 5786 | 34.48 | .113 0647 | 34.90 |
| 30 | 2.076 2462 | 33.69 | 2.088 4449 | 34.08 | 2.100 7855 | 34.48 | 2.113 2741 | 34.90 |
| 31 | .076 4484 | 33.70 | .088 6494 | 34.09 | .100 9924 | 34.49 | .113 4835 | 34.91 |
| 32 | .076 6507 | 33.71 | .088 8540 | 34.09 | .101 1993 | 34.50 | .113 6930 | 34.92 |
| 33 | .076 8529 | 33.71 | .089 0586 | 34.10 | .101 4063 | 34.50 | .113 9025 | 34.92 |
| 34 | .077 0552 | 33.72 | .089 2632 | 34.11 | .101 6134 | 34.51 | .114 1121 | 34.93 |
| 35 | 2.077 2575 | 33.73 | 2.089 4678 | 34.11 | 2.101 8204 | 34.52 | 2.114 3217 | 34.94 |
| 36 | .077 4599 | 33.73 | .089 6725 | 34.12 | .102 0276 | 34.52 | .114 5313 | 34.95 |
| 37 | .077 6623 | 33.74 | .089 8772 | 34.12 | .102 2347 | 34.53 | .114 7410 | 34.95 |
| 38 | .077 8647 | 33.74 | .090 0820 | 34.13 | .102 4419 | 34.54 | .114 9508 | 34.96 |
| 39 | .078 0672 | 33.75 | .090 2868 | 34.14 | .102 6492 | 34.54 | .115 1605 | 34.97 |
| 40 | 2.078 2697 | 33.76 | 2.090 4917 | 34.15 | 2.102 8564 | 34.55 | 2.115 3704 | 34.97 |
| 41 | .078 4723 | 33.76 | .090 6966 | 34.15 | .103 0638 | 34.56 | .115 5802 | 34.98 |
| 42 | .078 6749 | 33.77 | .090 9015 | 34.16 | .103 2711 | 34.56 | .115 7901 | 34.99 |
| 43 | .078 8775 | 33.78 | .091 1065 | 34.17 | .103 4785 | 34.57 | .116 0001 | 35.00 |
| 44 | .079 0802 | 33.78 | .091 3115 | 34.17 | .103 6860 | 34.58 | .116 2101 | 35.00 |
| 45 | 2.079 2829 | 33.79 | 2.091 5165 | 34.18 | 2.103 8935 | 34.59 | 2.116 4201 | 35.01 |
| 46 | .079 4857 | 33.80 | .091 7216 | 34.19 | .104 1010 | 34.59 | .116 6301 | 35.02 |
| 47 | .079 6885 | 33.80 | .091 9268 | 34.19 | .104 3086 | 34.60 | .116 8403 | 35.02 |
| 48 | .079 8913 | 33.81 | .092 1319 | 34.20 | .104 5162 | 34.61 | .117 0505 | 35.03 |
| 49 | .080 0942 | 33.81 | .092 3371 | 34.20 | .104 7239 | 34.61 | .117 2607 | 35.04 |
| 50 | 2.080 2971 | 33.82 | 2.092 5424 | 34.21 | 2.104 9316 | 34.62 | 2.117 4710 | 35.05 |
| 51 | .080 5000 | 33.83 | .092 7477 | 34.22 | .105 1393 | 34.63 | .117 6813 | 35.05 |
| 52 | .080 7030 | 33.83 | .092 9530 | 34.22 | .105 3471 | 34.63 | .117 8916 | 35.06 |
| 53 | .080 9060 | 33.84 | .093 1584 | 34.23 | .105 5549 | 34.64 | .118 1020 | 35.07 |
| 54 | .081 1091 | 33.85 | .093 3638 | 34.24 | .105 7628 | 34.65 | .118 3124 | 35.08 |
| 55 | 2.081 3122 | 33.85 | 2.093 5692 | 34.25 | 2.105 9707 | 34.66 | 2.118 5229 | 35.08 |
| 56 | .081 5153 | 33.86 | .093 7747 | 34.25 | .106 1786 | 34.66 | .118 7334 | 35.09 |
| 57 | .081 7185 | 33.87 | .093 9803 | 34.26 | .106 3866 | 34.67 | .118 9440 | 35.10 |
| 58 | .081 9217 | 33.87 | .094 1858 | 34.27 | .106 5947 | 34.68 | .119 1546 | 35.10 |
| 59 | .082 1249 | 33.88 | .094 3914 | 34.27 | .106 8027 | 34.68 | .119 3652 | 35.11 |
| 60 | 2.082 3282 | 33.88 | 2.094 5971 | 34.28 | 2.107 0109 | 34.69 | 2.119 5759 | 35.12 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 100° | | 101° | | 102° | | 103° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 2.119 5759 | 35.12 | 2.132 2989 | 35.57 | 2.145 1866 | 36.03 | 2.158 2460 | 36.52 |
| 1 | .119 7867 | 35.13 | .132 5123 | 35.57 | .145 4028 | 36.04 | .158 4652 | 36.53 |
| 2 | .119 9974 | 35.13 | .132 7258 | 35.58 | .145 6191 | 36.05 | .158 6844 | 36.54 |
| 3 | .120 2083 | 35.14 | .132 9393 | 35.59 | .145 8354 | 36.06 | .158 9036 | 36.55 |
| 4 | .120 4191 | 35.15 | .133 1529 | 35.60 | .146 0518 | 36.07 | .159 1229 | 36.55 |
| 5 | 2.120 6301 | 35.16 | 2.133 3665 | 35.61 | 2.146 2682 | 36.07 | 2.159 3423 | 36.56 |
| 6 | .120 8410 | 35.16 | .133 5802 | 35.61 | .146 4847 | 36.08 | .159 5617 | 36.57 |
| 7 | .121 0520 | 35.17 | .133 7939 | 35.62 | .146 7012 | 36.09 | .159 7811 | 36.58 |
| 8 | .121 2630 | 35.18 | .134 0076 | 35.63 | .146 9178 | 36.10 | .160 0006 | 36.59 |
| 9 | .121 4741 | 35.19 | .134 2214 | 35.64 | .147 1344 | 36.11 | .160 2202 | 36.60 |
| 10 | 2.121 6853 | 35.19 | 2.134 4352 | 35.64 | 2.147 3510 | 36.11 | 2.160 4398 | 36.60 |
| 11 | .121 8965 | 35.20 | .134 6491 | 35.65 | .147 5677 | 36.12 | .160 6594 | 36.61 |
| 12 | .122 1077 | 35.21 | .134 8631 | 35.66 | .147 7845 | 36.13 | .160 8791 | 36.62 |
| 13 | .122 3190 | 35.21 | .135 0770 | 35.67 | .148 0013 | 36.14 | .161 0989 | 36.63 |
| 14 | .122 5303 | 35.22 | .135 2910 | 35.67 | .148 2182 | 36.15 | .161 3187 | 36.64 |
| 15 | 2.122 7416 | 35.23 | 2.135 5051 | 35.68 | 2.148 4351 | 36.15 | 2.161 5385 | 36.65 |
| 16 | .122 9530 | 35.24 | .135 7192 | 35.69 | .148 6520 | 36.16 | .161 7584 | 36.65 |
| 17 | .123 1644 | 35.24 | .135 9334 | 35.70 | .148 8690 | 36.17 | .161 9784 | 36.66 |
| 18 | .123 3759 | 35.25 | .136 1476 | 35.71 | .149 0861 | 36.18 | .162 1984 | 36.67 |
| 19 | .123 5875 | 35.26 | .136 3619 | 35.71 | .149 3032 | 36.19 | .162 4185 | 36.68 |
| 20 | 2.123 7990 | 35.27 | 2.136 5762 | 35.72 | 2.149 5203 | 36.19 | 2.162 6386 | 36.69 |
| 21 | .124 0107 | 35.27 | .136 7905 | 35.73 | .149 7375 | 36.20 | .162 8587 | 36.70 |
| 22 | .124 2223 | 35.28 | .137 0049 | 35.74 | .149 9547 | 36.21 | .163 0789 | 36.70 |
| 23 | .124 4340 | 35.29 | .137 2193 | 35.74 | .150 1720 | 36.22 | .163 2992 | 36.71 |
| 24 | .124 6458 | 35.30 | .137 4338 | 35.75 | .150 3893 | 36.23 | .163 5195 | 36.72 |
| 25 | 2.124 8576 | 35.30 | 2.137 6484 | 35.76 | 2.150 6067 | 36.23 | 2.163 7398 | 36.73 |
| 26 | .125 0694 | 35.31 | .137 8630 | 35.77 | .150 8242 | 36.24 | .163 9602 | 36.74 |
| 27 | .125 2813 | 35.32 | .138 0776 | 35.77 | .151 0417 | 36.25 | .164 1807 | 36.74 |
| 28 | .125 4933 | 35.33 | .138 2922 | 35.78 | .151 2592 | 36.26 | .164 4012 | 36.75 |
| 29 | .125 7052 | 35.33 | .138 5070 | 35.79 | .151 4768 | 36.27 | .164 6218 | 36.76 |
| 30 | 2.125 9173 | 35.34 | 2.138 7217 | 35.80 | 2.151 6944 | 36.28 | 2.164 8424 | 36.77 |
| 31 | .126 1293 | 35.35 | .138 9365 | 35.81 | .151 9121 | 36.28 | .165 0630 | 36.78 |
| 32 | .126 3414 | 35.35 | .139 1514 | 35.81 | .152 1298 | 36.29 | .165 2837 | 36.79 |
| 33 | .126 5536 | 35.36 | .139 3663 | 35.82 | .152 3476 | 36.30 | .165 5045 | 36.80 |
| 34 | .126 7658 | 35.37 | .139 5813 | 35.83 | .152 5654 | 36.31 | .165 7253 | 36.81 |
| 35 | 2.126 9780 | 35.38 | 2.139 7963 | 35.84 | 2.152 7833 | 36.32 | 2.165 9462 | 36.81 |
| 36 | .127 1903 | 35.39 | .140 0113 | 35.84 | .153 0012 | 36.32 | .166 1671 | 36.82 |
| 37 | .127 4027 | 35.39 | .140 2264 | 35.85 | .153 2192 | 36.33 | .166 3881 | 36.83 |
| 38 | .127 6151 | 35.40 | .140 4415 | 35.86 | .153 4372 | 36.34 | .166 6091 | 36.84 |
| 39 | .127 8275 | 35.41 | .140 6567 | 35.87 | .153 6552 | 36.35 | .166 8301 | 36.85 |
| 40 | 2.128 0400 | 35.42 | 2.140 8720 | 35.88 | 2.153 8734 | 36.35 | 2.167 0513 | 36.86 |
| 41 | .128 2525 | 35.42 | .141 0873 | 35.88 | .154 0915 | 36.36 | .167 2724 | 36.87 |
| 42 | .128 4650 | 35.43 | .141 3026 | 35.89 | .154 3097 | 36.37 | .167 4936 | 36.87 |
| 43 | .128 6776 | 35.44 | .141 5180 | 35.90 | .154 5280 | 36.38 | .167 7149 | 36.88 |
| 44 | .128 8903 | 35.45 | .141 7334 | 35.91 | .154 7463 | 36.39 | .167 9362 | 36.89 |
| 45 | 2.129 1030 | 35.45 | 2.141 9489 | 35.92 | 2.154 9647 | 36.40 | 2.168 1576 | 36.90 |
| 46 | .129 3157 | 35.46 | .142 1644 | 35.92 | .155 1831 | 36.41 | .168 3790 | 36.91 |
| 47 | .129 5285 | 35.47 | .142 3799 | 35.93 | .155 4015 | 36.41 | .168 6005 | 36.92 |
| 48 | .129 7414 | 35.48 | .142 5955 | 35.94 | .155 6200 | 36.42 | .168 8220 | 36.93 |
| 49 | .129 9542 | 35.48 | .142 8112 | 35.95 | .155 8386 | 36.43 | .169 0436 | 36.93 |
| 50 | 2.130 1672 | 35.49 | 2.143 0269 | 35.96 | 2.156 0572 | 36.44 | 2.169 2652 | 36.94 |
| 51 | .130 3801 | 35.50 | .143 2427 | 35.96 | .156 2759 | 36.45 | .169 4869 | 36.95 |
| 52 | .130 5931 | 35.51 | .143 4585 | 35.97 | .156 4946 | 36.46 | .169 7087 | 36.96 |
| 53 | .130 8062 | 35.51 | .143 6743 | 35.98 | .156 7133 | 36.46 | .169 9304 | 36.97 |
| 54 | .131 0193 | 35.52 | .143 8902 | 35.99 | .156 9321 | 36.47 | .170 1523 | 36.98 |
| 55 | 2.131 2325 | 35.53 | 2.144 1062 | 36.00 | 2.157 1510 | 36.48 | 2.170 3742 | 36.99 |
| 56 | .131 4457 | 35.54 | .144 3222 | 36.00 | .157 3699 | 36.49 | .170 5961 | 36.99 |
| 57 | .131 6589 | 35.54 | .144 5382 | 36.01 | .157 5889 | 36.50 | .170 8181 | 37.00 |
| 58 | .131 8722 | 35.55 | .144 7543 | 36.02 | .157 8079 | 36.50 | .171 0401 | 37.01 |
| 59 | .132 0855 | 35.56 | .144 9704 | 36.03 | .158 0269 | 36.51 | .171 2622 | 37.02 |
| 60 | 2.132 2989 | 35.57 | 2.145 1866 | 36.03 | 2.158 2460 | 36.52 | 2.171 4844 | 37.03 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 104° | | 105° | | 106° | | 107° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.171 4844 | 37.03 | 2.184 9092 | 37.56 | 2.198 5282 | 38.11 | 2.212 3493 | 38.68 |
| 1 | .171 7066 | 37.04 | .185 1346 | 37.57 | .198 7568 | 38.12 | .212 5814 | 38.69 |
| 2 | .171 9288 | 37.05 | .185 3000 | 37.57 | .198 9856 | 38.13 | .212 8136 | 38.70 |
| 3 | .172 1511 | 37.05 | .185 5855 | 37.58 | .199 2144 | 38.14 | .213 0458 | 38.71 |
| 4 | .172 3735 | 37.06 | .185 8110 | 37.59 | .199 4432 | 38.14 | .213 2781 | 38.72 |
| 5 | 2.172 5959 | 37.07 | 2.186 0366 | 37.60 | 2.199 6721 | 38.15 | 2.213 5104 | 38.73 |
| 6 | .172 8184 | 37.08 | .186 2622 | 37.61 | .199 9010 | 38.16 | .213 7428 | 38.74 |
| 7 | .173 0409 | 37.09 | .186 4879 | 37.62 | .200 1300 | 38.17 | .213 9753 | 38.75 |
| 8 | .173 2634 | 37.10 | .186 7137 | 37.63 | .200 3591 | 38.18 | .214 2078 | 38.76 |
| 9 | .173 4860 | 37.11 | .186 9395 | 37.64 | .200 5882 | 38.19 | .214 4404 | 38.77 |
| 10 | 2.173 7087 | 37.12 | 2.187 1653 | 37.65 | 2.200 8174 | 38.20 | 2.214 6730 | 38.78 |
| 11 | .173 9314 | 37.12 | .187 3912 | 37.66 | .201 0467 | 38.21 | .214 9057 | 38.79 |
| 12 | .174 1542 | 37.13 | .187 6172 | 37.67 | .201 2760 | 38.22 | .215 1385 | 38.80 |
| 13 | .174 3770 | 37.14 | .187 8432 | 37.67 | .201 5053 | 38.23 | .215 3713 | 38.81 |
| 14 | .174 5999 | 37.15 | .188 0693 | 37.68 | .201 7347 | 38.24 | .215 6042 | 38.82 |
| 15 | 2.174 8228 | 37.16 | 2.188 2954 | 37.69 | 2.201 9642 | 38.25 | 2.215 8371 | 38.83 |
| 16 | .175 0458 | 37.17 | .188 5216 | 37.70 | .202 1937 | 38.26 | .216 0701 | 38.84 |
| 17 | .175 2688 | 37.18 | .188 7478 | 37.71 | .202 4233 | 38.27 | .216 3032 | 38.85 |
| 18 | .175 4919 | 37.18 | .188 9741 | 37.72 | .202 6529 | 38.28 | .216 5363 | 38.86 |
| 19 | .175 7150 | 37.19 | .189 2005 | 37.73 | .202 8826 | 38.29 | .216 7694 | 38.87 |
| 20 | 2.175 9382 | 37.20 | 2.189 4269 | 37.74 | 2.203 1123 | 38.30 | 2.217 0027 | 38.88 |
| 21 | .176 1615 | 37.21 | .189 6533 | 37.75 | .203 3421 | 38.31 | .217 2360 | 38.89 |
| 22 | .176 3848 | 37.22 | .189 8798 | 37.76 | .203 5720 | 38.31 | .217 4693 | 38.90 |
| 23 | .176 6081 | 37.23 | .190 1064 | 37.77 | .203 8019 | 38.32 | .217 7027 | 38.91 |
| 24 | .176 8315 | 37.24 | .190 3330 | 37.77 | .204 0319 | 38.33 | .217 9362 | 38.92 |
| 25 | 2.177 0550 | 37.25 | 2.190 5597 | 37.78 | 2.204 2619 | 38.34 | 2.218 1697 | 38.93 |
| 26 | .177 2785 | 37.25 | .190 7864 | 37.79 | .204 4920 | 38.35 | .218 4033 | 38.94 |
| 27 | .177 5020 | 37.26 | .191 0132 | 37.80 | .204 7222 | 38.36 | .218 6369 | 38.95 |
| 28 | .177 7256 | 37.27 | .191 2401 | 37.81 | .204 9524 | 38.37 | .218 8706 | 38.96 |
| 29 | .177 9493 | 37.28 | .191 4670 | 37.82 | .205 1826 | 38.38 | .219 1044 | 38.97 |
| 30 | 2.178 1730 | 37.29 | 2.191 6939 | 37.83 | 2.205 4129 | 38.39 | 2.219 3382 | 38.98 |
| 31 | .178 3968 | 37.30 | .191 9209 | 37.84 | .205 6433 | 38.40 | .219 5721 | 38.99 |
| 32 | .178 6206 | 37.31 | .192 1480 | 37.85 | .205 8737 | 38.41 | .219 8061 | 39.00 |
| 33 | .178 8445 | 37.32 | .192 3751 | 37.86 | .206 1042 | 38.42 | .220 0401 | 39.01 |
| 34 | .179 0684 | 37.33 | .192 6023 | 37.87 | .206 3348 | 38.43 | .220 2741 | 39.02 |
| 35 | 2.179 2924 | 37.33 | 2.192 8295 | 37.88 | 2.206 5654 | 38.44 | 2.220 5082 | 39.03 |
| 36 | .179 5164 | 37.34 | .193 0568 | 37.88 | .206 7961 | 38.45 | .220 7424 | 39.04 |
| 37 | .179 7405 | 37.35 | .193 2841 | 37.89 | .207 0268 | 38.46 | .220 9767 | 39.05 |
| 38 | .179 9646 | 37.36 | .193 5115 | 37.90 | .207 2575 | 38.47 | .221 2110 | 39.06 |
| 39 | .180 1888 | 37.37 | .193 7389 | 37.91 | .207 4884 | 38.48 | .221 4453 | 39.07 |
| 40 | 2.180 4131 | 37.38 | 2.193 9664 | 37.92 | 2.207 7193 | 38.49 | 2.221 6797 | 39.08 |
| 41 | .180 6374 | 37.39 | .194 1940 | 37.93 | .207 9502 | 38.50 | .221 9142 | 39.09 |
| 42 | .180 8617 | 37.40 | .194 4216 | 37.94 | .208 1812 | 38.51 | .222 1488 | 39.10 |
| 43 | .181 0861 | 37.41 | .194 6493 | 37.95 | .208 4123 | 38.52 | .222 3834 | 39.11 |
| 44 | .181 3106 | 37.41 | .194 8770 | 37.96 | .208 6434 | 38.53 | .222 6180 | 39.12 |
| 45 | 2.181 5351 | 37.42 | 2.195 1048 | 37.97 | 2.208 8746 | 38.54 | 2.222 8528 | 39.13 |
| 46 | .181 7597 | 37.43 | .195 3326 | 37.98 | .209 1058 | 38.54 | .223 0876 | 39.14 |
| 47 | .181 9843 | 37.44 | .195 5605 | 37.99 | .209 3371 | 38.55 | .223 3224 | 39.15 |
| 48 | .182 1089 | 37.45 | .195 7885 | 38.00 | .209 5685 | 38.56 | .223 5573 | 39.16 |
| 49 | .182 4337 | 37.46 | .196 0165 | 38.00 | .209 7999 | 38.57 | .223 7923 | 39.17 |
| 50 | 2.182 6584 | 37.47 | 2.196 2445 | 38.01 | 2.210 0314 | 38.58 | 2.224 0273 | 39.18 |
| 51 | .182 8833 | 37.48 | .196 4726 | 38.02 | .210 2629 | 38.59 | .224 2624 | 39.19 |
| 52 | .183 1082 | 37.49 | .196 7008 | 38.03 | .210 4945 | 38.60 | .224 4975 | 39.20 |
| 53 | .183 3331 | 37.49 | .196 9290 | 38.04 | .210 7261 | 38.61 | .224 7327 | 39.21 |
| 54 | .183 5581 | 37.50 | .197 1573 | 38.05 | .210 9578 | 38.62 | .224 9680 | 39.22 |
| 55 | 2.183 7831 | 37.51 | 2.197 3856 | 38.06 | 2.211 1896 | 38.63 | 2.225 2033 | 39.23 |
| 56 | .184 0082 | 37.52 | .197 6140 | 38.07 | .211 4214 | 38.64 | .225 4387 | 39.24 |
| 57 | .184 2334 | 37.53 | .197 8425 | 38.08 | .211 6533 | 38.65 | .225 6741 | 39.25 |
| 58 | .184 4586 | 37.54 | .198 0710 | 38.09 | .211 8852 | 38.66 | .225 9096 | 39.26 |
| 59 | .184 6839 | 37.55 | .198 2995 | 38.10 | .212 1172 | 38.67 | .226 1452 | 39.27 |
| 60 | 2.184 9092 | 37.56 | 2.198 5282 | 38.11 | 2.212 3493 | 38.68 | 2.226 3808 | 39.28 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 108° | | 109° | | 110° | | 111° | |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | .226 3808 | 39.28 | .240 6314 | 39.90 | .255 1099 | 40.54 | .269 8255 | 41.21 |
| 1 | .226 6165 | 39.29 | .240 8708 | 39.91 | .255 3532 | 40.55 | .270 0728 | 41.23 |
| 2 | .226 8523 | 39.30 | .241 1103 | 39.92 | .255 5965 | 40.56 | .270 3202 | 41.24 |
| 3 | .227 0881 | 39.31 | .241 3498 | 39.93 | .255 8399 | 40.58 | .270 5676 | 41.25 |
| 4 | .227 3240 | 39.32 | .241 5894 | 39.94 | .256 0834 | 40.59 | .270 8152 | 41.26 |
| 5 | .227 5599 | 39.33 | .241 8291 | 39.95 | .256 3270 | 40.60 | .271 0628 | 41.27 |
| 6 | .227 7959 | 39.34 | .242 0688 | 39.96 | .256 5706 | 40.61 | .271 3104 | 41.28 |
| 7 | .228 0320 | 39.35 | .242 3086 | 39.97 | .256 8143 | 40.62 | .271 5582 | 41.29 |
| 8 | .228 2681 | 39.36 | .242 5485 | 39.98 | .257 0580 | 40.63 | .271 8060 | 41.30 |
| 9 | .228 5043 | 39.37 | .242 7884 | 39.99 | .257 3019 | 40.64 | .272 0538 | 41.32 |
| 10 | .228 7405 | 39.38 | .243 0284 | 40.00 | .257 5458 | 40.65 | .272 3018 | 41.33 |
| 11 | .228 9768 | 39.39 | .243 2685 | 40.01 | .257 7897 | 40.66 | .272 5498 | 41.34 |
| 12 | .229 2131 | 39.40 | .243 5086 | 40.02 | .258 0337 | 40.68 | .272 7979 | 41.35 |
| 13 | .229 4496 | 39.41 | .243 7488 | 40.03 | .258 2778 | 40.69 | .273 0460 | 41.36 |
| 14 | .229 6861 | 39.42 | .243 9890 | 40.05 | .258 5220 | 40.70 | .273 2942 | 41.38 |
| 15 | .229 9226 | 39.43 | .244 2293 | 40.06 | .258 7662 | 40.71 | .273 5425 | 41.39 |
| 16 | .230 1592 | 39.44 | .244 4697 | 40.07 | .259 0105 | 40.72 | .273 7909 | 41.40 |
| 17 | .230 3959 | 39.45 | .244 7101 | 40.08 | .259 2548 | 40.73 | .274 0393 | 41.41 |
| 18 | .230 6326 | 39.46 | .244 9506 | 40.09 | .259 4992 | 40.74 | .274 2878 | 41.42 |
| 19 | .230 8694 | 39.47 | .245 1912 | 40.10 | .259 7437 | 40.75 | .274 5364 | 41.43 |
| 20 | .231 1063 | 39.48 | .245 4318 | 40.11 | .259 9883 | 40.76 | .274 7850 | 41.44 |
| 21 | .231 3432 | 39.49 | .245 6725 | 40.12 | .260 2329 | 40.78 | .275 0337 | 41.46 |
| 22 | .231 5802 | 39.50 | .245 9132 | 40.13 | .260 4776 | 40.79 | .275 2825 | 41.47 |
| 23 | .231 8172 | 39.51 | .246 1541 | 40.14 | .260 7223 | 40.80 | .275 5313 | 41.48 |
| 24 | .232 0543 | 39.52 | .246 3949 | 40.15 | .260 9671 | 40.81 | .275 7802 | 41.49 |
| 25 | .232 2915 | 39.53 | .246 6359 | 40.16 | .261 2120 | 40.82 | .276 0292 | 41.50 |
| 26 | .232 5287 | 39.54 | .246 8769 | 40.17 | .261 4570 | 40.83 | .276 2783 | 41.51 |
| 27 | .232 7660 | 39.55 | .247 1180 | 40.18 | .261 7020 | 40.84 | .276 5274 | 41.53 |
| 28 | .233 0033 | 39.56 | .247 3591 | 40.19 | .261 9471 | 40.85 | .276 7766 | 41.54 |
| 29 | .233 2407 | 39.57 | .247 6003 | 40.21 | .262 1922 | 40.86 | .277 0258 | 41.55 |
| 30 | .233 4782 | 39.58 | .247 8416 | 40.22 | .262 4374 | 40.88 | .277 2752 | 41.56 |
| 31 | .233 7157 | 39.59 | .248 0829 | 40.23 | .262 6827 | 40.89 | .277 5246 | 41.57 |
| 32 | .233 9533 | 39.60 | .248 3243 | 40.24 | .262 9281 | 40.90 | .277 7740 | 41.58 |
| 33 | .234 1910 | 39.61 | .248 5658 | 40.25 | .263 1735 | 40.91 | .278 0236 | 41.60 |
| 34 | .234 4287 | 39.63 | .248 8073 | 40.26 | .263 4190 | 40.92 | .278 2732 | 41.61 |
| 35 | .234 6665 | 39.64 | .249 0489 | 40.27 | .263 6645 | 40.93 | .278 5229 | 41.62 |
| 36 | .234 9043 | 39.65 | .249 2906 | 40.28 | .263 9102 | 40.94 | .278 7726 | 41.63 |
| 37 | .235 1422 | 39.66 | .249 5323 | 40.29 | .264 1559 | 40.95 | .279 0224 | 41.64 |
| 38 | .235 3802 | 39.67 | .249 7741 | 40.30 | .264 4016 | 40.96 | .279 2723 | 41.65 |
| 39 | .235 6183 | 39.68 | .250 0159 | 40.31 | .264 6474 | 40.98 | .279 5223 | 41.67 |
| 40 | .235 8563 | 39.69 | .250 2578 | 40.32 | .264 8933 | 40.99 | .279 7723 | 41.68 |
| 41 | .236 0945 | 39.70 | .250 4998 | 40.34 | .265 1393 | 41.00 | .280 0224 | 41.69 |
| 42 | .236 3327 | 39.71 | .250 7419 | 40.35 | .265 3853 | 41.01 | .280 2726 | 41.70 |
| 43 | .236 5710 | 39.72 | .250 9840 | 40.36 | .265 6314 | 41.02 | .280 5228 | 41.71 |
| 44 | .236 8093 | 39.73 | .251 2262 | 40.37 | .265 8776 | 41.03 | .280 7731 | 41.72 |
| 45 | .237 0478 | 39.74 | .251 4684 | 40.38 | .266 1238 | 41.04 | .281 0235 | 41.74 |
| 46 | .237 2862 | 39.75 | .251 7107 | 40.39 | .266 3701 | 41.06 | .281 2740 | 41.75 |
| 47 | .237 5247 | 39.76 | .251 9531 | 40.40 | .266 6165 | 41.07 | .281 5245 | 41.76 |
| 48 | .237 7633 | 39.77 | .252 1955 | 40.41 | .266 8629 | 41.08 | .281 7751 | 41.77 |
| 49 | .238 0020 | 39.78 | .252 4380 | 40.42 | .267 1094 | 41.09 | .282 0258 | 41.78 |
| 50 | .238 2407 | 39.79 | .252 6806 | 40.43 | .267 3560 | 41.10 | .282 2765 | 41.80 |
| 51 | .238 4795 | 39.80 | .252 9232 | 40.44 | .267 6026 | 41.11 | .282 5273 | 41.81 |
| 52 | .238 7284 | 39.81 | .253 1659 | 40.46 | .267 8493 | 41.12 | .282 7782 | 41.82 |
| 53 | .238 9673 | 39.82 | .253 4087 | 40.47 | .268 0961 | 41.13 | .283 0291 | 41.83 |
| 54 | .239 1962 | 39.83 | .253 6515 | 40.48 | .268 3430 | 41.15 | .283 2801 | 41.84 |
| 55 | .239 4353 | 39.84 | .253 8944 | 40.49 | .268 5899 | 41.16 | .283 5312 | 41.85 |
| 56 | .239 6744 | 39.86 | .254 1374 | 40.50 | .268 8369 | 41.17 | .283 7824 | 41.87 |
| 57 | .239 9235 | 39.87 | .254 3804 | 40.51 | .269 0839 | 41.18 | .284 0336 | 41.88 |
| 58 | .240 1528 | 39.88 | .254 6235 | 40.52 | .269 3310 | 41.19 | .284 2849 | 41.89 |
| 59 | .240 3921 | 39.89 | .254 8666 | 40.53 | .269 5782 | 41.20 | .284 5363 | 41.90 |
| 60 | .240 6314 | 39.90 | .255 1099 | 40.54 | .269 8255 | 41.21 | .284 7878 | 41.91 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 112° | | 113° | | 114° | | 115° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 2.284 7878 | 41.91 | 2.300 0067 | 42.64 | 2.315 4927 | 43.40 | 2.331 2564 | 44.18 |
| 1 | .285 0393 | 41.93 | .300 2626 | 42.65 | .315 7531 | 43.41 | .331 5216 | 44.20 |
| 2 | .285 2909 | 41.94 | .300 5186 | 42.67 | .316 0136 | 43.42 | .331 7868 | 44.21 |
| 3 | .285 5425 | 41.95 | .300 7746 | 42.68 | .316 2742 | 43.44 | .332 0521 | 44.22 |
| 4 | .285 7943 | 41.96 | .301 0307 | 42.69 | .316 5348 | 43.45 | .332 3175 | 44.24 |
| 5 | 2.286 0461 | 41.97 | 2.301 2869 | 42.70 | 2.316 7956 | 43.46 | 2.332 5830 | 44.25 |
| 6 | .286 2979 | 41.99 | .301 5431 | 42.72 | .317 0564 | 43.47 | .332 8485 | 44.26 |
| 7 | .286 5499 | 42.00 | .301 7995 | 42.73 | .317 3173 | 43.49 | .333 1141 | 44.28 |
| 8 | .286 8019 | 42.01 | .302 0559 | 42.74 | .317 5782 | 43.50 | .333 3799 | 44.29 |
| 9 | .287 0540 | 42.02 | .302 3123 | 42.75 | .317 8393 | 43.51 | .333 6456 | 44.31 |
| 10 | 2.287 3062 | 42.03 | 2.302 5689 | 42.76 | 2.318 1004 | 43.53 | 2.333 9115 | 44.32 |
| 11 | .287 5584 | 42.04 | .302 8255 | 42.78 | .318 3616 | 43.54 | .334 1775 | 44.33 |
| 12 | .287 8107 | 42.06 | .303 0822 | 42.79 | .318 6229 | 43.55 | .334 4435 | 44.34 |
| 13 | .288 0631 | 42.07 | .303 3390 | 42.80 | .318 8842 | 43.56 | .334 7096 | 44.36 |
| 14 | .288 3155 | 42.08 | .303 5958 | 42.81 | .319 1456 | 43.58 | .334 9758 | 44.37 |
| 15 | 2.288 5680 | 42.09 | 2.303 8528 | 42.83 | 2.319 4072 | 43.59 | 2.335 2421 | 44.39 |
| 16 | .288 8206 | 42.10 | .304 1098 | 42.84 | .319 6687 | 43.60 | .335 5084 | 44.40 |
| 17 | .289 0733 | 42.12 | .304 3668 | 42.85 | .319 9304 | 43.62 | .335 7749 | 44.41 |
| 18 | .289 3260 | 42.13 | .304 6240 | 42.86 | .320 1921 | 43.63 | .336 0414 | 44.43 |
| 19 | .289 5788 | 42.14 | .304 8812 | 42.88 | .320 4540 | 43.64 | .336 3080 | 44.44 |
| 20 | 2.289 8317 | 42.15 | 2.305 1385 | 42.89 | 2.320 7159 | 43.66 | 2.336 5747 | 44.45 |
| 21 | .290 0847 | 42.16 | .305 3959 | 42.90 | .320 9778 | 43.67 | .336 8414 | 44.47 |
| 22 | .290 3377 | 42.18 | .305 6533 | 42.91 | .321 2399 | 43.68 | .337 1083 | 44.48 |
| 23 | .290 5908 | 42.19 | .305 9109 | 42.93 | .321 5020 | 43.69 | .337 3752 | 44.49 |
| 24 | .290 8440 | 42.20 | .306 1685 | 42.94 | .321 7642 | 43.70 | .337 6422 | 44.51 |
| 25 | 2.291 0972 | 42.21 | 2.306 4261 | 42.95 | 2.322 0265 | 43.72 | 2.337 9093 | 44.52 |
| 26 | .291 3505 | 42.22 | .306 6839 | 42.96 | .322 2889 | 43.73 | .338 1765 | 44.53 |
| 27 | .291 6039 | 42.24 | .306 9417 | 42.98 | .322 5513 | 43.75 | .338 4437 | 44.55 |
| 28 | .291 8574 | 42.25 | .307 1996 | 42.99 | .322 8139 | 43.76 | .338 7111 | 44.56 |
| 29 | .292 1109 | 42.26 | .307 4576 | 43.00 | .323 0765 | 43.77 | .338 9785 | 44.58 |
| 30 | 2.292 3645 | 42.27 | 2.307 7157 | 43.02 | 2.323 3391 | 43.79 | 2.339 2460 | 44.59 |
| 31 | .292 6182 | 42.29 | .307 9738 | 43.03 | .323 6019 | 43.80 | .339 5135 | 44.60 |
| 32 | .292 8719 | 42.30 | .308 2320 | 43.04 | .323 8647 | 43.81 | .339 7812 | 44.62 |
| 33 | .293 1258 | 42.31 | .308 4903 | 43.05 | .324 1277 | 43.83 | .340 0490 | 44.63 |
| 34 | .293 3797 | 42.32 | .308 7486 | 43.07 | .324 3907 | 43.84 | .340 3168 | 44.64 |
| 35 | 2.293 6336 | 42.33 | 2.309 0071 | 43.08 | 2.324 6537 | 43.85 | 2.340 5847 | 44.66 |
| 36 | .293 8877 | 42.35 | .309 2656 | 43.09 | .324 9169 | 43.87 | .340 8527 | 44.67 |
| 37 | .294 1418 | 42.36 | .309 5242 | 43.10 | .325 1801 | 43.88 | .341 1207 | 44.69 |
| 38 | .294 3960 | 42.37 | .309 7828 | 43.12 | .325 4434 | 43.89 | .341 3889 | 44.70 |
| 39 | .294 6503 | 42.38 | .310 0416 | 43.13 | .325 7068 | 43.91 | .341 6571 | 44.71 |
| 40 | 2.294 9046 | 42.40 | 2.310 3004 | 43.14 | 2.325 9703 | 43.92 | 2.341 9255 | 44.73 |
| 41 | .295 1590 | 42.41 | .310 5593 | 43.15 | .326 2339 | 43.93 | .342 1939 | 44.74 |
| 42 | .295 4135 | 42.42 | .310 8182 | 43.17 | .326 4975 | 43.94 | .342 4623 | 44.75 |
| 43 | .295 6680 | 42.43 | .311 0773 | 43.18 | .326 7612 | 43.96 | .342 7309 | 44.77 |
| 44 | .295 9227 | 42.44 | .311 3364 | 43.19 | .327 0250 | 43.97 | .342 9995 | 44.78 |
| 45 | 2.296 1774 | 42.46 | 2.311 5956 | 43.21 | 2.327 2889 | 43.98 | 2.343 2683 | 44.80 |
| 46 | .296 4321 | 42.47 | .311 8549 | 43.22 | .327 5528 | 44.00 | .343 5371 | 44.81 |
| 47 | .296 6870 | 42.48 | .312 1142 | 43.23 | .327 8168 | 44.01 | .343 8060 | 44.82 |
| 48 | .296 9419 | 42.49 | .312 3736 | 43.24 | .328 0809 | 44.02 | .344 0750 | 44.84 |
| 49 | .297 1969 | 42.51 | .312 6331 | 43.26 | .328 3451 | 44.04 | .344 3440 | 44.85 |
| 50 | 2.297 4520 | 42.52 | 2.312 8927 | 43.27 | 2.328 6094 | 44.05 | 2.344 6132 | 44.86 |
| 51 | .297 7071 | 42.53 | .313 1524 | 43.28 | .328 8737 | 44.06 | .344 8824 | 44.88 |
| 52 | .297 9623 | 42.54 | .313 4121 | 43.29 | .329 1382 | 44.08 | .345 1517 | 44.89 |
| 53 | .298 2176 | 42.55 | .313 6719 | 43.31 | .329 4027 | 44.09 | .345 4211 | 44.91 |
| 54 | .298 4730 | 42.57 | .313 9318 | 43.32 | .329 6672 | 44.10 | .345 6906 | 44.92 |
| 55 | 2.298 7284 | 42.58 | 2.314 1917 | 43.33 | 2.329 9319 | 44.12 | 2.345 9601 | 44.93 |
| 56 | .298 9839 | 42.59 | .314 4518 | 43.35 | .330 1967 | 44.13 | .346 2298 | 44.95 |
| 57 | .299 2395 | 42.60 | .314 7119 | 43.36 | .330 4615 | 44.14 | .346 4995 | 44.96 |
| 58 | .299 4952 | 42.61 | .314 9721 | 43.37 | .330 7264 | 44.16 | .346 7693 | 44.97 |
| 59 | .299 7509 | 42.63 | .315 2323 | 43.38 | .330 9914 | 44.17 | .347 0392 | 44.99 |
| 60 | 2.300 0067 | 42.64 | 2.315 4927 | 43.40 | 2.331 2564 | 44.18 | 2.347 3092 | 45.00 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 116° | | 117° | | 118° | | 119° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.347 3092 | 45.00 | 2.363 6626 | 45.86 | 2.380 3290 | 46.74 | 2.397 3210 | 47.66 |
| 1 | .347 5792 | 45.02 | .363 9378 | 45.87 | .380 6095 | 46.76 | .397 6070 | 47.68 |
| 2 | .347 8494 | 45.03 | .364 2131 | 45.88 | .380 8901 | 46.77 | .397 8931 | 47.70 |
| 3 | .348 1196 | 45.04 | .364 4885 | 45.90 | .381 1708 | 46.79 | .398 1794 | 47.71 |
| 4 | .348 3899 | 45.06 | .364 7639 | 45.91 | .381 4515 | 46.80 | .398 4657 | 47.73 |
| 5 | 2.348 6603 | 45.07 | 2.365 0394 | 45.93 | 2.381 7324 | 46.82 | 2.398 7521 | 47.74 |
| 6 | .348 9308 | 45.09 | .365 3150 | 45.94 | .382 0133 | 46.83 | .399 0386 | 47.76 |
| 7 | .349 2014 | 45.10 | .365 5907 | 45.96 | .382 2944 | 46.85 | .399 3252 | 47.77 |
| 8 | .349 4720 | 45.11 | .365 8665 | 45.97 | .382 5755 | 46.86 | .399 6119 | 47.79 |
| 9 | .349 7428 | 45.13 | .366 1423 | 45.99 | .382 8567 | 46.88 | .399 8987 | 47.81 |
| 10 | 2.350 0136 | 45.14 | 2.366 4183 | 46.00 | 2.383 1380 | 46.89 | 2.400 1856 | 47.82 |
| 11 | .350 2845 | 45.16 | .366 6944 | 46.01 | .383 4194 | 46.91 | .400 4725 | 47.84 |
| 12 | .350 5554 | 45.17 | .366 9705 | 46.03 | .383 7009 | 46.92 | .400 7596 | 47.85 |
| 13 | .350 8265 | 45.18 | .367 2467 | 46.04 | .383 9825 | 46.94 | .401 0468 | 47.87 |
| 14 | .351 0977 | 45.20 | .367 5230 | 46.06 | .384 2642 | 46.95 | .401 3340 | 47.89 |
| 15 | 2.351 3689 | 45.21 | 2.367 7994 | 46.07 | 2.384 5460 | 46.97 | 2.401 6214 | 47.90 |
| 16 | .351 6402 | 45.23 | .368 0759 | 46.09 | .384 8278 | 46.98 | .401 9088 | 47.92 |
| 17 | .351 9116 | 45.24 | .368 3525 | 46.10 | .385 1098 | 46.99 | .402 1964 | 47.93 |
| 18 | .352 1831 | 45.25 | .368 6291 | 46.12 | .385 3918 | 47.01 | .402 4840 | 47.95 |
| 19 | .352 4547 | 45.27 | .368 9059 | 46.13 | .385 6739 | 47.03 | .402 7718 | 47.97 |
| 20 | 2.352 7263 | 45.28 | 2.369 1827 | 46.15 | 2.385 9562 | 47.05 | 2.403 0596 | 47.98 |
| 21 | .352 9981 | 45.30 | .369 4596 | 46.16 | .386 2385 | 47.06 | .403 3475 | 48.00 |
| 22 | .353 2699 | 45.31 | .369 7367 | 46.18 | .386 5209 | 47.08 | .403 6356 | 48.01 |
| 23 | .353 5418 | 45.33 | .370 0138 | 46.19 | .386 8034 | 47.09 | .403 9237 | 48.03 |
| 24 | .353 8138 | 45.34 | .370 2909 | 46.21 | .387 0860 | 47.11 | .404 2119 | 48.04 |
| 25 | 2.354 0859 | 45.35 | 2.370 5682 | 46.22 | 2.387 3687 | 47.12 | 2.404 5002 | 48.06 |
| 26 | .354 3581 | 45.37 | .370 8456 | 46.24 | .387 6514 | 47.14 | .404 7886 | 48.08 |
| 27 | .354 6303 | 45.38 | .371 1230 | 46.25 | .387 9343 | 47.15 | .405 0771 | 48.09 |
| 28 | .354 9027 | 45.40 | .371 4006 | 46.26 | .388 2173 | 47.17 | .405 3657 | 48.11 |
| 29 | .355 1751 | 45.41 | .371 6782 | 46.28 | .388 5003 | 47.18 | .405 6544 | 48.12 |
| 30 | 2.355 4476 | 45.42 | 2.371 9559 | 46.29 | 2.388 7835 | 47.20 | 2.405 9432 | 48.14 |
| 31 | .355 7202 | 45.44 | .372 2337 | 46.31 | .389 0667 | 47.21 | .406 2321 | 48.16 |
| 32 | .355 9928 | 45.45 | .372 5116 | 46.32 | .389 3500 | 47.23 | .406 5211 | 48.17 |
| 33 | .356 2666 | 45.47 | .372 7896 | 46.34 | .389 6335 | 47.24 | .406 8102 | 48.19 |
| 34 | .356 5385 | 45.48 | .373 0677 | 46.35 | .389 9170 | 47.26 | .407 0993 | 48.20 |
| 35 | 2.356 8114 | 45.50 | 2.373 3459 | 46.37 | 2.390 2006 | 47.28 | 2.407 3886 | 48.22 |
| 36 | .357 0844 | 45.51 | .373 6241 | 46.38 | .390 4843 | 47.29 | .407 6780 | 48.24 |
| 37 | .357 3575 | 45.52 | .373 9024 | 46.40 | .390 7681 | 47.31 | .407 9674 | 48.25 |
| 38 | .357 6307 | 45.54 | .374 1809 | 46.41 | .391 0519 | 47.32 | .408 2570 | 48.27 |
| 39 | .357 9040 | 45.55 | .374 4594 | 46.43 | .391 3359 | 47.34 | .408 5467 | 48.28 |
| 40 | 2.358 1773 | 45.57 | 2.374 7380 | 46.44 | 2.391 6200 | 47.35 | 2.408 8364 | 48.30 |
| 41 | .358 4508 | 45.58 | .375 0167 | 46.46 | .392 9042 | 47.37 | .409 1263 | 48.32 |
| 42 | .358 7243 | 45.60 | .375 2955 | 46.47 | .392 1884 | 47.38 | .409 4162 | 48.33 |
| 43 | .358 9979 | 45.61 | .375 5744 | 46.49 | .392 4728 | 47.40 | .409 7063 | 48.35 |
| 44 | .359 2716 | 45.62 | .375 8533 | 46.50 | .392 7572 | 47.41 | .409 9964 | 48.37 |
| 45 | 2.359 5454 | 45.64 | 2.376 1324 | 46.51 | 2.393 0417 | 47.43 | 2.410 2866 | 48.38 |
| 46 | .359 8193 | 45.65 | .376 4115 | 46.53 | .393 3264 | 47.45 | .410 5770 | 48.40 |
| 47 | .360 0933 | 45.67 | .376 6908 | 46.55 | .393 6111 | 47.46 | .410 8674 | 48.41 |
| 48 | .360 3673 | 45.68 | .376 9701 | 46.56 | .393 8959 | 47.48 | .411 1579 | 48.43 |
| 49 | .360 6415 | 45.70 | .377 2495 | 46.58 | .394 1808 | 47.49 | .411 4486 | 48.45 |
| 50 | 2.360 9157 | 45.71 | 2.377 5290 | 46.59 | 2.394 4658 | 47.51 | 2.411 7393 | 48.46 |
| 51 | .361 1900 | 45.72 | .377 8086 | 46.60 | .394 7509 | 47.52 | .412 0301 | 48.48 |
| 52 | .361 4644 | 45.74 | .378 0883 | 46.62 | .395 0361 | 47.54 | .412 3210 | 48.49 |
| 53 | .361 7389 | 45.75 | .378 3681 | 46.64 | .395 3214 | 47.55 | .412 6120 | 48.51 |
| 54 | .362 0134 | 45.77 | .378 6479 | 46.65 | .395 6067 | 47.57 | .412 9031 | 48.53 |
| 55 | 2.362 2881 | 45.78 | 2.378 9279 | 46.67 | 2.395 8922 | 47.59 | 2.413 1944 | 48.54 |
| 56 | .362 5628 | 45.80 | .379 2079 | 46.68 | .396 1778 | 47.60 | .413 4857 | 48.56 |
| 57 | .362 8376 | 45.81 | .379 4881 | 46.70 | .396 4634 | 47.62 | .413 7771 | 48.58 |
| 58 | .363 1126 | 45.82 | .379 7683 | 46.71 | .396 7492 | 47.63 | .414 0686 | 48.59 |
| 59 | .363 3876 | 45.84 | .380 0486 | 46.73 | .397 0350 | 47.65 | .414 3602 | 48.61 |
| 60 | 2.363 6626 | 45.86 | 2.380 3290 | 46.74 | 2.397 3210 | 47.66 | 2.414 6519 | 48.62 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 120° | | 121° | | 122° | | 123° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 2.414 6519 | 48.62 | 2.432 3356 | 49.62 | 2.450 3868 | 50.67 | 2.468 8205 | 51.75 |
| 1 | .414 9437 | 48.64 | .432 6334 | 49.64 | .450 6908 | 50.68 | .469 1311 | 51.77 |
| 2 | .415 2356 | 48.66 | .432 9313 | 49.66 | .450 9950 | 50.70 | .469 4418 | 51.79 |
| 3 | .415 5276 | 48.67 | .433 2293 | 49.68 | .451 2992 | 50.72 | .469 7526 | 51.81 |
| 4 | .415 8197 | 48.69 | .433 5274 | 49.69 | .451 6036 | 50.74 | .470 0634 | 51.82 |
| 5 | 2.416 1119 | 48.71 | 2.433 8257 | 49.71 | 2.451 9081 | 50.75 | 2.470 3744 | 51.84 |
| 6 | .416 4042 | 48.72 | .434 1240 | 49.73 | .452 2127 | 50.77 | .470 6856 | 51.86 |
| 7 | .416 6965 | 48.74 | .434 4224 | 49.74 | .452 5174 | 50.79 | .470 9968 | 51.88 |
| 8 | .416 9890 | 48.76 | .434 7209 | 49.76 | .452 8222 | 50.81 | .471 3081 | 51.90 |
| 9 | .417 2816 | 48.77 | .435 0195 | 49.78 | .453 1271 | 50.83 | .471 6196 | 51.92 |
| 10 | 2.417 5743 | 48.79 | 2.435 3182 | 49.80 | 2.453 4321 | 50.84 | 2.471 9311 | 51.94 |
| 11 | .417 8671 | 48.81 | .435 6171 | 49.81 | .453 7372 | 50.86 | .472 2428 | 51.95 |
| 12 | .418 1600 | 48.82 | .435 9160 | 49.83 | .454 0424 | 50.88 | .472 5546 | 51.97 |
| 13 | .418 4529 | 48.84 | .436 2150 | 49.85 | .454 3477 | 50.90 | .472 8665 | 51.99 |
| 14 | .418 7460 | 48.85 | .436 5141 | 49.86 | .454 6532 | 50.92 | .473 1785 | 52.01 |
| 15 | 2.419 0392 | 48.87 | 2.436 8134 | 49.88 | 2.454 9587 | 50.93 | 2.473 4906 | 52.03 |
| 16 | .419 3325 | 48.89 | .437 1127 | 49.90 | .455 2644 | 50.95 | .473 8028 | 52.05 |
| 17 | .419 6258 | 48.90 | .437 4122 | 49.92 | .455 5701 | 50.97 | .474 1152 | 52.07 |
| 18 | .419 9193 | 48.92 | .437 7117 | 49.93 | .455 8760 | 50.99 | .474 4276 | 52.09 |
| 19 | .420 2129 | 48.94 | .438 0114 | 49.95 | .456 1820 | 51.00 | .474 7402 | 52.10 |
| 20 | 2.420 5066 | 48.95 | 2.438 3111 | 49.97 | 2.456 4881 | 51.02 | 2.475 0529 | 52.12 |
| 21 | .420 8003 | 48.97 | .438 6110 | 49.98 | .456 7943 | 51.04 | .475 3657 | 52.14 |
| 22 | .421 0942 | 48.99 | .438 9109 | 50.00 | .457 1006 | 51.06 | .475 6786 | 52.16 |
| 23 | .421 3882 | 49.00 | .439 2110 | 50.02 | .457 4070 | 51.08 | .475 9916 | 52.18 |
| 24 | .421 6822 | 49.02 | .439 5112 | 50.04 | .457 7135 | 51.09 | .476 3047 | 52.20 |
| 25 | 2.421 9764 | 49.03 | 2.439 8114 | 50.05 | 2.458 0201 | 51.11 | 2.476 6180 | 52.22 |
| 26 | .422 2707 | 49.05 | .440 1118 | 50.07 | .458 3268 | 51.13 | .476 9313 | 52.23 |
| 27 | .422 5650 | 49.07 | .440 4123 | 50.09 | .458 6337 | 51.15 | .477 2448 | 52.25 |
| 28 | .422 8595 | 49.09 | .440 7129 | 50.11 | .458 9406 | 51.17 | .477 5584 | 52.27 |
| 29 | .423 1541 | 49.10 | .441 0136 | 50.12 | .459 2477 | 51.18 | .477 8721 | 52.29 |
| 30 | 2.423 4488 | 49.12 | 2.441 3143 | 50.14 | 2.459 5548 | 51.20 | 2.478 1859 | 52.31 |
| 31 | .423 7435 | 49.14 | .441 6152 | 50.16 | .459 8621 | 51.22 | .478 4998 | 52.33 |
| 32 | .424 0384 | 49.15 | .441 9162 | 50.18 | .460 1695 | 51.24 | .478 8138 | 52.35 |
| 33 | .424 3334 | 49.17 | .442 2173 | 50.19 | .460 4770 | 51.26 | .479 1280 | 52.37 |
| 34 | .424 6284 | 49.19 | .442 5185 | 50.21 | .460 7846 | 51.28 | .479 4422 | 52.39 |
| 35 | 2.424 9236 | 49.20 | 2.442 8199 | 50.23 | 2.461 0923 | 51.29 | 2.479 7566 | 52.40 |
| 36 | .425 2189 | 49.22 | .443 1213 | 50.24 | .461 4001 | 51.31 | .480 0711 | 52.42 |
| 37 | .425 5142 | 49.24 | .443 4228 | 50.26 | .461 7080 | 51.33 | .480 3857 | 52.44 |
| 38 | .425 8097 | 49.25 | .443 7244 | 50.28 | .462 0161 | 51.35 | .480 7004 | 52.46 |
| 39 | .426 1053 | 49.27 | .444 0261 | 50.30 | .462 3242 | 51.37 | .481 0152 | 52.48 |
| 40 | 2.426 4010 | 49.29 | 2.444 3280 | 50.31 | 2.462 6325 | 51.38 | 2.481 3301 | 52.50 |
| 41 | .426 6967 | 49.30 | .444 6299 | 50.33 | .462 9408 | 51.40 | .481 6452 | 52.52 |
| 42 | .426 9926 | 49.32 | .444 9320 | 50.35 | .463 2493 | 51.42 | .481 9604 | 52.54 |
| 43 | .427 2886 | 49.34 | .445 2341 | 50.37 | .463 5579 | 51.44 | .482 2756 | 52.56 |
| 44 | .427 5847 | 49.35 | .445 5364 | 50.38 | .463 8666 | 51.46 | .482 5910 | 52.58 |
| 45 | 2.427 8808 | 49.37 | 2.445 8387 | 50.40 | 2.464 1754 | 51.48 | 2.482 9065 | 52.59 |
| 46 | .428 1771 | 49.39 | .446 1412 | 50.42 | .464 4843 | 51.49 | .483 2222 | 52.61 |
| 47 | .428 4735 | 49.40 | .446 4437 | 50.44 | .464 7933 | 51.51 | .483 5379 | 52.63 |
| 48 | .428 7700 | 49.42 | .446 7464 | 50.45 | .465 1024 | 51.53 | .483 8537 | 52.65 |
| 49 | .429 0665 | 49.44 | .447 0492 | 50.47 | .465 4116 | 51.55 | .484 1697 | 52.67 |
| 50 | 2.429 3632 | 49.46 | 2.447 3521 | 50.49 | 2.465 7210 | 51.57 | 2.484 4858 | 52.69 |
| 51 | .429 6600 | 49.47 | .447 6551 | 50.51 | .466 0305 | 51.59 | .484 8020 | 52.71 |
| 52 | .429 9569 | 49.49 | .447 9582 | 50.53 | .466 3400 | 51.60 | .485 1183 | 52.73 |
| 53 | .430 2539 | 49.51 | .448 2614 | 50.54 | .466 6497 | 51.62 | .485 4347 | 52.75 |
| 54 | .430 5510 | 49.52 | .448 5647 | 50.56 | .466 9595 | 51.64 | .485 7513 | 52.77 |
| 55 | 2.430 8482 | 49.54 | 2.448 8681 | 50.58 | 2.467 2694 | 51.66 | 2.486 0679 | 52.78 |
| 56 | .431 1455 | 49.56 | .449 1716 | 50.60 | .467 5794 | 51.68 | .486 3847 | 52.80 |
| 57 | .431 4428 | 49.57 | .449 4753 | 50.61 | .467 8895 | 51.70 | .486 7016 | 52.82 |
| 58 | .431 7403 | 49.59 | .449 7790 | 50.63 | .468 1997 | 51.71 | .487 0186 | 52.84 |
| 59 | .432 0379 | 49.61 | .450 0828 | 50.65 | .468 5101 | 51.73 | .487 3357 | 52.86 |
| 60 | 2.432 3356 | 49.62 | 2.450 3868 | 50.67 | 2.468 8205 | 51.75 | 2.487 6529 | 52.88 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit

| v. | 124° | | | 125° | | | 126° | | | 127° | | |
|----|------------|-----------|--|------------|-----------|--|------------|-----------|--|------------|-----------|--|
| | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | |
| 0 | 2.487 6529 | 52.88 | | 2.506 9006 | 54.06 | | 2.526 5813 | 55.29 | | 2.546 7135 | 56.57 | |
| 1 | .487 9702 | 52.90 | | .507 2251 | 54.08 | | .526 9131 | 55.31 | | .547 0530 | 56.59 | |
| 2 | .488 2877 | 52.92 | | .507 5496 | 54.10 | | .527 2450 | 55.33 | | .547 3926 | 56.61 | |
| 3 | .488 6053 | 52.94 | | .507 8742 | 54.12 | | .527 5771 | 55.35 | | .547 7323 | 56.63 | |
| 4 | .488 9230 | 52.96 | | .508 1990 | 54.14 | | .527 9092 | 55.37 | | .548 0722 | 56.65 | |
| 5 | 2.489 2408 | 52.98 | | 2.508 5239 | 54.16 | | 2.528 2415 | 55.39 | | 2.548 4122 | 56.68 | |
| 6 | .489 5587 | 53.00 | | .508 8489 | 54.18 | | .528 5739 | 55.41 | | .548 7523 | 56.70 | |
| 7 | .489 8767 | 53.02 | | .509 1741 | 54.20 | | .528 9065 | 55.43 | | .549 0926 | 56.72 | |
| 8 | .490 1949 | 53.03 | | .509 4993 | 54.22 | | .529 2391 | 55.45 | | .549 4330 | 56.74 | |
| 9 | .490 5132 | 53.05 | | .509 8247 | 54.24 | | .529 5719 | 55.48 | | .549 7735 | 56.76 | |
| 10 | 2.490 8315 | 53.07 | | 2.510 1502 | 54.26 | | 2.529 9048 | 55.50 | | 2.550 1141 | 56.79 | |
| 11 | .491 1500 | 53.09 | | .510 4758 | 54.28 | | .530 2379 | 55.52 | | .550 4549 | 56.81 | |
| 12 | .491 4686 | 53.11 | | .510 8016 | 54.30 | | .530 5710 | 55.54 | | .550 7958 | 56.83 | |
| 13 | .491 7874 | 53.13 | | .511 1274 | 54.32 | | .530 9043 | 55.56 | | .551 1369 | 56.85 | |
| 14 | .492 1063 | 53.15 | | .511 4534 | 54.34 | | .531 2378 | 55.58 | | .551 4781 | 56.87 | |
| 15 | 2.492 4252 | 53.17 | | 2.511 7795 | 54.36 | | 2.531 5713 | 55.60 | | 2.551 8194 | 56.90 | |
| 16 | .492 7443 | 53.19 | | .512 1057 | 54.38 | | .531 9050 | 55.62 | | .552 1608 | 56.92 | |
| 17 | .493 0635 | 53.21 | | .512 4321 | 54.40 | | .532 2388 | 55.64 | | .552 5024 | 56.94 | |
| 18 | .493 3828 | 53.23 | | .512 7586 | 54.42 | | .532 5727 | 55.67 | | .552 8441 | 56.96 | |
| 19 | .493 7023 | 53.25 | | .513 0852 | 54.44 | | .532 9068 | 55.69 | | .553 1859 | 56.98 | |
| 20 | 2.494 0218 | 53.27 | | 2.513 4119 | 54.46 | | 2.533 2410 | 55.71 | | 2.553 5279 | 57.01 | |
| 21 | .494 3415 | 53.29 | | .513 7387 | 54.48 | | .533 5753 | 55.73 | | .553 8700 | 57.03 | |
| 22 | .494 6613 | 53.31 | | .514 0657 | 54.50 | | .533 9097 | 55.75 | | .554 2122 | 57.05 | |
| 23 | .494 9812 | 53.33 | | .514 3927 | 54.52 | | .534 2443 | 55.77 | | .554 5546 | 57.07 | |
| 24 | .495 3012 | 53.35 | | .514 7199 | 54.54 | | .534 5790 | 55.79 | | .554 8971 | 57.10 | |
| 25 | 2.495 6213 | 53.37 | | 2.515 0473 | 54.56 | | 2.534 9138 | 55.81 | | 2.555 2398 | 57.12 | |
| 26 | .495 9416 | 53.39 | | .515 3747 | 54.58 | | .535 2487 | 55.84 | | .555 5825 | 57.14 | |
| 27 | .496 2619 | 53.41 | | .515 7023 | 54.60 | | .535 5838 | 55.86 | | .555 9254 | 57.16 | |
| 28 | .496 5824 | 53.42 | | .516 0300 | 54.63 | | .535 9190 | 55.88 | | .556 2685 | 57.18 | |
| 29 | .496 9032 | 53.44 | | .516 3578 | 54.65 | | .536 2543 | 55.90 | | .556 6116 | 57.21 | |
| 30 | 2.497 2238 | 53.46 | | 2.516 6857 | 54.67 | | 2.536 5898 | 55.92 | | 2.556 9549 | 57.23 | |
| 31 | .497 5446 | 53.48 | | .517 0138 | 54.69 | | .536 9254 | 55.94 | | .557 2984 | 57.25 | |
| 32 | .497 8656 | 53.50 | | .517 3420 | 54.71 | | .537 2611 | 55.96 | | .557 6420 | 57.27 | |
| 33 | .498 1867 | 53.52 | | .517 6703 | 54.73 | | .537 5970 | 55.98 | | .557 9857 | 57.29 | |
| 34 | .498 5079 | 53.54 | | .517 9987 | 54.75 | | .537 9329 | 56.01 | | .558 3295 | 57.32 | |
| 35 | 2.498 8292 | 53.56 | | 2.518 3273 | 54.77 | | 2.538 2690 | 56.03 | | 2.558 6735 | 57.34 | |
| 36 | .499 1506 | 53.58 | | .518 6559 | 54.79 | | .538 6052 | 56.05 | | .559 0176 | 57.36 | |
| 37 | .499 4721 | 53.60 | | .518 9847 | 54.81 | | .538 9416 | 56.07 | | .559 3618 | 57.38 | |
| 38 | .499 7938 | 53.62 | | .519 3137 | 54.83 | | .539 2781 | 56.09 | | .559 7062 | 57.41 | |
| 39 | .500 1156 | 53.64 | | .519 6427 | 54.85 | | .539 6147 | 56.11 | | .560 0507 | 57.43 | |
| 40 | 2.500 4375 | 53.66 | | 2.519 9719 | 54.87 | | 2.539 9514 | 56.13 | | 2.560 3953 | 57.45 | |
| 41 | .500 7595 | 53.68 | | .520 3012 | 54.89 | | .540 2883 | 56.15 | | .560 7401 | 57.47 | |
| 42 | .501 0817 | 53.70 | | .520 6306 | 54.91 | | .540 6253 | 56.18 | | .561 0850 | 57.50 | |
| 43 | .501 4039 | 53.72 | | .520 9601 | 54.93 | | .540 9625 | 56.20 | | .561 4301 | 57.52 | |
| 44 | .501 7263 | 53.74 | | .521 2898 | 54.95 | | .541 2997 | 56.22 | | .561 7753 | 57.54 | |
| 45 | 2.502 0488 | 53.76 | | 2.521 6196 | 54.97 | | 2.541 6371 | 56.24 | | 2.562 1206 | 57.56 | |
| 46 | .502 3714 | 53.78 | | .521 9495 | 54.99 | | .541 9746 | 56.26 | | .562 4660 | 57.59 | |
| 47 | .502 6942 | 53.80 | | .522 2795 | 55.02 | | .542 3123 | 56.29 | | .562 8116 | 57.61 | |
| 48 | .503 0170 | 53.82 | | .522 6097 | 55.04 | | .542 6500 | 56.31 | | .563 1574 | 57.63 | |
| 49 | .503 3400 | 53.84 | | .522 9400 | 55.06 | | .542 9880 | 56.33 | | .563 5032 | 57.65 | |
| 50 | 2.503 6631 | 53.86 | | 2.523 2704 | 55.08 | | 2.543 3260 | 56.35 | | 2.563 8492 | 57.68 | |
| 51 | .503 9863 | 53.88 | | .523 6009 | 55.10 | | .543 6641 | 56.37 | | .564 1953 | 57.70 | |
| 52 | .504 3096 | 53.90 | | .523 9316 | 55.12 | | .544 0024 | 56.39 | | .564 5416 | 57.72 | |
| 53 | .504 6331 | 53.92 | | .524 2624 | 55.14 | | .544 3409 | 56.42 | | .564 8880 | 57.74 | |
| 54 | .504 9567 | 53.94 | | .524 5933 | 55.16 | | .544 6794 | 56.44 | | .565 2345 | 57.77 | |
| 55 | 2.505 2804 | 53.96 | | 2.524 9243 | 55.18 | | 2.545 0181 | 56.46 | | 2.565 5812 | 57.79 | |
| 56 | .505 6042 | 53.98 | | .525 2555 | 55.20 | | .545 3569 | 56.48 | | .565 9280 | 57.81 | |
| 57 | .505 9282 | 54.00 | | .525 5867 | 55.22 | | .545 6959 | 56.50 | | .566 2750 | 57.84 | |
| 58 | .506 2522 | 54.02 | | .525 9181 | 55.24 | | .546 0350 | 56.52 | | .566 6221 | 57.86 | |
| 59 | .506 5763 | 54.04 | | .526 2497 | 55.26 | | .546 3742 | 56.55 | | .566 9693 | 57.88 | |
| 60 | 2.506 9006 | 54.06 | | 2.526 5813 | 55.29 | | 2.546 7135 | 56.57 | | 2.567 3166 | 57.90 | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 128° | | 129° | | 130° | | 131° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.567 3166 | 57.90 | 2.588 4112 | 59.30 | 2.610 0188 | 60.75 | 2.632 1622 | 62.28 |
| 1 | .567 6641 | 57.93 | .588 7670 | 59.32 | .610 3834 | 60.78 | .632 5360 | 62.30 |
| 2 | .568 0117 | 57.95 | .589 1230 | 59.35 | .610 7481 | 60.80 | .632 9099 | 62.33 |
| 3 | .568 3595 | 57.97 | .589 4792 | 59.37 | .611 1130 | 60.83 | .633 2839 | 62.35 |
| 4 | .568 7074 | 57.99 | .589 8355 | 59.39 | .611 4781 | 60.85 | .633 6581 | 62.38 |
| 5 | 2.569 0554 | 58.02 | 2.590 1919 | 59.42 | 2.611 8433 | 60.88 | 2.634 0325 | 62.41 |
| 6 | .569 4036 | 58.04 | .590 5485 | 59.44 | .612 2086 | 60.90 | .634 4070 | 62.43 |
| 7 | .569 7519 | 58.06 | .590 9052 | 59.47 | .612 5741 | 60.93 | .634 7817 | 62.46 |
| 8 | .570 1004 | 58.09 | .591 2620 | 59.49 | .612 9397 | 60.95 | .635 1565 | 62.48 |
| 9 | .570 4490 | 58.11 | .591 6190 | 59.51 | .613 3055 | 60.98 | .635 5315 | 62.51 |
| 10 | 2.570 7977 | 58.13 | 2.591 9762 | 59.54 | 2.613 6715 | 61.00 | 2.635 9066 | 62.54 |
| 11 | .571 1405 | 58.15 | .592 3335 | 59.56 | .614 0376 | 61.03 | .636 2819 | 62.56 |
| 12 | .571 4955 | 58.18 | .592 6909 | 59.58 | .614 4038 | 61.05 | .636 6573 | 62.59 |
| 13 | .571 8447 | 58.20 | .593 0485 | 59.61 | .614 7702 | 61.08 | .637 0329 | 62.61 |
| 14 | .572 1939 | 58.22 | .593 4062 | 59.63 | .615 1368 | 61.10 | .637 4087 | 62.64 |
| 15 | 2.572 5434 | 58.25 | 2.593 7641 | 59.66 | 2.615 5035 | 61.13 | 2.637 7846 | 62.67 |
| 16 | .572 8929 | 58.27 | .594 1221 | 59.68 | .615 8703 | 61.15 | .638 1607 | 62.69 |
| 17 | .573 2426 | 58.29 | .594 4803 | 59.70 | .616 2373 | 61.18 | .638 5369 | 62.72 |
| 18 | .573 5924 | 58.32 | .594 8386 | 59.73 | .616 6045 | 61.20 | .638 9133 | 62.75 |
| 19 | .573 9424 | 58.34 | .595 1970 | 59.75 | .616 9718 | 61.23 | .639 2899 | 62.77 |
| 20 | 2.574 2925 | 58.36 | 2.595 5556 | 59.78 | 2.617 3392 | 61.25 | 2.639 6666 | 62.80 |
| 21 | .574 6427 | 58.38 | .595 9143 | 59.80 | .617 7068 | 61.28 | .640 0435 | 62.82 |
| 22 | .574 9931 | 58.41 | .596 2732 | 59.82 | .618 0746 | 61.30 | .640 4205 | 62.85 |
| 23 | .575 3436 | 58.43 | .596 6322 | 59.85 | .618 4425 | 61.33 | .640 7977 | 62.88 |
| 24 | .575 6943 | 58.45 | .596 9914 | 59.87 | .618 8105 | 61.36 | .641 1750 | 62.90 |
| 25 | 2.576 0451 | 58.48 | 2.597 3507 | 59.90 | 2.619 1787 | 61.38 | 2.641 5525 | 62.93 |
| 26 | .576 3960 | 58.50 | .597 7102 | 59.92 | .619 5471 | 61.41 | .641 9302 | 62.96 |
| 27 | .576 7471 | 58.52 | .598 0698 | 59.95 | .619 9156 | 61.44 | .642 3080 | 62.98 |
| 28 | .577 0983 | 58.55 | .598 4295 | 59.97 | .620 2843 | 61.46 | .642 6860 | 63.01 |
| 29 | .577 4496 | 58.57 | .598 7894 | 59.99 | .620 6531 | 61.48 | .643 0641 | 63.04 |
| 30 | 2.577 8011 | 58.59 | 2.599 1494 | 60.02 | 2.621 0220 | 61.51 | 2.643 4424 | 63.06 |
| 31 | .578 1528 | 58.62 | .599 5096 | 60.04 | .621 3911 | 61.53 | .643 8209 | 63.09 |
| 32 | .578 5045 | 58.64 | .599 8699 | 60.07 | .621 7604 | 61.56 | .644 1995 | 63.12 |
| 33 | .578 8564 | 58.66 | .600 2304 | 60.09 | .622 1298 | 61.58 | .644 5783 | 63.14 |
| 34 | .579 2085 | 58.69 | .600 5910 | 60.12 | .622 4994 | 61.61 | .644 9572 | 63.17 |
| 35 | 2.579 5607 | 58.71 | 2.600 9518 | 60.14 | 2.622 8691 | 61.63 | 2.645 3363 | 63.19 |
| 36 | .579 9130 | 58.73 | .601 3127 | 60.16 | .623 2390 | 61.66 | .645 7155 | 63.22 |
| 37 | .580 2655 | 58.76 | .601 6738 | 60.19 | .623 6091 | 61.68 | .646 0949 | 63.25 |
| 38 | .580 6181 | 58.78 | .602 0350 | 60.21 | .623 9793 | 61.71 | .646 4745 | 63.27 |
| 39 | .580 9708 | 58.80 | .602 3963 | 60.24 | .624 3496 | 61.74 | .646 8542 | 63.30 |
| 40 | 2.581 3237 | 58.83 | 2.602 7578 | 60.26 | 2.624 7201 | 61.76 | 2.647 2341 | 63.33 |
| 41 | .581 6768 | 58.85 | .603 1195 | 60.29 | .625 0907 | 61.79 | .647 6142 | 63.35 |
| 42 | .582 0299 | 58.87 | .603 4813 | 60.31 | .625 4615 | 61.81 | .647 9944 | 63.38 |
| 43 | .582 3832 | 58.90 | .603 8432 | 60.34 | .625 8325 | 61.84 | .648 3748 | 63.41 |
| 44 | .582 8267 | 58.92 | .604 2053 | 60.36 | .626 2036 | 61.86 | .648 7553 | 63.44 |
| 45 | 2.583 0903 | 58.94 | 2.604 5675 | 60.38 | 2.626 5748 | 61.89 | 2.649 1360 | 63.46 |
| 46 | .583 4440 | 58.97 | .604 9299 | 60.41 | .626 9462 | 61.91 | .649 5168 | 63.49 |
| 47 | .583 7979 | 58.99 | .605 2924 | 60.43 | .627 3178 | 61.94 | .649 8978 | 63.52 |
| 48 | .584 1519 | 59.01 | .605 6551 | 60.46 | .627 6895 | 61.97 | .650 2790 | 63.54 |
| 49 | .584 5061 | 59.04 | .606 0179 | 60.48 | .628 0614 | 61.99 | .650 6603 | 63.57 |
| 50 | 2.584 8604 | 59.06 | 2.606 3809 | 60.51 | 2.628 4334 | 62.02 | 2.651 0418 | 63.60 |
| 51 | .585 2148 | 59.09 | .606 7440 | 60.53 | .628 8056 | 62.04 | .651 4235 | 63.62 |
| 52 | .585 5694 | 59.11 | .607 1073 | 60.56 | .629 1780 | 62.07 | .651 8053 | 63.65 |
| 53 | .585 9241 | 59.13 | .607 4707 | 60.58 | .629 5505 | 62.09 | .652 1873 | 63.68 |
| 54 | .586 2790 | 59.16 | .607 8343 | 60.61 | .629 9231 | 62.12 | .652 5695 | 63.70 |
| 55 | 2.586 6340 | 59.18 | 2.608 1980 | 60.63 | 2.630 2959 | 62.15 | 2.652 9518 | 63.73 |
| 56 | .586 9891 | 59.20 | .608 5618 | 60.66 | .630 6689 | 62.17 | .653 3342 | 63.76 |
| 57 | .587 3444 | 59.23 | .608 9258 | 60.68 | .631 0420 | 62.20 | .653 7168 | 63.79 |
| 58 | .587 6999 | 59.25 | .609 2901 | 60.70 | .631 4152 | 62.22 | .654 0996 | 63.81 |
| 59 | .588 0555 | 59.27 | .609 6544 | 60.73 | .631 7887 | 62.25 | .654 4826 | 63.84 |
| 60 | 2.588 4112 | 59.30 | 2.610 0188 | 60.75 | 2.632 1622 | 62.28 | 2.654 8657 | 63.87 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 132° | | 133° | | 134° | | 135° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 2.654 8657 | 63.87 | 2.678 1547 | 65.53 | 2.702 0562 | 67.27 | 2.726 5990 | 69.09 |
| 1 | .655 2490 | 63.89 | .678 5480 | 65.56 | .702 4600 | 67.30 | .727 0137 | 69.12 |
| 2 | .655 6324 | 63.92 | .678 9414 | 65.59 | .702 8638 | 67.33 | .727 4285 | 69.15 |
| 3 | .656 0160 | 63.95 | .679 3350 | 65.61 | .703 2679 | 67.36 | .727 8435 | 69.19 |
| 4 | .656 3998 | 63.97 | .679 7288 | 65.64 | .703 6721 | 67.39 | .728 2587 | 69.22 |
| 5 | 2.656 7837 | 64.00 | 2.680 1227 | 65.67 | 2.704 0766 | 67.42 | 2.728 6741 | 69.25 |
| 6 | .657 1678 | 64.03 | .680 5168 | 65.70 | .704 4812 | 67.45 | .729 0897 | 69.28 |
| 7 | .657 5521 | 64.06 | .680 9111 | 65.73 | .704 8860 | 67.48 | .729 5055 | 69.31 |
| 8 | .657 9365 | 64.08 | .681 3056 | 65.76 | .705 2909 | 67.51 | .729 9215 | 69.34 |
| 9 | .658 3211 | 64.11 | .681 7002 | 65.79 | .705 6961 | 67.54 | .730 3376 | 69.37 |
| 10 | 2.658 7058 | 64.14 | 2.682 0950 | 65.81 | 2.706 1014 | 67.57 | 2.730 7539 | 69.40 |
| 11 | .659 0907 | 64.17 | .682 4900 | 65.84 | .706 5069 | 67.60 | .731 1705 | 69.44 |
| 12 | .659 4758 | 64.19 | .682 8851 | 65.87 | .706 9126 | 67.63 | .731 5872 | 69.47 |
| 13 | .659 8611 | 64.22 | .683 2804 | 65.90 | .707 3184 | 67.66 | .732 0041 | 69.50 |
| 14 | .660 2465 | 64.25 | .683 6759 | 65.93 | .707 7244 | 67.69 | .732 4212 | 69.53 |
| 15 | 2.660 6320 | 64.28 | 2.684 0716 | 65.96 | 2.708 1307 | 67.72 | 2.732 8385 | 69.56 |
| 16 | .661 0178 | 64.30 | .684 4674 | 65.99 | .708 5371 | 67.75 | .733 2559 | 69.59 |
| 17 | .661 4037 | 64.33 | .684 8634 | 66.01 | .708 9436 | 67.78 | .733 6736 | 69.62 |
| 18 | .661 7897 | 64.36 | .685 2596 | 66.04 | .709 3504 | 67.81 | .734 0914 | 69.66 |
| 19 | .662 1760 | 64.38 | .685 6559 | 66.07 | .709 7573 | 67.84 | .734 5094 | 69.69 |
| 20 | 2.662 5623 | 64.41 | 2.686 0524 | 66.10 | 2.710 1645 | 67.87 | 2.734 9277 | 69.72 |
| 21 | .662 9489 | 64.44 | .686 4491 | 66.13 | .710 5718 | 67.90 | .735 3461 | 69.75 |
| 22 | .663 3356 | 64.47 | .686 8460 | 66.16 | .710 9792 | 67.93 | .735 7647 | 69.78 |
| 23 | .663 7225 | 64.49 | .687 2430 | 66.19 | .711 3869 | 67.96 | .736 1835 | 69.81 |
| 24 | .664 1096 | 64.52 | .687 6402 | 66.22 | .711 7947 | 67.99 | .736 6025 | 69.85 |
| 25 | 2.664 4968 | 64.55 | 2.688 0376 | 66.25 | 2.712 2028 | 68.02 | 2.737 0216 | 69.88 |
| 26 | .664 8842 | 64.57 | .688 4352 | 66.27 | .712 6110 | 68.05 | .737 4410 | 69.91 |
| 27 | .665 2717 | 64.60 | .688 8329 | 66.30 | .713 0194 | 68.08 | .737 8605 | 69.94 |
| 28 | .665 6594 | 64.63 | .689 2308 | 66.33 | .713 4279 | 68.11 | .738 2803 | 69.97 |
| 29 | .666 0473 | 64.66 | .689 6289 | 66.36 | .713 8367 | 68.14 | .738 7002 | 70.00 |
| 30 | 2.666 4354 | 64.69 | 2.690 0272 | 66.39 | 2.714 2456 | 68.17 | 2.739 1203 | 70.04 |
| 31 | .666 8236 | 64.72 | .690 4256 | 66.42 | .714 6547 | 68.20 | .739 5406 | 70.07 |
| 32 | .667 2120 | 64.74 | .690 8242 | 66.45 | .715 0640 | 68.23 | .739 9612 | 70.10 |
| 33 | .667 6005 | 64.77 | .691 2230 | 66.48 | .715 4735 | 68.26 | .740 3819 | 70.13 |
| 34 | .667 9892 | 64.80 | .691 6219 | 66.51 | .715 8832 | 68.29 | .740 8027 | 70.16 |
| 35 | 2.668 3781 | 64.83 | 2.692 0210 | 66.54 | 2.716 2930 | 68.32 | 2.741 2238 | 70.20 |
| 36 | .668 7672 | 64.86 | .692 4203 | 66.56 | .716 7031 | 68.35 | .741 6451 | 70.23 |
| 37 | .669 1564 | 64.88 | .692 8198 | 66.59 | .717 1133 | 68.38 | .742 0666 | 70.26 |
| 38 | .669 5457 | 64.91 | .693 2194 | 66.62 | .717 5237 | 68.41 | .742 4882 | 70.29 |
| 39 | .669 9353 | 64.94 | .693 6193 | 66.65 | .717 9342 | 68.44 | .742 9101 | 70.32 |
| 40 | 2.670 3250 | 64.97 | 2.694 0193 | 66.68 | 2.718 3450 | 68.48 | 2.743 3321 | 70.36 |
| 41 | .670 7149 | 65.00 | .694 4194 | 66.71 | .718 7560 | 68.51 | .743 7543 | 70.39 |
| 42 | .671 1050 | 65.02 | .694 8198 | 66.74 | .719 1671 | 68.54 | .744 1768 | 70.42 |
| 43 | .671 4952 | 65.05 | .695 2203 | 66.77 | .719 5784 | 68.57 | .744 5994 | 70.45 |
| 44 | .671 8856 | 65.08 | .695 6210 | 66.80 | .719 9899 | 68.60 | .745 0222 | 70.48 |
| 45 | 2.672 2761 | 65.11 | 2.696 0219 | 66.83 | 2.720 4016 | 68.63 | 2.745 4452 | 70.52 |
| 46 | .672 6668 | 65.13 | .696 4229 | 66.86 | .720 8135 | 68.66 | .745 8684 | 70.55 |
| 47 | .673 0577 | 65.16 | .696 8242 | 66.89 | .721 2255 | 68.69 | .746 2918 | 70.58 |
| 48 | .673 4488 | 65.19 | .697 2256 | 66.92 | .721 6377 | 68.72 | .746 7154 | 70.61 |
| 49 | .673 8400 | 65.22 | .697 6272 | 66.95 | .722 0502 | 68.75 | .747 1391 | 70.65 |
| 50 | 2.674 2314 | 65.25 | 2.698 0289 | 66.97 | 2.722 4628 | 68.78 | 2.747 5631 | 70.68 |
| 51 | .674 6230 | 65.28 | .698 4308 | 67.00 | .722 8756 | 68.81 | .747 9873 | 70.71 |
| 52 | .675 0147 | 65.30 | .698 8330 | 67.03 | .723 2885 | 68.84 | .748 4116 | 70.74 |
| 53 | .675 4066 | 65.33 | .699 2353 | 67.06 | .723 7017 | 68.88 | .748 8362 | 70.78 |
| 54 | .675 7987 | 65.36 | .699 6377 | 67.09 | .724 1150 | 68.91 | .749 2609 | 70.81 |
| 55 | 2.676 1909 | 65.39 | 2.700 0404 | 67.12 | 2.724 5286 | 68.94 | 2.749 6859 | 70.84 |
| 56 | .676 5833 | 65.42 | .700 4432 | 67.15 | .724 9423 | 68.97 | .750 1110 | 70.87 |
| 57 | .676 9759 | 65.44 | .700 8462 | 67.18 | .725 3562 | 69.00 | .750 5364 | 70.90 |
| 58 | .677 3687 | 65.47 | .701 2494 | 67.21 | .725 7703 | 69.03 | .750 9619 | 70.94 |
| 59 | .677 7616 | 65.50 | .701 6527 | 67.24 | .726 1846 | 69.06 | .751 3876 | 70.97 |
| 60 | 2.678 1547 | 65.53 | 2.702 0562 | 67.27 | 2.726 5990 | 69.09 | 2.751 8135 | 71.00 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 136° | | | 137° | | | 138° | | | 139° | | |
|----|------------|-----------|--|------------|-----------|--|------------|-----------|--|------------|-----------|--|
| | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | |
| 0' | 2.751 8135 | 71.00 | | 2.777 7322 | 73.01 | | 2.804 3895 | 75.11 | | 2.831 8224 | 77.32 | |
| 1 | .752 2396 | 71.03 | | .778 1703 | 73.04 | | .804 8403 | 75.14 | | .832 2864 | 77.35 | |
| 2 | .752 6659 | 71.07 | | .778 6087 | 73.07 | | .805 2912 | 75.18 | | .832 7506 | 77.39 | |
| 3 | .753 0925 | 71.10 | | .779 0472 | 73.11 | | .805 7424 | 75.21 | | .833 2151 | 77.43 | |
| 4 | .753 5192 | 71.13 | | .779 4859 | 73.14 | | .806 1938 | 75.25 | | .833 6798 | 77.47 | |
| 5 | 2.753 9461 | 71.17 | | 2.779 9249 | 73.18 | | 2.806 6454 | 75.29 | | 2.834 1447 | 77.50 | |
| 6 | .754 3732 | 71.20 | | .780 3641 | 73.21 | | .807 0973 | 75.32 | | .834 6098 | 77.54 | |
| 7 | .754 8004 | 71.23 | | .780 8034 | 73.24 | | .807 5493 | 75.36 | | .835 0752 | 77.58 | |
| 8 | .755 2279 | 71.26 | | .781 2430 | 73.28 | | .808 0016 | 75.40 | | .835 5408 | 77.62 | |
| 9 | .755 6556 | 71.30 | | .781 6828 | 73.31 | | .808 4541 | 75.43 | | .836 0066 | 77.66 | |
| 10 | 2.756 0835 | 71.33 | | 2.782 1228 | 73.35 | | 2.808 9068 | 75.47 | | 2.836 4727 | 77.69 | |
| 11 | .756 5116 | 71.36 | | .782 5630 | 73.38 | | .809 3597 | 75.50 | | .836 9390 | 77.73 | |
| 12 | .756 9399 | 71.40 | | .783 0034 | 73.42 | | .809 8128 | 75.54 | | .837 4055 | 77.77 | |
| 13 | .757 3683 | 71.43 | | .783 4440 | 73.45 | | .810 2662 | 75.58 | | .837 8722 | 77.81 | |
| 14 | .757 7970 | 71.46 | | .783 8848 | 73.49 | | .810 7197 | 75.61 | | .838 3392 | 77.85 | |
| 15 | 2.758 2259 | 71.49 | | 2.784 3258 | 73.53 | | 2.811 1735 | 75.65 | | 2.838 8064 | 77.89 | |
| 16 | .758 6549 | 71.53 | | .784 7671 | 73.56 | | .811 6275 | 75.69 | | .839 2738 | 77.92 | |
| 17 | .759 0842 | 71.56 | | .785 2085 | 73.59 | | .812 0817 | 75.72 | | .839 7414 | 77.96 | |
| 18 | .759 5137 | 71.59 | | .785 6502 | 73.63 | | .812 5362 | 75.76 | | .840 2093 | 78.00 | |
| 19 | .759 9433 | 71.63 | | .786 0920 | 73.66 | | .812 9908 | 75.79 | | .840 6774 | 78.04 | |
| 20 | 2.760 3732 | 71.66 | | 2.786 5341 | 73.70 | | 2.813 4457 | 75.83 | | 2.841 1458 | 78.08 | |
| 21 | .760 8032 | 71.69 | | .786 9764 | 73.73 | | .813 9008 | 75.87 | | .841 6144 | 78.11 | |
| 22 | .761 2335 | 71.73 | | .787 4189 | 73.76 | | .814 3561 | 75.90 | | .842 0832 | 78.15 | |
| 23 | .761 6639 | 71.76 | | .787 8615 | 73.80 | | .814 8117 | 75.94 | | .842 5522 | 78.19 | |
| 24 | .762 0946 | 71.79 | | .788 3044 | 73.83 | | .815 2674 | 75.98 | | .843 0215 | 78.23 | |
| 25 | 2.762 5255 | 71.83 | | 2.788 7476 | 73.87 | | 2.815 7234 | 76.01 | | 2.843 4909 | 78.27 | |
| 26 | .762 9565 | 71.86 | | .789 1909 | 73.90 | | .816 1796 | 76.05 | | .843 9607 | 78.31 | |
| 27 | .763 3878 | 71.89 | | .789 6344 | 73.94 | | .816 6360 | 76.09 | | .844 4306 | 78.35 | |
| 28 | .763 8192 | 71.93 | | .790 0781 | 73.97 | | .817 0927 | 76.12 | | .844 9008 | 78.38 | |
| 29 | .764 2509 | 71.96 | | .790 5221 | 74.01 | | .817 5495 | 76.16 | | .845 3712 | 78.42 | |
| 30 | 2.764 6827 | 71.99 | | 2.790 9662 | 74.04 | | 2.818 0066 | 76.20 | | 2.845 8419 | 78.46 | |
| 31 | .765 1148 | 72.03 | | .791 4106 | 74.08 | | .818 4639 | 76.23 | | .846 3128 | 78.50 | |
| 32 | .765 5470 | 72.06 | | .791 8552 | 74.11 | | .818 9214 | 76.27 | | .846 7839 | 78.54 | |
| 33 | .765 9795 | 72.09 | | .792 3000 | 74.15 | | .819 3792 | 76.31 | | .847 2553 | 78.58 | |
| 34 | .766 4121 | 72.13 | | .792 7450 | 74.18 | | .819 8371 | 76.34 | | .847 7268 | 78.62 | |
| 35 | 2.766 8450 | 72.16 | | 2.793 1902 | 74.22 | | 2.820 2953 | 76.38 | | 2.848 1986 | 78.66 | |
| 36 | .767 2781 | 72.19 | | .793 6356 | 74.25 | | .820 7537 | 76.42 | | .848 6707 | 78.69 | |
| 37 | .767 7113 | 72.23 | | .794 0813 | 74.29 | | .821 2123 | 76.46 | | .849 1430 | 78.73 | |
| 38 | .768 1448 | 72.26 | | .794 5271 | 74.32 | | .821 6712 | 76.49 | | .849 6155 | 78.77 | |
| 39 | .768 5784 | 72.29 | | .794 9731 | 74.36 | | .822 1302 | 76.53 | | .850 0882 | 78.81 | |
| 40 | 2.769 0123 | 72.33 | | 2.795 4194 | 74.40 | | 2.822 5895 | 76.57 | | 2.850 5612 | 78.85 | |
| 41 | .769 4464 | 72.36 | | .795 8659 | 74.43 | | .823 0491 | 76.60 | | .851 0344 | 78.89 | |
| 42 | .769 8806 | 72.39 | | .796 3126 | 74.47 | | .823 5088 | 76.64 | | .851 5079 | 78.93 | |
| 43 | .770 3151 | 72.43 | | .796 7595 | 74.50 | | .823 9688 | 76.68 | | .851 9816 | 78.97 | |
| 44 | .770 7498 | 72.46 | | .797 2066 | 74.54 | | .824 4289 | 76.72 | | .852 4555 | 79.01 | |
| 45 | 2.771 1846 | 72.50 | | 2.797 6539 | 74.58 | | 2.824 8894 | 76.75 | | 2.852 9297 | 79.05 | |
| 46 | .771 6197 | 72.53 | | .798 1015 | 74.61 | | .825 3500 | 76.79 | | .853 4041 | 79.08 | |
| 47 | .772 0550 | 72.56 | | .798 5492 | 74.64 | | .825 8108 | 76.83 | | .853 8787 | 79.12 | |
| 48 | .772 4905 | 72.60 | | .798 9972 | 74.68 | | .826 2719 | 76.87 | | .854 3535 | 79.16 | |
| 49 | .772 9262 | 72.63 | | .799 4454 | 74.71 | | .826 7332 | 76.90 | | .854 8286 | 79.20 | |
| 50 | 2.773 3621 | 72.67 | | 2.799 8938 | 74.75 | | 2.827 1947 | 76.94 | | 2.855 3040 | 79.24 | |
| 51 | .773 7982 | 72.70 | | .800 3424 | 74.79 | | .827 6565 | 76.98 | | .855 7795 | 79.28 | |
| 52 | .774 2344 | 72.73 | | .800 7912 | 74.82 | | .828 1185 | 77.01 | | .856 2553 | 79.32 | |
| 53 | .774 6709 | 72.77 | | .801 2402 | 74.86 | | .828 5807 | 77.05 | | .856 7314 | 79.36 | |
| 54 | .775 1077 | 72.80 | | .801 6895 | 74.89 | | .829 0431 | 77.09 | | .857 2077 | 79.40 | |
| 55 | 2.775 5446 | 72.84 | | 2.802 1390 | 74.93 | | 2.829 5058 | 77.13 | | 2.857 6842 | 79.44 | |
| 56 | .775 9817 | 72.87 | | .802 5886 | 74.96 | | .829 9686 | 77.16 | | .858 1609 | 79.48 | |
| 57 | .776 4190 | 72.90 | | .803 0385 | 75.00 | | .830 4317 | 77.20 | | .858 6379 | 79.52 | |
| 58 | .776 8565 | 72.94 | | .803 4886 | 75.04 | | .830 8951 | 77.24 | | .859 1151 | 79.56 | |
| 59 | .777 2942 | 72.97 | | .803 9390 | 75.08 | | .831 3586 | 77.28 | | .859 5926 | 79.60 | |
| 60 | 2.777 7322 | 73.01 | | 2.804 3895 | 75.11 | | 2.831 8224 | 77.32 | | 2.860 0703 | 79.64 | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 140° | | | 141° | | | 142° | | | 143° | | |
|----|------------|-----------|--|------------|-----------|--|------------|-----------|--|------------|-----------|--|
| | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | |
| 0 | 2.860 0703 | 79.64 | | 2.889 1754 | 82.08 | | 2.919 1831 | 84.65 | | 2.950 1420 | 87.37 | |
| 1 | .860 5482 | 79.68 | | .889 6680 | 82.12 | | .919 6911 | 84.70 | | .950 6664 | 87.41 | |
| 2 | .861 0264 | 79.72 | | .890 1609 | 82.16 | | .920 1994 | 84.74 | | .951 1910 | 87.46 | |
| 3 | .861 5048 | 79.76 | | .890 6540 | 82.20 | | .920 7080 | 84.78 | | .951 7159 | 87.50 | |
| 4 | .861 9835 | 79.80 | | .891 1473 | 82.25 | | .921 2169 | 84.83 | | .952 2411 | 87.55 | |
| 5 | 2.862 4624 | 79.84 | | 2.891 6409 | 82.29 | | 2.921 7260 | 84.87 | | 2.952 7665 | 87.60 | |
| 6 | .862 9415 | 79.88 | | .892 1348 | 82.33 | | .922 2353 | 84.92 | | .953 2923 | 87.65 | |
| 7 | .863 4209 | 79.92 | | .892 6289 | 82.37 | | .922 7450 | 84.96 | | .953 8183 | 87.69 | |
| 8 | .863 9005 | 79.96 | | .893 1233 | 82.41 | | .923 2549 | 85.01 | | .954 3446 | 87.74 | |
| 9 | .864 3803 | 80.00 | | .893 6179 | 82.46 | | .923 7650 | 85.05 | | .954 8711 | 87.79 | |
| 10 | 2.864 8604 | 80.04 | | 2.894 1127 | 82.50 | | 2.924 2755 | 85.10 | | 2.955 3980 | 87.83 | |
| 11 | .865 3408 | 80.08 | | .894 6078 | 82.54 | | .924 7862 | 85.14 | | .955 9251 | 87.88 | |
| 12 | .865 8213 | 80.12 | | .895 1032 | 82.58 | | .925 2972 | 85.18 | | .956 4525 | 87.93 | |
| 13 | .866 3021 | 80.16 | | .895 5989 | 82.63 | | .925 8084 | 85.23 | | .956 9802 | 87.97 | |
| 14 | .866 7832 | 80.20 | | .896 0948 | 82.67 | | .926 3199 | 85.27 | | .957 5082 | 88.02 | |
| 15 | 2.867 2645 | 80.24 | | 2.896 5909 | 82.71 | | 2.926 8317 | 85.32 | | 2.958 0365 | 88.07 | |
| 16 | .867 7460 | 80.28 | | .897 0873 | 82.75 | | .927 3437 | 85.36 | | .958 5651 | 88.11 | |
| 17 | .868 2278 | 80.32 | | .897 5839 | 82.79 | | .927 8560 | 85.41 | | .959 0939 | 88.16 | |
| 18 | .868 7098 | 80.36 | | .898 0808 | 82.84 | | .928 3686 | 85.45 | | .959 6230 | 88.21 | |
| 19 | .869 1921 | 80.40 | | .898 5780 | 82.88 | | .928 8814 | 85.50 | | .960 1524 | 88.26 | |
| 20 | 2.869 6746 | 80.44 | | 2.899 0754 | 82.92 | | 2.929 3945 | 85.54 | | 2.960 6821 | 88.30 | |
| 21 | .870 1573 | 80.48 | | .899 5730 | 82.96 | | .929 9079 | 85.59 | | .961 2120 | 88.35 | |
| 22 | .870 6403 | 80.52 | | .900 0709 | 83.01 | | .930 4216 | 85.63 | | .961 7423 | 88.40 | |
| 23 | .871 1235 | 80.56 | | .900 5691 | 83.05 | | .930 9355 | 85.68 | | .962 2728 | 88.45 | |
| 24 | .871 6070 | 80.60 | | .901 0675 | 83.09 | | .931 4497 | 85.72 | | .962 7036 | 88.49 | |
| 25 | 2.872 0907 | 80.64 | | 2.901 5662 | 83.13 | | 2.931 9641 | 85.77 | | 2.963 3347 | 88.54 | |
| 26 | .872 5747 | 80.68 | | .902 0651 | 83.18 | | .932 4788 | 85.81 | | .963 8661 | 88.59 | |
| 27 | .873 0589 | 80.72 | | .902 5643 | 83.22 | | .932 9938 | 85.86 | | .964 3978 | 88.64 | |
| 28 | .873 5433 | 80.76 | | .903 0638 | 83.26 | | .933 5091 | 85.91 | | .964 9297 | 88.68 | |
| 29 | .874 0280 | 80.80 | | .903 5635 | 83.31 | | .934 0247 | 85.95 | | .965 4620 | 88.73 | |
| 30 | 2.874 5129 | 80.84 | | 2.904 0635 | 83.35 | | 2.934 5405 | 85.99 | | 2.965 9945 | 88.78 | |
| 31 | .874 9981 | 80.88 | | .904 5637 | 83.39 | | .935 0565 | 86.04 | | .966 5273 | 88.83 | |
| 32 | .875 4835 | 80.92 | | .905 0642 | 83.43 | | .935 5729 | 86.08 | | .967 0604 | 88.87 | |
| 33 | .875 9692 | 80.96 | | .905 5649 | 83.48 | | .936 0895 | 86.13 | | .967 5938 | 88.92 | |
| 34 | .876 4551 | 81.01 | | .906 0659 | 83.52 | | .936 6064 | 86.17 | | .968 1275 | 88.97 | |
| 35 | 2.876 9413 | 81.05 | | 2.906 5672 | 83.56 | | 2.937 1236 | 86.22 | | 2.968 6615 | 89.02 | |
| 36 | .877 4277 | 81.09 | | .907 0687 | 83.61 | | .937 6410 | 86.26 | | .969 1957 | 89.07 | |
| 37 | .877 9143 | 81.13 | | .907 5704 | 83.65 | | .938 1587 | 86.31 | | .969 7303 | 89.12 | |
| 38 | .878 4012 | 81.17 | | .908 0725 | 83.69 | | .938 6767 | 86.35 | | .970 2651 | 89.17 | |
| 39 | .878 8883 | 81.21 | | .908 5748 | 83.74 | | .939 1950 | 86.40 | | .970 8002 | 89.21 | |
| 40 | 2.879 3757 | 81.25 | | 2.909 0773 | 83.78 | | 2.939 7135 | 86.45 | | 2.971 3356 | 89.26 | |
| 41 | .879 8633 | 81.29 | | .909 5801 | 83.82 | | .940 2323 | 86.49 | | .971 8713 | 89.31 | |
| 42 | .880 3512 | 81.33 | | .910 0832 | 83.87 | | .940 7514 | 86.54 | | .972 4073 | 89.36 | |
| 43 | .880 8393 | 81.37 | | .910 5865 | 83.91 | | .941 2708 | 86.58 | | .972 9436 | 89.40 | |
| 44 | .881 3277 | 81.42 | | .911 0901 | 83.95 | | .941 7904 | 86.63 | | .973 4801 | 89.45 | |
| 45 | 2.881 8163 | 81.46 | | 2.911 5940 | 83.99 | | 2.942 3103 | 86.67 | | 2.974 0170 | 89.50 | |
| 46 | .882 3052 | 81.50 | | .912 0981 | 84.04 | | .942 8305 | 86.72 | | .974 5541 | 89.55 | |
| 47 | .882 7943 | 81.54 | | .912 6024 | 84.08 | | .943 3510 | 86.77 | | .975 0916 | 89.60 | |
| 48 | .883 2837 | 81.58 | | .913 1070 | 84.13 | | .943 8717 | 86.81 | | .975 6293 | 89.65 | |
| 49 | .883 7733 | 81.62 | | .913 6119 | 84.17 | | .944 3927 | 86.86 | | .976 1673 | 89.69 | |
| 50 | 2.884 2631 | 81.66 | | 2.914 1171 | 84.22 | | 2.944 9140 | 86.90 | | 2.976 7056 | 89.74 | |
| 51 | .884 7532 | 81.70 | | .914 6225 | 84.26 | | .945 4355 | 86.95 | | .977 2442 | 89.79 | |
| 52 | .885 2436 | 81.75 | | .915 1282 | 84.30 | | .945 9574 | 87.00 | | .977 7831 | 89.84 | |
| 53 | .885 7342 | 81.79 | | .915 6341 | 84.34 | | .946 4795 | 87.04 | | .978 3223 | 89.89 | |
| 54 | .886 2251 | 81.83 | | .916 1403 | 84.39 | | .947 0019 | 87.09 | | .978 8618 | 89.94 | |
| 55 | 2.886 7162 | 81.87 | | 2.916 6468 | 84.43 | | 2.947 5245 | 87.13 | | 2.979 4015 | 89.99 | |
| 56 | .887 2075 | 81.91 | | .917 1535 | 84.48 | | .948 0475 | 87.18 | | .979 9416 | 90.03 | |
| 57 | .887 6991 | 81.95 | | .917 6605 | 84.52 | | .948 5707 | 87.23 | | .980 4820 | 90.08 | |
| 58 | .888 1910 | 81.99 | | .918 1678 | 84.56 | | .949 0942 | 87.27 | | .981 1226 | 90.13 | |
| 59 | .888 6831 | 82.04 | | .918 6753 | 84.61 | | .949 6180 | 87.32 | | .981 6636 | 90.18 | |
| 60 | 2.889 1754 | 82.08 | | 2.919 1831 | 84.65 | | 2.950 1420 | 87.37 | | 2.982 1048 | 90.23 | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 144° | | 145° | | 146° | | 147° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 2.982 1048 | 90.23 | 3.015 1281 | 93.26 | 3.049 2733 | 96.47 | 3.084 6070 | 99.87 |
| 1 | .982 6463 | 90.28 | .015 6878 | 93.31 | .049 8522 | 96.52 | .085 2064 | 99.92 |
| 2 | .983 1882 | 90.33 | .016 2478 | 93.36 | .050 4315 | 96.58 | .085 8061 | 99.98 |
| 3 | .983 7303 | 90.38 | .016 8082 | 93.42 | .051 0112 | 96.63 | .086 4062 | 100.04 |
| 4 | .984 2727 | 90.43 | .017 3688 | 93.47 | .051 5911 | 96.69 | .087 0066 | 100.10 |
| 5 | 2.984 8154 | 90.48 | 3.017 9298 | 93.52 | 3.052 1714 | 96.74 | 3.087 6073 | 100.16 |
| 6 | .985 3584 | 90.53 | .018 4911 | 93.57 | .052 7520 | 96.80 | .088 2085 | 100.22 |
| 7 | .985 9017 | 90.58 | .019 0526 | 93.62 | .053 3329 | 96.85 | .088 8099 | 100.28 |
| 8 | .986 4453 | 90.63 | .019 6145 | 93.68 | .053 9142 | 96.91 | .089 4118 | 100.33 |
| 9 | .986 9892 | 90.67 | .020 1768 | 93.73 | .054 4959 | 96.96 | .090 0140 | 100.39 |
| 10 | 2.987 5334 | 90.72 | 3.020 7393 | 93.78 | 3.055 0778 | 97.01 | 3.090 6165 | 100.45 |
| 11 | .988 0779 | 90.77 | .021 3021 | 93.83 | .055 6601 | 97.07 | .091 2194 | 100.51 |
| 12 | .988 6227 | 90.82 | .021 8653 | 93.89 | .056 2427 | 97.13 | .091 8226 | 100.57 |
| 13 | .989 1678 | 90.87 | .022 4288 | 93.94 | .056 8256 | 97.19 | .092 4262 | 100.63 |
| 14 | .989 7132 | 90.92 | .022 9926 | 93.99 | .057 4089 | 97.24 | .093 0302 | 100.69 |
| 15 | 2.990 2589 | 90.97 | 3.023 5567 | 94.04 | 3.057 9925 | 97.30 | 3.093 6345 | 100.75 |
| 16 | .990 8049 | 91.02 | .024 1211 | 94.10 | .058 5765 | 97.35 | .094 2392 | 100.81 |
| 17 | .991 3512 | 91.07 | .024 6859 | 94.15 | .059 1608 | 97.41 | .094 8442 | 100.87 |
| 18 | .991 8977 | 91.12 | .025 2509 | 94.20 | .059 7454 | 97.47 | .095 4496 | 100.93 |
| 19 | .992 4446 | 91.17 | .025 8163 | 94.26 | .060 3304 | 97.52 | .096 0553 | 100.98 |
| 20 | 2.992 9918 | 91.22 | 3.026 3820 | 94.31 | 3.060 9157 | 97.58 | 3.096 6614 | 101.04 |
| 21 | .993 5393 | 91.27 | .026 9480 | 94.36 | .061 5013 | 97.63 | .097 2678 | 101.10 |
| 22 | .994 0871 | 91.32 | .027 5143 | 94.41 | .062 0873 | 97.69 | .097 8746 | 101.16 |
| 23 | .994 6351 | 91.37 | .028 0810 | 94.47 | .062 6736 | 97.75 | .098 4818 | 101.22 |
| 24 | .995 1835 | 91.42 | .028 6479 | 94.52 | .063 2602 | 97.80 | .099 0893 | 101.28 |
| 25 | 2.995 7322 | 91.47 | 3.029 2152 | 94.57 | 3.063 8472 | 97.86 | 3.099 6972 | 101.34 |
| 26 | .996 2812 | 91.52 | .029 7828 | 94.63 | .064 4345 | 97.91 | .100 3054 | 101.40 |
| 27 | .996 8305 | 91.57 | .030 3507 | 94.68 | .065 0222 | 97.97 | .100 9140 | 101.46 |
| 28 | .997 3801 | 91.62 | .030 9190 | 94.73 | .065 6101 | 98.03 | .101 5230 | 101.52 |
| 29 | .997 9300 | 91.67 | .031 4875 | 94.79 | .066 1985 | 98.08 | .102 1323 | 101.58 |
| 30 | 2.998 4802 | 91.72 | 3.032 0564 | 94.84 | 3.066 7872 | 98.14 | 3.102 7420 | 101.64 |
| 31 | .999 0307 | 91.77 | .032 6256 | 94.89 | .067 3762 | 98.20 | .103 3520 | 101.70 |
| 32 | .999 5815 | 91.82 | .033 1951 | 94.94 | .067 9655 | 98.25 | .103 9624 | 101.76 |
| 33 | 3.000 1326 | 91.87 | .033 7650 | 95.00 | .068 5552 | 98.31 | .104 5732 | 101.82 |
| 34 | .000 6840 | 91.93 | .034 3351 | 95.05 | .069 1453 | 98.37 | .105 1843 | 101.88 |
| 35 | 3.001 2357 | 91.98 | 3.034 9056 | 95.11 | 3.069 7357 | 98.42 | 3.105 7958 | 101.94 |
| 36 | .001 7877 | 92.03 | .035 4764 | 95.16 | .070 3264 | 98.48 | .106 4076 | 102.00 |
| 37 | .002 3400 | 92.08 | .036 0475 | 95.22 | .070 9174 | 98.54 | .107 0198 | 102.07 |
| 38 | .002 8926 | 92.13 | .036 6190 | 95.27 | .071 5088 | 98.60 | .107 6324 | 102.13 |
| 39 | .003 4456 | 92.18 | .037 1908 | 95.32 | .072 1006 | 98.65 | .108 2454 | 102.19 |
| 40 | 3.003 9988 | 92.23 | 3.037 7629 | 95.38 | 3.072 6927 | 98.71 | 3.108 8587 | 102.25 |
| 41 | .004 5523 | 92.28 | .038 3353 | 95.43 | .073 2851 | 98.77 | .109 4723 | 102.31 |
| 42 | .005 1062 | 92.33 | .038 9080 | 95.48 | .073 8779 | 98.82 | .110 0864 | 102.37 |
| 43 | .005 6603 | 92.38 | .039 4811 | 95.54 | .074 4710 | 98.88 | .110 7008 | 102.43 |
| 44 | .006 2148 | 92.44 | .040 0545 | 95.60 | .075 0645 | 98.94 | .111 3155 | 102.49 |
| 45 | 3.006 7696 | 92.49 | 3.040 6282 | 95.65 | 3.075 6583 | 99.00 | 3.111 9306 | 102.55 |
| 46 | .007 3246 | 92.54 | .041 2023 | 95.70 | .076 2524 | 99.05 | .112 5461 | 102.61 |
| 47 | .007 8800 | 92.59 | .041 7767 | 95.76 | .076 8469 | 99.11 | .113 1620 | 102.67 |
| 48 | .008 4357 | 92.64 | .042 3514 | 95.81 | .077 4418 | 99.17 | .113 7782 | 102.73 |
| 49 | .008 9917 | 92.69 | .042 9264 | 95.86 | .078 0370 | 99.23 | .114 3948 | 102.80 |
| 50 | 3.009 5480 | 92.74 | 3.043 5017 | 95.92 | 3.078 6325 | 99.28 | 3.115 0118 | 102.86 |
| 51 | .010 1046 | 92.79 | .044 0774 | 95.97 | .079 2284 | 99.34 | .115 6291 | 102.92 |
| 52 | .010 6615 | 92.85 | .044 6534 | 96.03 | .079 8246 | 99.40 | .116 2468 | 102.98 |
| 53 | .011 2188 | 92.90 | .045 2297 | 96.08 | .080 4212 | 99.46 | .116 8649 | 103.04 |
| 54 | .011 7763 | 92.95 | .045 8064 | 96.14 | .081 0181 | 99.52 | .117 4833 | 103.10 |
| 55 | 3.012 3342 | 93.00 | 3.046 3834 | 96.19 | 3.081 6154 | 99.57 | 3.118 1022 | 103.16 |
| 56 | .012 8923 | 93.05 | .046 9607 | 96.25 | .082 2130 | 99.63 | .118 7213 | 103.23 |
| 57 | .013 4508 | 93.10 | .047 5383 | 96.30 | .082 8110 | 99.69 | .119 3409 | 103.29 |
| 58 | .014 0096 | 93.16 | .048 1163 | 96.36 | .083 4093 | 99.75 | .119 9608 | 103.35 |
| 59 | .014 5687 | 93.21 | .048 6946 | 96.41 | .084 0080 | 99.81 | .120 5811 | 103.41 |
| 60 | 3.015 1281 | 93.26 | 3.049 2733 | 96.47 | 3.084 6070 | 99.87 | 3.121 2018 | 103.48 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 148° | | | 149° | | | 150° | | | 151° | | |
|----|------------|-----------|------------|------------|------------|--------|------------|-----------|--|------------|-----------|--|
| | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | |
| 0' | 3.121 2018 | 103.48 | | 3.159 1367 | 107.31 | | 3.198 4984 | 111.41 | | 3.239 3820 | 115.77 | |
| 1 | .121 8228 | 103.54 | .159 7808 | 107.38 | .199 1671 | 111.48 | .240 0768 | 115.85 | | | | |
| 2 | .122 4442 | 103.60 | .160 4253 | 107.45 | .199 8361 | 111.55 | .240 7722 | 115.92 | | | | |
| 3 | .123 0660 | 103.66 | .161 0702 | 107.51 | .200 5056 | 111.62 | .241 4680 | 116.00 | | | | |
| 4 | .123 6882 | 103.72 | .161 7154 | 107.58 | .201 1755 | 111.69 | .242 1642 | 116.08 | | | | |
| 5 | 3.124 3107 | 103.79 | 3.162 3611 | 107.65 | 3.201 8459 | 111.76 | 3.242 8608 | 116.15 | | | | |
| 6 | .124 9336 | 103.85 | .163 0072 | 107.71 | .202 5166 | 111.83 | .243 5580 | 116.23 | | | | |
| 7 | .125 5569 | 103.91 | .163 6536 | 107.78 | .203 1878 | 111.90 | .244 2556 | 116.30 | | | | |
| 8 | .126 1805 | 103.97 | .164 3005 | 107.85 | .203 8594 | 111.97 | .244 9536 | 116.38 | | | | |
| 9 | .126 8045 | 104.04 | .164 9478 | 107.91 | .204 5315 | 112.04 | .245 6521 | 116.45 | | | | |
| 10 | 3.127 4289 | 104.10 | 3.165 5955 | 107.98 | 3.205 2040 | 112.11 | 3.246 3511 | 116.53 | | | | |
| 11 | .128 0537 | 104.16 | .166 2435 | 108.04 | .205 8769 | 112.18 | .247 0505 | 116.61 | | | | |
| 12 | .128 6789 | 104.22 | .166 8920 | 108.11 | .206 5502 | 112.26 | .247 7503 | 116.68 | | | | |
| 13 | .129 3044 | 104.29 | .167 5409 | 108.18 | .207 2239 | 112.33 | .248 4507 | 116.76 | | | | |
| 14 | .129 9303 | 104.35 | .168 1901 | 108.25 | .207 8981 | 112.40 | .249 1515 | 116.84 | | | | |
| 15 | 3.130 5566 | 104.41 | 3.168 8398 | 108.31 | 3.208 5727 | 112.47 | 3.249 8527 | 116.91 | | | | |
| 16 | .131 1833 | 104.48 | .169 4899 | 108.38 | .209 2478 | 112.54 | .250 5544 | 116.99 | | | | |
| 17 | .131 8103 | 104.54 | .170 1404 | 108.45 | .209 9232 | 112.61 | .251 2566 | 117.07 | | | | |
| 18 | .132 4377 | 104.60 | .170 7913 | 108.51 | .210 5991 | 112.69 | .251 9592 | 117.14 | | | | |
| 19 | .133 0655 | 104.67 | .171 4426 | 108.58 | .211 2755 | 112.76 | .252 6623 | 117.22 | | | | |
| 20 | 3.133 6937 | 104.73 | 3.172 0942 | 108.65 | 3.211 9522 | 112.83 | 3.253 3658 | 117.30 | | | | |
| 21 | .134 3223 | 104.79 | .172 7463 | 108.72 | .212 6294 | 112.90 | .254 0698 | 117.37 | | | | |
| 22 | .134 9512 | 104.86 | .173 3988 | 108.78 | .213 3070 | 112.97 | .254 7743 | 117.45 | | | | |
| 23 | .135 5805 | 104.92 | .174 0517 | 108.85 | .213 9851 | 113.05 | .255 4792 | 117.53 | | | | |
| 24 | .136 2102 | 104.98 | .174 7051 | 108.92 | .214 6636 | 113.12 | .256 1846 | 117.60 | | | | |
| 25 | 3.136 8403 | 105.05 | 3.175 3588 | 108.99 | 3.215 3425 | 113.19 | 3.256 8905 | 117.68 | | | | |
| 26 | .137 4708 | 105.11 | .176 0129 | 109.06 | .216 0219 | 113.26 | .257 5968 | 117.76 | | | | |
| 27 | .138 1016 | 105.17 | .176 6674 | 109.12 | .216 7017 | 113.34 | .258 3036 | 117.84 | | | | |
| 28 | .138 7329 | 105.24 | .177 3224 | 109.19 | .217 3819 | 113.41 | .259 0109 | 117.91 | | | | |
| 29 | .139 3645 | 105.30 | .177 9777 | 109.26 | .218 0626 | 113.48 | .259 7186 | 117.99 | | | | |
| 30 | 3.139 9965 | 105.36 | 3.178 6335 | 109.33 | 3.218 7437 | 113.55 | 3.260 4268 | 118.07 | | | | |
| 31 | .140 6289 | 105.43 | .179 2897 | 109.40 | .219 4252 | 113.63 | .261 1354 | 118.15 | | | | |
| 32 | .141 2616 | 105.49 | .179 9462 | 109.46 | .220 1072 | 113.70 | .261 8446 | 118.23 | | | | |
| 33 | .141 8948 | 105.55 | .180 6032 | 109.53 | .220 7896 | 113.77 | .262 5542 | 118.30 | | | | |
| 34 | .142 5283 | 105.62 | .181 2606 | 109.60 | .221 4724 | 113.84 | .263 2642 | 118.38 | | | | |
| 35 | 3.143 1622 | 105.68 | 3.181 9184 | 109.67 | 3.222 1557 | 113.92 | 3.263 9747 | 118.46 | | | | |
| 36 | .143 7965 | 105.75 | .182 5766 | 109.74 | .222 8395 | 113.99 | .264 6857 | 118.54 | | | | |
| 37 | .144 4312 | 105.81 | .183 2353 | 109.81 | .223 5236 | 114.06 | .265 3972 | 118.62 | | | | |
| 38 | .145 0663 | 105.87 | .183 8943 | 109.87 | .224 2082 | 114.14 | .266 1091 | 118.70 | | | | |
| 39 | .145 7018 | 105.94 | .184 5538 | 109.94 | .224 8933 | 114.21 | .266 8216 | 118.77 | | | | |
| 40 | 3.146 3376 | 106.00 | 3.185 2136 | 110.01 | 3.225 5788 | 114.28 | 3.267 5345 | 118.85 | | | | |
| 41 | .146 9739 | 106.07 | .185 8739 | 110.08 | .226 2647 | 114.36 | .268 2478 | 118.93 | | | | |
| 42 | .147 6105 | 106.14 | .186 5346 | 110.15 | .226 9511 | 114.43 | .268 9616 | 119.01 | | | | |
| 43 | .148 2475 | 106.20 | .187 1957 | 110.22 | .227 6379 | 114.51 | .269 6759 | 119.09 | | | | |
| 44 | .148 8849 | 106.27 | .187 8572 | 110.29 | .228 3252 | 114.58 | .270 3907 | 119.17 | | | | |
| 45 | 3.149 5227 | 106.33 | 3.188 5192 | 110.36 | 3.229 0129 | 114.65 | 3.271 1060 | 119.25 | | | | |
| 46 | .150 1609 | 106.40 | .189 1815 | 110.43 | .229 7010 | 114.73 | .271 8217 | 119.33 | | | | |
| 47 | .150 7995 | 106.46 | .189 8443 | 110.50 | .230 3896 | 114.80 | .272 5379 | 119.41 | | | | |
| 48 | .151 4385 | 106.53 | .190 5075 | 110.57 | .231 0786 | 114.88 | .273 2546 | 119.49 | | | | |
| 49 | .152 0778 | 106.59 | .191 1711 | 110.64 | .231 7681 | 114.95 | .273 9717 | 119.57 | | | | |
| 50 | 3.152 7176 | 106.66 | 3.191 8351 | 110.71 | 3.232 4581 | 115.03 | 3.274 6894 | 119.65 | | | | |
| 51 | .153 3577 | 106.72 | .192 4996 | 110.77 | .233 1484 | 115.10 | .275 4075 | 119.73 | | | | |
| 52 | .153 9983 | 106.79 | .193 1644 | 110.84 | .233 8392 | 115.17 | .276 1261 | 119.81 | | | | |
| 53 | .154 6392 | 106.85 | .193 8297 | 110.91 | .234 5305 | 115.25 | .276 8452 | 119.89 | | | | |
| 54 | .155 2805 | 106.92 | .194 4954 | 110.98 | .235 2222 | 115.32 | .277 5647 | 119.97 | | | | |
| 55 | 3.155 9222 | 106.99 | 3.195 1615 | 111.05 | 3.235 9144 | 115.40 | 3.278 2848 | 120.05 | | | | |
| 56 | .156 5643 | 107.05 | .195 8281 | 111.12 | .236 6070 | 115.47 | .279 0053 | 120.13 | | | | |
| 57 | .157 2068 | 107.12 | .196 4950 | 111.19 | .237 3001 | 115.55 | .279 7263 | 120.21 | | | | |
| 58 | .157 8497 | 107.18 | .197 1624 | 111.26 | .237 9956 | 115.62 | .280 4477 | 120.29 | | | | |
| 59 | .158 4930 | 107.25 | .197 8302 | 111.34 | .238 6876 | 115.70 | .281 1697 | 120.37 | | | | |
| 60 | 3.159 1367 | 107.31 | 3.198 4984 | 111.41 | 3.239 3820 | 115.77 | 3.281 8921 | 120.45 | | | | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 152° | | 153° | | 154° | | 155° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 3.281 8921 | 120.45 | 3.326 1448 | 125.46 | 3.372 2684 | 130.85 | 3.420 4064 | 136.66 |
| 1 | .282 6151 | 120.53 | .326 8978 | 125.55 | .373 0538 | 130.94 | .421 2266 | 136.76 |
| 2 | .283 3385 | 120.61 | .327 6513 | 125.63 | .373 8397 | 131.04 | .422 0475 | 136.86 |
| 3 | .284 0624 | 120.69 | .328 4054 | 125.72 | .374 6262 | 131.13 | .422 8690 | 136.96 |
| 4 | .284 7868 | 120.77 | .329 1600 | 125.81 | .375 4133 | 131.22 | .423 6910 | 137.06 |
| 5 | 3.285 5116 | 120.85 | 3.329 9151 | 125.89 | 3.376 2009 | 131.32 | 3.424 5137 | 137.16 |
| 6 | .286 2370 | 120.93 | .330 6707 | 125.98 | .376 9890 | 131.41 | .425 3370 | 137.26 |
| 7 | .286 9628 | 121.01 | .331 4268 | 126.07 | .377 7778 | 131.50 | .426 1609 | 137.37 |
| 8 | .287 6891 | 121.10 | .332 1835 | 126.16 | .378 5671 | 131.60 | .426 9854 | 137.47 |
| 9 | .288 4160 | 121.18 | .332 9407 | 126.24 | .379 3570 | 131.69 | .427 8105 | 137.57 |
| 10 | 3.289 1433 | 121.26 | 3.333 6984 | 126.33 | 3.380 1474 | 131.79 | 3.428 6362 | 137.67 |
| 11 | .289 8711 | 121.34 | .334 4567 | 126.42 | .380 9384 | 131.88 | .429 4626 | 137.77 |
| 12 | .290 5993 | 121.42 | .335 2154 | 126.51 | .381 7300 | 131.98 | .430 2895 | 137.88 |
| 13 | .291 3281 | 121.50 | .335 9747 | 126.59 | .382 5221 | 132.07 | .431 1171 | 137.98 |
| 14 | .292 0574 | 121.59 | .336 7346 | 126.68 | .383 3148 | 132.16 | .431 9452 | 138.08 |
| 15 | 3.292 7872 | 121.67 | 3.337 4949 | 126.77 | 3.384 1081 | 132.26 | 3.432 7740 | 138.18 |
| 16 | .293 5174 | 121.75 | .338 2558 | 126.86 | .384 9019 | 132.35 | .433 6034 | 138.29 |
| 17 | .294 2481 | 121.83 | .339 0172 | 126.95 | .385 6963 | 132.45 | .434 4334 | 138.39 |
| 18 | .294 9794 | 121.91 | .339 7792 | 127.03 | .386 4913 | 132.54 | .435 2641 | 138.49 |
| 19 | .295 7111 | 122.00 | .340 5417 | 127.12 | .387 2869 | 132.64 | .436 0953 | 138.59 |
| 20 | 3.296 4433 | 122.08 | 3.341 3047 | 127.21 | 3.388 0830 | 132.73 | 3.436 9272 | 138.70 |
| 21 | .297 1761 | 122.16 | .342 0682 | 127.30 | .388 8797 | 132.83 | .437 7597 | 138.80 |
| 22 | .297 9093 | 122.24 | .342 8323 | 127.39 | .389 6770 | 132.93 | .438 5928 | 138.90 |
| 23 | .298 6430 | 122.33 | .343 5969 | 127.48 | .390 4749 | 133.02 | .439 4266 | 139.01 |
| 24 | .299 3772 | 122.41 | .344 3620 | 127.57 | .391 2733 | 133.12 | .440 2609 | 139.11 |
| 25 | 3.300 1119 | 122.49 | 3.345 1277 | 127.66 | 3.392 0723 | 133.22 | 3.441 0959 | 139.22 |
| 26 | .300 8471 | 122.58 | .345 8939 | 127.75 | .392 8719 | 133.31 | .441 9315 | 139.32 |
| 27 | .301 5828 | 122.66 | .346 6606 | 127.84 | .393 6720 | 133.41 | .442 7677 | 139.42 |
| 28 | .302 3190 | 122.74 | .347 4279 | 127.93 | .394 4728 | 133.50 | .443 6046 | 139.53 |
| 29 | .303 0557 | 122.83 | .348 1958 | 128.02 | .395 2741 | 133.60 | .444 4421 | 139.63 |
| 30 | 3.303 7929 | 122.91 | 3.348 9641 | 128.11 | 3.396 0760 | 133.70 | 3.445 2802 | 139.74 |
| 31 | .304 5306 | 122.99 | .349 7330 | 128.19 | .396 8785 | 133.79 | .446 1189 | 139.84 |
| 32 | .305 2688 | 123.08 | .350 5024 | 128.28 | .397 6815 | 133.89 | .446 9583 | 139.95 |
| 33 | .306 0075 | 123.16 | .351 2724 | 128.37 | .398 4852 | 133.99 | .447 7983 | 140.05 |
| 34 | .306 7468 | 123.24 | .352 0429 | 128.46 | .399 2894 | 134.09 | .448 6389 | 140.16 |
| 35 | 3.307 4865 | 123.33 | 3.352 8140 | 128.55 | 3.400 0942 | 134.19 | 3.449 4802 | 140.26 |
| 36 | .308 2267 | 123.41 | .353 5856 | 128.65 | .400 8996 | 134.28 | .450 3221 | 140.37 |
| 37 | .308 9674 | 123.50 | .354 3577 | 128.74 | .401 7056 | 134.38 | .451 1646 | 140.47 |
| 38 | .309 7086 | 123.58 | .355 1304 | 128.83 | .402 5122 | 134.48 | .452 0077 | 140.57 |
| 39 | .310 4504 | 123.66 | .355 9037 | 128.92 | .403 3193 | 134.57 | .452 8515 | 140.68 |
| 40 | 3.311 1926 | 123.75 | 3.356 6774 | 129.01 | 3.404 1270 | 134.67 | 3.453 6959 | 140.79 |
| 41 | .311 9354 | 123.83 | .357 4517 | 129.10 | .404 9354 | 134.77 | .454 5410 | 140.90 |
| 42 | .312 6786 | 123.92 | .358 2266 | 129.19 | .405 7443 | 134.87 | .455 3867 | 141.00 |
| 43 | .313 4224 | 124.00 | .359 0020 | 129.28 | .406 5538 | 134.97 | .456 2330 | 141.11 |
| 44 | .314 1667 | 124.09 | .359 7780 | 129.37 | .407 3639 | 135.07 | .457 0800 | 141.21 |
| 45 | 3.314 9115 | 124.17 | 3.360 5545 | 129.46 | 3.408 1746 | 135.16 | 3.457 9276 | 141.32 |
| 46 | .315 6567 | 124.26 | .361 3316 | 129.56 | .408 9859 | 135.26 | .458 7759 | 141.43 |
| 47 | .316 4025 | 124.34 | .362 1092 | 129.65 | .409 7977 | 135.36 | .459 6248 | 141.54 |
| 48 | .317 1489 | 124.43 | .362 8873 | 129.74 | .410 6102 | 135.46 | .460 4743 | 141.64 |
| 49 | .317 8957 | 124.51 | .363 6660 | 129.83 | .411 4233 | 135.56 | .461 3245 | 141.75 |
| 50 | 3.318 6430 | 124.60 | 3.364 4453 | 129.92 | 3.412 2369 | 135.66 | 3.462 1753 | 141.86 |
| 51 | .319 3909 | 124.68 | .365 2251 | 130.01 | .413 0512 | 135.76 | .463 0268 | 141.97 |
| 52 | .320 1392 | 124.77 | .366 0055 | 130.11 | .413 8660 | 135.86 | .463 8789 | 142.07 |
| 53 | .320 8881 | 124.86 | .366 7864 | 130.20 | .414 6815 | 135.96 | .464 7317 | 142.18 |
| 54 | .321 6375 | 124.94 | .367 5679 | 130.29 | .415 4975 | 136.06 | .465 5851 | 142.29 |
| 55 | 3.322 3874 | 125.03 | 3.368 3499 | 130.38 | 3.416 3142 | 136.16 | 3.466 4392 | 142.40 |
| 56 | .323 1379 | 125.11 | .369 1325 | 130.48 | .417 1314 | 136.26 | .467 2939 | 142.51 |
| 57 | .323 8888 | 125.20 | .369 9156 | 130.57 | .417 9492 | 136.36 | .468 1492 | 142.61 |
| 58 | .324 6403 | 125.29 | .370 6993 | 130.66 | .418 7677 | 136.46 | .469 0052 | 142.72 |
| 59 | .325 3923 | 125.37 | .371 4830 | 130.76 | .419 5867 | 136.56 | .469 8619 | 142.83 |
| 60 | 3.326 1448 | 125.46 | 3.372 2684 | 130.85 | 3.420 4064 | 136.66 | 3.470 7192 | 142.94 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 156° | | 157° | | 158° | | 159° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 3.470 7192 | 142.94 | 3.523 3875 | 149.75 | 3.578 6154 | 157.17 | 3.636 6351 | 165.28 |
| 1 | .471 5772 | 143.05 | .524 2864 | 149.87 | .579 5588 | 157.30 | .637 6272 | 165.42 |
| 2 | .472 4358 | 143.16 | .525 1860 | 149.99 | .580 5030 | 157.43 | .638 6202 | 165.56 |
| 3 | .473 2951 | 143.27 | .526 0863 | 150.11 | .581 4480 | 157.56 | .639 6140 | 165.71 |
| 4 | .474 1550 | 143.38 | .526 9873 | 150.23 | .582 3937 | 157.69 | .640 6087 | 165.85 |
| 5 | 3.475 0156 | 143.49 | 3.527 8890 | 150.35 | 3.583 3403 | 157.82 | 3.641 6042 | 165.99 |
| 6 | .475 8769 | 143.60 | .528 7915 | 150.47 | .584 2876 | 157.95 | .642 6006 | 166.13 |
| 7 | .476 7388 | 143.71 | .529 6947 | 150.59 | .585 2357 | 158.08 | .643 5978 | 166.28 |
| 8 | .477 6014 | 143.82 | .530 5985 | 150.71 | .586 1846 | 158.21 | .644 5959 | 166.42 |
| 9 | .478 4646 | 143.93 | .531 5031 | 150.83 | .587 1342 | 158.34 | .645 5948 | 166.56 |
| 10 | 3.479 3285 | 144.04 | 3.532 4085 | 150.95 | 3.588 0847 | 158.47 | 3.646 5946 | 166.71 |
| 11 | .480 1931 | 144.15 | .533 3145 | 151.07 | .589 0359 | 158.61 | .647 5953 | 166.85 |
| 12 | .481 0583 | 144.26 | .534 2213 | 151.19 | .589 9880 | 158.74 | .648 5968 | 166.99 |
| 13 | .481 9242 | 144.37 | .535 1288 | 151.31 | .590 9408 | 158.87 | .649 5992 | 167.14 |
| 14 | .482 7907 | 144.48 | .536 0370 | 151.43 | .591 8944 | 159.00 | .650 6025 | 167.28 |
| 15 | 3.483 6579 | 144.59 | 3.536 9459 | 151.55 | 3.592 8488 | 159.13 | 3.651 6066 | 167.42 |
| 16 | .484 5258 | 144.70 | .537 8556 | 151.67 | .593 8040 | 159.26 | .652 6116 | 167.57 |
| 17 | .485 3944 | 144.81 | .538 7660 | 151.79 | .594 7600 | 159.40 | .653 6175 | 167.72 |
| 18 | .486 2636 | 144.93 | .539 6771 | 151.91 | .595 7167 | 159.53 | .654 6242 | 167.86 |
| 19 | .487 1335 | 145.04 | .540 5890 | 152.04 | .596 6743 | 159.66 | .655 6318 | 168.01 |
| 20 | 3.488 0040 | 145.15 | 3.541 5015 | 152.16 | 3.597 6327 | 159.79 | 3.656 6403 | 168.15 |
| 21 | .488 8752 | 145.26 | .542 4148 | 152.28 | .598 5919 | 159.93 | .657 6497 | 168.30 |
| 22 | .489 7472 | 145.37 | .543 3289 | 152.40 | .599 5518 | 160.06 | .658 6599 | 168.45 |
| 23 | .490 6198 | 145.49 | .544 2436 | 152.52 | .600 5126 | 160.19 | .659 6710 | 168.59 |
| 24 | .491 4930 | 145.60 | .545 1591 | 152.65 | .601 4742 | 160.33 | .660 6830 | 168.74 |
| 25 | 3.492 3670 | 145.71 | 3.546 0754 | 152.77 | 3.602 4365 | 160.46 | 3.661 6959 | 168.89 |
| 26 | .493 2416 | 145.82 | .546 9924 | 152.89 | .603 3997 | 160.60 | .662 7096 | 169.03 |
| 27 | .494 1168 | 145.94 | .547 9101 | 153.01 | .604 3637 | 160.73 | .663 7243 | 169.18 |
| 28 | .494 9928 | 146.05 | .548 8285 | 153.14 | .605 3285 | 160.87 | .664 7398 | 169.33 |
| 29 | .495 8695 | 146.16 | .549 7477 | 153.26 | .606 2941 | 161.00 | .665 7562 | 169.48 |
| 30 | 3.496 7468 | 146.28 | 3.550 6677 | 153.38 | 3.607 2605 | 161.14 | 3.666 7735 | 169.62 |
| 31 | .497 6248 | 146.39 | .551 5883 | 153.51 | .608 2277 | 161.27 | .667 7917 | 169.77 |
| 32 | .498 5035 | 146.50 | .552 5097 | 153.63 | .609 1957 | 161.41 | .668 8108 | 169.92 |
| 33 | .499 3828 | 146.62 | .553 4319 | 153.75 | .610 1646 | 161.54 | .669 8308 | 170.07 |
| 34 | .500 2629 | 146.73 | .554 3548 | 153.88 | .611 1342 | 161.68 | .670 8516 | 170.22 |
| 35 | 3.501 1436 | 146.85 | 3.555 2785 | 154.00 | 3.612 1047 | 161.81 | 3.671 8734 | 170.37 |
| 36 | .502 0250 | 146.96 | .556 2029 | 154.13 | .613 0760 | 161.95 | .672 8961 | 170.52 |
| 37 | .502 9071 | 147.08 | .557 1280 | 154.25 | .614 0481 | 162.09 | .673 9196 | 170.67 |
| 38 | .503 7899 | 147.19 | .558 0539 | 154.38 | .615 0210 | 162.22 | .674 9441 | 170.82 |
| 39 | .504 6734 | 147.31 | .558 9806 | 154.50 | .615 9948 | 162.36 | .675 9694 | 170.97 |
| 40 | 3.505 5576 | 147.42 | 3.559 9080 | 154.63 | 3.616 9693 | 162.50 | 3.676 9957 | 171.12 |
| 41 | .506 4425 | 147.54 | .560 8361 | 154.75 | .617 9447 | 162.63 | .678 0228 | 171.27 |
| 42 | .507 3280 | 147.65 | .561 7650 | 154.88 | .618 9209 | 162.77 | .679 0509 | 171.42 |
| 43 | .508 2143 | 147.77 | .562 6947 | 155.01 | .619 8980 | 162.91 | .680 0799 | 171.57 |
| 44 | .509 1012 | 147.88 | .563 6251 | 155.13 | .620 8758 | 163.05 | .681 1098 | 171.72 |
| 45 | 3.509 9889 | 148.00 | 3.564 5562 | 155.26 | 3.621 8545 | 163.18 | 3.682 1406 | 171.87 |
| 46 | .510 8772 | 148.11 | .565 4882 | 155.38 | .622 8340 | 163.32 | .683 1723 | 172.03 |
| 47 | .511 7662 | 148.23 | .566 4209 | 155.51 | .623 8144 | 163.46 | .684 2049 | 172.18 |
| 48 | .512 6560 | 148.34 | .567 3543 | 155.64 | .624 7956 | 163.60 | .685 2384 | 172.33 |
| 49 | .513 5464 | 148.46 | .568 2885 | 155.76 | .625 7776 | 163.74 | .686 2728 | 172.48 |
| 50 | 3.514 4375 | 148.58 | 3.569 2235 | 155.89 | 3.626 7604 | 163.88 | 3.687 3082 | 172.64 |
| 51 | .515 3294 | 148.70 | .570 1592 | 156.02 | .627 7441 | 164.02 | .688 3445 | 172.79 |
| 52 | .516 2219 | 148.81 | .571 0957 | 156.15 | .628 7287 | 164.16 | .689 3817 | 172.94 |
| 53 | .517 1151 | 148.93 | .572 0330 | 156.27 | .629 7140 | 164.30 | .690 4198 | 173.10 |
| 54 | .518 0090 | 149.05 | .572 9710 | 156.40 | .630 7002 | 164.44 | .691 4588 | 173.25 |
| 55 | 3.518 9037 | 149.17 | 3.573 9098 | 156.53 | 3.631 6873 | 164.58 | 3.692 4988 | 173.40 |
| 56 | .519 7990 | 149.28 | .574 8494 | 156.66 | .632 6751 | 164.72 | .693 5397 | 173.56 |
| 57 | .520 6951 | 149.40 | .575 7897 | 156.79 | .633 6638 | 164.86 | .694 5815 | 173.71 |
| 58 | .521 5918 | 149.52 | .576 7308 | 156.92 | .634 6534 | 165.00 | .695 6243 | 173.87 |
| 59 | .522 4893 | 149.64 | .577 6727 | 157.04 | .635 6438 | 165.14 | .696 6680 | 174.02 |
| 60 | 3.523 3875 | 149.75 | 3.578 6154 | 157.17 | 3.636 6351 | 165.28 | 3.697 7126 | 174.18 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 160° | | 161° | | 162° | | 163° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 3.697 7126 | 174.18 | 3.762 1539 | 183.99 | 3.830 3147 | 194.87 | 3.902 6107 | 207.00 |
| 1 | .698 7581 | 174.34 | .763 2584 | 184.16 | .831 4845 | 195.06 | .903 8534 | 207.21 |
| 2 | .699 8046 | 174.49 | .764 3639 | 184.34 | .832 6554 | 195.25 | .905 0973 | 207.43 |
| 3 | .700 8520 | 174.65 | .765 4704 | 184.51 | .833 8275 | 195.44 | .906 3425 | 207.64 |
| 4 | .701 9003 | 174.80 | .766 5780 | 184.68 | .835 0008 | 195.64 | .907 5890 | 207.86 |
| 5 | 3.702 9496 | 174.96 | 3.767 6867 | 184.86 | 3.836 1752 | 195.83 | 3.908 8368 | 208.08 |
| 6 | .703 9999 | 175.12 | .768 7963 | 185.03 | .837 3508 | 196.02 | .910 0859 | 208.29 |
| 7 | .705 0511 | 175.28 | .769 9070 | 185.20 | .838 5275 | 196.22 | .911 3363 | 208.51 |
| 8 | .706 1032 | 175.43 | .771 0187 | 185.38 | .839 7054 | 196.41 | .912 5880 | 208.72 |
| 9 | .707 1562 | 175.59 | .772 1315 | 185.55 | .840 8844 | 196.60 | .913 8410 | 208.94 |
| 10 | 3.708 2102 | 175.75 | 3.773 2454 | 185.73 | 3.842 0646 | 196.80 | 3.915 0953 | 209.16 |
| 11 | .709 2652 | 175.91 | .774 3603 | 185.90 | .843 2460 | 196.99 | .916 3509 | 209.38 |
| 12 | .710 3211 | 176.07 | .775 4762 | 186.08 | .844 4286 | 197.19 | .917 6078 | 209.60 |
| 13 | .711 3780 | 176.22 | .776 5932 | 186.25 | .845 6123 | 197.38 | .918 8661 | 209.81 |
| 14 | .712 4358 | 176.38 | .777 7112 | 186.43 | .846 7972 | 197.58 | .920 1256 | 210.03 |
| 15 | 3.713 4946 | 176.54 | 3.778 8303 | 186.60 | 3.847 9833 | 197.78 | 3.921 3865 | 210.25 |
| 16 | .714 5543 | 176.70 | .779 9505 | 186.78 | .849 1705 | 197.97 | .922 6487 | 210.48 |
| 17 | .715 6150 | 176.86 | .781 0717 | 186.96 | .850 3589 | 198.17 | .923 9122 | 210.70 |
| 18 | .716 6766 | 177.02 | .782 1940 | 187.14 | .851 5486 | 198.37 | .925 1770 | 210.92 |
| 19 | .717 7392 | 177.18 | .783 3174 | 187.31 | .852 7394 | 198.57 | .926 4432 | 211.14 |
| 20 | 3.718 8028 | 177.34 | 3.784 4418 | 187.49 | 3.853 9314 | 198.76 | 3.927 7107 | 211.36 |
| 21 | .719 8673 | 177.50 | .785 5672 | 187.67 | .855 1245 | 198.96 | .928 9795 | 211.58 |
| 22 | .720 9328 | 177.66 | .786 6938 | 187.85 | .856 3189 | 199.16 | .930 2497 | 211.81 |
| 23 | .721 9993 | 177.83 | .787 8214 | 188.03 | .857 5145 | 199.36 | .931 5212 | 212.03 |
| 24 | .723 0668 | 178.00 | .788 9501 | 188.21 | .858 7112 | 199.56 | .932 7940 | 212.25 |
| 25 | 3.724 1352 | 178.15 | 3.790 0799 | 188.39 | 3.859 9092 | 199.76 | 3.934 0682 | 212.48 |
| 26 | .725 2045 | 178.31 | .791 2108 | 188.57 | .861 1084 | 199.96 | .935 3438 | 212.70 |
| 27 | .726 2749 | 178.47 | .792 3427 | 188.75 | .862 3087 | 200.16 | .936 6207 | 212.93 |
| 28 | .727 3462 | 178.63 | .793 4757 | 188.93 | .863 5103 | 200.36 | .937 8989 | 213.15 |
| 29 | .728 4185 | 178.80 | .794 6098 | 189.11 | .864 7131 | 200.56 | .939 1785 | 213.38 |
| 30 | 3.729 4918 | 178.96 | 3.795 7450 | 189.29 | 3.865 9171 | 200.77 | 3.940 4595 | 213.61 |
| 31 | .730 5661 | 179.13 | .796 8812 | 189.47 | .867 1223 | 200.97 | .941 7418 | 213.83 |
| 32 | .731 6413 | 179.29 | .798 0186 | 189.65 | .868 3287 | 201.17 | .943 0254 | 214.06 |
| 33 | .732 7176 | 179.45 | .799 1571 | 189.83 | .869 5363 | 201.37 | .944 3105 | 214.29 |
| 34 | .733 7948 | 179.62 | .800 2966 | 190.01 | .870 7452 | 201.58 | .945 5969 | 214.52 |
| 35 | 3.734 8730 | 179.78 | 3.801 4372 | 190.20 | 3.871 9552 | 201.78 | 3.946 8847 | 214.74 |
| 36 | .735 9522 | 179.95 | .802 5790 | 190.38 | .873 1665 | 201.98 | .948 1738 | 214.97 |
| 37 | .737 0324 | 180.11 | .803 7218 | 190.56 | .874 3791 | 202.19 | .949 4644 | 215.20 |
| 38 | .738 1136 | 180.28 | .804 8657 | 190.65 | .875 5928 | 202.39 | .950 7563 | 215.43 |
| 39 | .739 1957 | 180.45 | .806 0108 | 190.93 | .876 8078 | 202.60 | .952 0496 | 215.66 |
| 40 | 3.740 2789 | 180.61 | 3.807 1569 | 191.11 | 3.878 0240 | 202.80 | 3.953 3443 | 216.90 |
| 41 | .741 3631 | 180.78 | .808 3041 | 191.30 | .879 2414 | 203.01 | .954 6403 | 216.13 |
| 42 | .742 4482 | 180.94 | .809 4525 | 191.48 | .880 4601 | 203.22 | .955 9378 | 216.36 |
| 43 | .743 5344 | 181.11 | .810 6020 | 191.67 | .881 6800 | 203.42 | .957 2366 | 216.59 |
| 44 | .744 6216 | 181.28 | .811 7525 | 191.86 | .882 9012 | 203.63 | .958 5369 | 216.82 |
| 45 | 3.745 7097 | 181.45 | 3.812 9042 | 192.04 | 3.884 1236 | 203.84 | 3.959 8385 | 217.06 |
| 46 | .746 7989 | 181.61 | .814 0570 | 192.23 | .885 3473 | 204.05 | .961 1416 | 217.29 |
| 47 | .747 8891 | 181.78 | .815 2110 | 192.41 | .886 5722 | 204.26 | .962 4460 | 217.53 |
| 48 | .748 9803 | 181.95 | .816 3660 | 192.60 | .887 7983 | 204.46 | .963 7519 | 217.76 |
| 49 | .750 0725 | 182.12 | .817 5222 | 192.79 | .889 0257 | 204.67 | .965 0592 | 218.00 |
| 50 | 3.751 1657 | 182.29 | 3.818 6795 | 192.98 | 3.890 2544 | 204.88 | 3.966 3678 | 218.23 |
| 51 | .752 2599 | 182.46 | .819 8379 | 193.16 | .891 4843 | 205.09 | .967 6779 | 218.47 |
| 52 | .753 3552 | 182.63 | .820 9974 | 193.35 | .892 7155 | 205.31 | .968 9895 | 218.70 |
| 53 | .754 4514 | 182.80 | .822 1581 | 193.54 | .893 9480 | 205.52 | .970 3024 | 218.94 |
| 54 | .755 5487 | 182.97 | .823 3199 | 193.73 | .895 1817 | 205.73 | .971 6168 | 219.18 |
| 55 | 3.756 6470 | 183.14 | 3.824 4829 | 193.92 | 3.896 4167 | 205.94 | 3.972 9326 | 219.42 |
| 56 | .757 7464 | 183.31 | .825 6470 | 194.11 | .897 6529 | 206.15 | .974 2498 | 219.66 |
| 57 | .758 8467 | 183.48 | .826 8122 | 194.30 | .898 8905 | 206.36 | .975 5684 | 219.90 |
| 58 | .759 9481 | 183.65 | .827 9785 | 194.49 | .900 1293 | 206.57 | .976 8885 | 220.13 |
| 59 | .761 0505 | 183.82 | .829 1460 | 194.68 | .901 3694 | 206.79 | .978 2100 | 220.37 |
| 60 | 3.762 1539 | 183.99 | 3.830 3147 | 194.87 | 3.902 6107 | 207.00 | 3.979 5330 | 220.61 |

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 164° | | 165° | | 166° | | 167° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0 | 3.979 5330 | 220.62 | 4.061 6673 | 236.01 | 4.149 7198 | 253.57 | 4.244 5537 | 273.78 |
| 1 | .980 8574 | 220.86 | .063 0842 | 236.28 | .151 2422 | 253.88 | .246 1975 | 274.14 |
| 2 | .982 1833 | 221.10 | .064 5027 | 236.56 | .152 7664 | 254.19 | .247 8434 | 274.51 |
| 3 | .983 5106 | 221.34 | .065 9229 | 236.83 | .154 2925 | 254.51 | .249 4916 | 274.87 |
| 4 | .984 8394 | 221.58 | .067 3447 | 237.11 | .155 8205 | 254.83 | .251 1419 | 275.24 |
| 5 | 3.986 1696 | 221.83 | 4.068 7682 | 237.39 | 4.157 3504 | 255.14 | 4.252 7944 | 275.60 |
| 6 | .987 5013 | 222.07 | .070 1933 | 237.66 | .158 8822 | 255.46 | .254 4491 | 275.97 |
| 7 | .988 8345 | 222.31 | .071 6201 | 237.94 | .160 4159 | 255.78 | .256 1061 | 276.34 |
| 8 | .990 1691 | 222.56 | .073 0486 | 238.22 | .161 9515 | 256.10 | .257 7652 | 276.71 |
| 9 | .991 5051 | 222.80 | .074 4787 | 238.50 | .163 4891 | 256.42 | .259 4266 | 277.08 |
| 10 | 3.992 8427 | 223.05 | 4.075 9106 | 238.78 | 4.165 0285 | 256.74 | 4.261 0902 | 277.45 |
| 11 | .994 1817 | 223.29 | .077 3441 | 239.06 | .166 5699 | 257.06 | .262 7560 | 277.82 |
| 12 | .995 5222 | 223.54 | .078 7792 | 239.34 | .168 1132 | 257.38 | .264 4240 | 278.20 |
| 13 | .996 8642 | 223.79 | .080 2161 | 239.62 | .169 6585 | 257.70 | .266 0943 | 278.57 |
| 14 | .998 2077 | 224.03 | .081 6546 | 239.90 | .171 2056 | 258.02 | .267 7669 | 278.95 |
| 15 | 3.999 5527 | 224.28 | 4.083 0948 | 240.18 | 4.172 7547 | 258.35 | 4.269 4417 | 279.32 |
| 16 | 4.000 8991 | 224.53 | .084 5368 | 240.46 | .174 3058 | 258.67 | .271 1187 | 279.70 |
| 17 | .002 2471 | 224.78 | .085 9804 | 240.75 | .175 8588 | 259.00 | .272 7981 | 280.08 |
| 18 | .003 5965 | 225.03 | .087 4257 | 241.03 | .177 4138 | 259.33 | .274 4797 | 280.46 |
| 19 | .004 9474 | 225.28 | .088 8728 | 241.32 | .178 9707 | 259.65 | .276 1635 | 280.84 |
| 20 | 4.006 2999 | 225.53 | 4.090 3215 | 241.60 | 4.180 5296 | 259.98 | 4.277 8497 | 281.22 |
| 21 | .007 6538 | 225.78 | .091 7720 | 241.89 | .182 0905 | 260.31 | .279 5381 | 281.60 |
| 22 | .009 0093 | 226.04 | .093 2242 | 242.08 | .183 6534 | 260.64 | .281 2289 | 281.98 |
| 23 | .010 3663 | 226.29 | .094 6781 | 242.56 | .185 2182 | 260.97 | .282 9219 | 282.36 |
| 24 | .011 7248 | 226.54 | .096 1337 | 242.75 | .186 7850 | 261.30 | .284 6173 | 282.75 |
| 25 | 4.013 0848 | 226.79 | 4.097 5911 | 243.04 | 4.188 3538 | 261.63 | 4.286 3149 | 283.14 |
| 26 | .014 4463 | 227.05 | .099 0502 | 243.33 | .189 9246 | 261.96 | .288 0149 | 283.52 |
| 27 | .015 8093 | 227.30 | .100 5110 | 243.62 | .191 4974 | 262.30 | .289 7172 | 283.91 |
| 28 | .017 1739 | 227.55 | .101 9736 | 243.91 | .193 0722 | 262.63 | .291 4218 | 284.30 |
| 29 | .018 5400 | 227.81 | .103 4379 | 244.20 | .194 6490 | 262.97 | .293 1288 | 284.69 |
| 30 | 4.019 9077 | 228.06 | 4.104 9040 | 244.49 | 4.196 2278 | 263.30 | 4.294 8381 | 285.08 |
| 31 | .021 2769 | 228.32 | .106 3718 | 244.78 | .197 8086 | 263.64 | .296 5498 | 285.47 |
| 32 | .022 6476 | 228.58 | .107 8414 | 245.08 | .199 3915 | 263.98 | .298 2638 | 285.87 |
| 33 | .024 0199 | 228.84 | .109 3127 | 245.37 | .200 9764 | 264.32 | .299 9802 | 286.26 |
| 34 | .025 3937 | 229.09 | .110 7858 | 245.67 | .202 5633 | 264.66 | .301 6990 | 286.66 |
| 35 | 4.026 7691 | 229.35 | 4.112 2607 | 245.96 | 4.204 1523 | 265.00 | 4.303 4201 | 287.05 |
| 36 | .028 1460 | 229.62 | .113 7374 | 246.26 | .205 7433 | 265.34 | .305 1436 | 287.45 |
| 37 | .029 5245 | 229.88 | .115 2158 | 246.55 | .207 3393 | 265.68 | .306 8695 | 287.85 |
| 38 | .030 9045 | 230.14 | .116 6960 | 246.85 | .208 9314 | 266.02 | .308 5978 | 288.25 |
| 39 | .032 2861 | 230.40 | .118 1780 | 247.15 | .210 5286 | 266.37 | .310 3285 | 288.65 |
| 40 | 4.033 6693 | 230.66 | 4.119 6618 | 247.45 | 4.212 1278 | 266.71 | 4.312 0616 | 289.05 |
| 41 | .035 0540 | 230.92 | .121 1474 | 247.75 | .213 7291 | 267.06 | .313 7971 | 289.45 |
| 42 | .036 4404 | 231.18 | .122 6348 | 248.05 | .215 3325 | 267.40 | .315 5350 | 289.86 |
| 43 | .037 8283 | 231.45 | .124 1239 | 248.35 | .216 9379 | 267.75 | .317 2753 | 290.26 |
| 44 | .039 2177 | 231.71 | .125 6149 | 248.65 | .218 5455 | 268.10 | .319 0181 | 290.67 |
| 45 | 4.040 6088 | 231.97 | 4.127 1077 | 248.95 | 4.220 1551 | 268.44 | 4.320 7633 | 291.07 |
| 46 | .042 0015 | 232.24 | .128 6023 | 249.25 | .221 7668 | 268.79 | .322 5110 | 291.48 |
| 47 | .043 3957 | 232.51 | .130 0988 | 249.56 | .223 3806 | 269.14 | .324 2611 | 291.89 |
| 48 | .044 7915 | 232.77 | .131 5970 | 249.86 | .224 9965 | 269.50 | .326 0137 | 292.30 |
| 49 | .046 1890 | 233.04 | .133 0971 | 250.17 | .226 6146 | 269.85 | .327 7688 | 292.71 |
| 50 | 4.047 5880 | 233.31 | 4.134 5990 | 250.47 | 4.228 2347 | 270.20 | 4.329 5263 | 293.13 |
| 51 | .048 9887 | 233.57 | .136 1028 | 250.78 | .229 8570 | 270.55 | .331 2863 | 293.54 |
| 52 | .050 3909 | 233.84 | .137 6084 | 251.08 | .231 4814 | 270.91 | .333 0487 | 293.95 |
| 53 | .051 7948 | 234.11 | .139 1158 | 251.39 | .233 1079 | 271.27 | .334 8137 | 294.37 |
| 54 | .053 2003 | 234.38 | .140 6251 | 251.70 | .234 7366 | 271.62 | .336 5812 | 294.79 |
| 55 | 4.054 6074 | 234.65 | 4.142 1362 | 252.01 | 4.236 3674 | 271.98 | 4.338 3511 | 295.20 |
| 56 | .056 0161 | 234.92 | .143 6492 | 252.32 | .238 0003 | 272.34 | .340 1236 | 295.62 |
| 57 | .057 4264 | 235.19 | .145 1641 | 252.63 | .239 6354 | 272.70 | .341 8986 | 296.04 |
| 58 | .058 8384 | 235.46 | .146 6808 | 252.94 | .241 2727 | 273.06 | .343 6762 | 296.47 |
| 59 | .060 2520 | 235.73 | .148 1994 | 253.25 | .242 9121 | 273.42 | .345 4562 | 296.89 |
| 60 | 4.061 6673 | 236.01 | 4.149 7198 | 253.57 | 4.244 5537 | 273.78 | 4.347 2388 | 297.31 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 168° | | | 169° | | | 170° | | | 171° | | |
|----|------------|-----------|--|------------|-----------|--|------------|-----------|--|------------|-----------|--|
| | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | | log M. | Diff. 1". | |
| 0' | 4.347 2388 | 297.31 | | 4.459 1242 | 325.07 | | 4.581 9445 | 358.31 | | 4.717 9835 | 398.87 | |
| 1 | .349 0240 | 297.74 | | .461 0761 | 325.57 | | .584 0962 | 358.92 | | .720 3790 | 399.62 | |
| 2 | .350 8117 | 298.16 | | .463 0311 | 326.08 | | .586 2516 | 359.53 | | .722 7790 | 400.38 | |
| 3 | .352 6019 | 298.59 | | .464 9891 | 326.59 | | .588 4106 | 360.15 | | .725 1835 | 401.14 | |
| 4 | .354 3948 | 299.02 | | .466 9501 | 327.10 | | .590 5734 | 360.76 | | .727 5926 | 401.90 | |
| 5 | 4.356 1902 | 299.45 | | 4.468 9142 | 327.61 | | 4.592 7398 | 361.38 | | 4.730 0063 | 402.66 | |
| 6 | .357 9882 | 299.88 | | .470 8814 | 328.12 | | .594 9100 | 362.00 | | .732 4245 | 403.43 | |
| 7 | .359 7888 | 300.31 | | .472 8517 | 328.64 | | .597 0838 | 362.62 | | .734 8474 | 404.19 | |
| 8 | .361 5919 | 300.75 | | .474 8250 | 329.15 | | .599 2615 | 363.25 | | .737 2749 | 404.96 | |
| 9 | .363 3977 | 301.18 | | .476 8015 | 329.67 | | .601 4428 | 363.88 | | .739 7070 | 405.74 | |
| 10 | 4.365 2061 | 301.62 | | 4.478 7811 | 330.19 | | 4.603 6280 | 364.50 | | 4.742 1438 | 406.52 | |
| 11 | .367 0171 | 302.05 | | .480 7637 | 330.71 | | .605 8169 | 365.14 | | .744 5852 | 407.30 | |
| 12 | .368 8308 | 302.49 | | .482 7495 | 331.23 | | .608 0096 | 365.77 | | .747 0314 | 408.08 | |
| 13 | .370 6470 | 302.93 | | .484 7385 | 331.75 | | .610 2061 | 366.40 | | .749 4822 | 408.87 | |
| 14 | .372 4659 | 303.37 | | .486 7306 | 332.28 | | .612 4064 | 367.04 | | .751 9378 | 409.66 | |
| 15 | 4.374 2875 | 303.81 | | 4.488 7258 | 332.81 | | 4.614 6106 | 367.68 | | 4.754 3981 | 410.45 | |
| 16 | .376 1117 | 304.26 | | .490 7242 | 333.33 | | .616 8186 | 368.32 | | .756 8632 | 411.24 | |
| 17 | .377 9386 | 304.70 | | .492 7258 | 333.86 | | .619 0304 | 368.96 | | .759 3330 | 412.04 | |
| 18 | .379 7681 | 305.15 | | .494 7306 | 334.40 | | .621 2461 | 369.61 | | .761 8077 | 412.84 | |
| 19 | .381 6003 | 305.59 | | .496 7386 | 334.93 | | .623 4657 | 370.26 | | .764 2872 | 413.65 | |
| 20 | 4.383 4352 | 306.04 | | 4.498 7498 | 335.46 | | 4.625 6892 | 370.91 | | 4.766 7715 | 414.46 | |
| 21 | .385 2728 | 306.49 | | .500 7642 | 336.00 | | .627 9166 | 371.56 | | .769 2606 | 415.27 | |
| 22 | .387 1131 | 306.94 | | .502 7818 | 336.54 | | .630 1480 | 372.21 | | .771 7547 | 416.08 | |
| 23 | .388 9561 | 307.39 | | .504 8026 | 337.08 | | .632 3832 | 372.87 | | .774 2536 | 416.90 | |
| 24 | .390 8019 | 307.85 | | .506 8267 | 337.62 | | .634 6224 | 373.53 | | .776 7574 | 417.72 | |
| 25 | 4.392 6503 | 308.30 | | 4.508 8541 | 338.16 | | 4.636 8656 | 374.19 | | 4.779 2662 | 418.54 | |
| 26 | .394 5015 | 308.76 | | .510 8847 | 338.71 | | .639 1127 | 374.86 | | .781 7799 | 419.37 | |
| 27 | .396 3554 | 309.21 | | .512 9186 | 339.26 | | .641 3639 | 375.52 | | .784 2986 | 420.20 | |
| 28 | .398 2121 | 309.67 | | .514 9558 | 339.80 | | .643 6190 | 376.19 | | .786 8222 | 421.03 | |
| 29 | .400 0715 | 310.13 | | .516 9962 | 340.35 | | .645 8781 | 376.86 | | .789 3509 | 421.86 | |
| 30 | 4.401 9337 | 310.59 | | 4.519 0400 | 340.91 | | 4.648 1413 | 377.53 | | 4.791 8846 | 422.70 | |
| 31 | .403 7986 | 311.06 | | .521 0871 | 341.46 | | .650 4085 | 378.21 | | .794 4233 | 423.54 | |
| 32 | .405 6662 | 311.52 | | .523 1376 | 342.02 | | .652 6798 | 378.89 | | .796 9671 | 424.39 | |
| 33 | .407 5368 | 311.99 | | .525 1913 | 342.57 | | .654 9552 | 379.57 | | .799 5160 | 425.24 | |
| 34 | .409 4102 | 312.45 | | .527 2484 | 343.13 | | .657 2346 | 380.25 | | .802 0700 | 426.09 | |
| 35 | 4.411 2863 | 312.92 | | 4.529 3089 | 343.69 | | 4.659 5182 | 380.93 | | 4.804 6291 | 426.95 | |
| 36 | .413 1652 | 313.39 | | .531 3728 | 344.26 | | .661 8059 | 381.62 | | .807 1934 | 427.81 | |
| 37 | .415 0469 | 313.86 | | .533 4400 | 344.82 | | .664 0977 | 382.31 | | .809 7628 | 428.67 | |
| 38 | .416 9315 | 314.33 | | .535 5106 | 345.39 | | .666 3936 | 383.00 | | .812 3374 | 429.53 | |
| 39 | .418 8189 | 314.80 | | .537 5846 | 345.95 | | .668 6937 | 383.70 | | .814 9172 | 430.40 | |
| 40 | 4.420 7091 | 315.28 | | 4.539 6620 | 346.52 | | 4.670 9980 | 384.39 | | 4.817 5022 | 431.28 | |
| 41 | .422 6022 | 315.75 | | .541 7429 | 347.09 | | .673 3064 | 385.09 | | .820 0925 | 432.15 | |
| 42 | .424 4982 | 316.23 | | .543 8272 | 347.67 | | .675 6191 | 385.80 | | .822 6881 | 433.03 | |
| 43 | .426 3970 | 316.71 | | .545 9149 | 348.24 | | .677 9360 | 386.50 | | .825 2889 | 433.91 | |
| 44 | .428 2987 | 317.19 | | .548 0061 | 348.82 | | .680 2571 | 387.21 | | .827 8950 | 434.80 | |
| 45 | 4.430 2033 | 317.67 | | 4.550 1007 | 349.40 | | 4.682 5825 | 387.92 | | 4.830 5065 | 435.69 | |
| 46 | .432 1108 | 318.16 | | .552 1989 | 349.98 | | .684 9121 | 388.63 | | .833 1234 | 436.59 | |
| 47 | .434 0212 | 318.64 | | .554 3005 | 350.56 | | .687 2460 | 389.34 | | .835 7456 | 437.48 | |
| 48 | .435 9345 | 319.13 | | .556 4056 | 351.15 | | .689 5842 | 390.06 | | .838 3732 | 438.38 | |
| 49 | .437 8507 | 319.61 | | .558 5143 | 351.73 | | .691 9268 | 390.78 | | .841 0062 | 439.29 | |
| 50 | 4.439 7698 | 320.10 | | 4.560 6264 | 352.32 | | 4.694 2736 | 391.50 | | 4.843 6446 | 440.20 | |
| 51 | .441 6919 | 320.59 | | .562 7421 | 352.91 | | .696 6248 | 392.23 | | .846 2886 | 441.11 | |
| 52 | .443 6169 | 321.08 | | .564 8614 | 353.50 | | .698 9803 | 392.96 | | .848 9380 | 442.03 | |
| 53 | .445 5449 | 321.58 | | .566 9842 | 354.10 | | .701 3402 | 393.68 | | .851 5929 | 442.95 | |
| 54 | .447 4758 | 322.07 | | .569 1106 | 354.69 | | .703 7046 | 394.42 | | .854 2533 | 443.87 | |
| 55 | 4.449 4097 | 322.57 | | 4.571 2405 | 355.29 | | 4.706 0733 | 395.15 | | 4.856 9193 | 444.80 | |
| 56 | .451 3466 | 323.06 | | .573 3741 | 355.89 | | .708 4464 | 395.89 | | .859 5909 | 445.73 | |
| 57 | .453 2865 | 323.56 | | .575 5113 | 356.49 | | .710 8240 | 396.63 | | .862 2680 | 446.66 | |
| 58 | .455 2294 | 324.06 | | .577 6521 | 357.10 | | .713 2060 | 397.38 | | .864 9508 | 447.60 | |
| 59 | .457 1753 | 324.56 | | .579 7965 | 357.70 | | .715 5925 | 398.12 | | .867 6392 | 448.54 | |
| 60 | 4.459 1242 | 325.07 | | 4.581 9445 | 358.31 | | 4.717 9835 | 398.87 | | 4.870 3333 | 449.49 | |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 172° | | 173° | | 174° | | 175° | |
|----|------------|-----------|------------|-----------|------------|-----------|------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 4.870 3333 | 449.49 | 5.043 3285 | 514.47 | 5.243 3165 | 601.00 | 5.480 1373 | 722.00 |
| 1 | .873 0311 | 450.44 | .046 4191 | 515.71 | .246 9276 | 602.69 | .484 4765 | 724.42 |
| 2 | .875 7386 | 451.39 | .049 5171 | 516.96 | .250 5488 | 604.38 | .488 8304 | 726.87 |
| 3 | .878 4499 | 452.35 | .052 6226 | 518.21 | .254 1802 | 606.08 | .493 1989 | 729.33 |
| 4 | .881 1668 | 453.31 | .055 7356 | 519.47 | .257 8218 | 607.80 | .497 5823 | 731.80 |
| 5 | 4.883 8896 | 454.28 | 5.058 8562 | 520.73 | 5.261 4738 | 609.53 | 5.501 9806 | 734.30 |
| 6 | .886 6182 | 455.25 | .061 9843 | 522.00 | .265 1361 | 611.26 | .506 3939 | 736.81 |
| 7 | .889 3526 | 456.23 | .065 1202 | 523.28 | .268 8089 | 613.00 | .510 8223 | 739.33 |
| 8 | .892 0929 | 457.20 | .068 2637 | 524.56 | .272 4922 | 614.75 | .515 2659 | 741.87 |
| 9 | .894 8391 | 458.19 | .071 4149 | 525.85 | .276 1860 | 616.52 | .519 7248 | 744.44 |
| 10 | 4.897 5912 | 459.17 | 5.074 5738 | 527.14 | 5.279 8904 | 618.29 | 5.524 1992 | 747.02 |
| 11 | .900 3492 | 460.16 | .077 7406 | 528.44 | .283 6055 | 620.08 | .528 6890 | 749.61 |
| 12 | .903 1132 | 461.16 | .080 9151 | 529.75 | .287 3313 | 621.87 | .533 1946 | 752.23 |
| 13 | .905 8831 | 462.16 | .084 0976 | 531.06 | .291 0680 | 623.67 | .537 7158 | 754.86 |
| 14 | .908 6591 | 463.16 | .087 2879 | 532.38 | .294 8154 | 625.49 | .542 2529 | 757.51 |
| 15 | 4.911 4411 | 464.17 | 5.090 4862 | 533.71 | 5.298 5738 | 627.31 | 5.546 8060 | 760.18 |
| 16 | .914 2291 | 465.18 | .093 6924 | 535.04 | .302 3432 | 629.15 | .551 3751 | 762.87 |
| 17 | .917 0233 | 466.20 | .096 9067 | 536.38 | .306 1237 | 631.00 | .555 9605 | 765.58 |
| 18 | .919 8235 | 467.22 | .100 1290 | 537.73 | .309 9152 | 632.85 | .560 5621 | 768.31 |
| 19 | .922 6299 | 468.25 | .103 3594 | 539.08 | .313 7179 | 634.72 | .565 1802 | 771.05 |
| 20 | 4.925 4425 | 469.28 | 5.106 5980 | 540.44 | 5.317 5319 | 636.60 | 5.569 8148 | 773.82 |
| 21 | .928 2612 | 470.31 | .109 8447 | 541.81 | .321 3571 | 638.49 | .574 4661 | 776.61 |
| 22 | .931 0862 | 471.35 | .113 0997 | 543.18 | .325 1938 | 640.39 | .579 1341 | 779.41 |
| 23 | .933 9174 | 472.39 | .116 3629 | 544.56 | .329 0418 | 642.30 | .583 8190 | 782.24 |
| 24 | .936 7549 | 473.44 | .119 6344 | 545.95 | .332 9014 | 644.23 | .588 5210 | 785.08 |
| 25 | 4.939 5987 | 474.49 | 5.122 9143 | 547.34 | 5.336 7726 | 646.16 | 5.593 2401 | 787.95 |
| 26 | .942 4489 | 475.55 | .126 2026 | 548.74 | .340 6554 | 648.11 | .597 9764 | 790.84 |
| 27 | .945 3053 | 476.61 | .129 4992 | 550.15 | .344 5499 | 650.07 | .602 7302 | 793.75 |
| 28 | .948 1682 | 477.68 | .132 8044 | 551.57 | .348 4562 | 652.04 | .607 5014 | 796.68 |
| 29 | .951 0375 | 478.75 | .136 1181 | 552.99 | .352 3744 | 654.02 | .612 2903 | 799.63 |
| 30 | 4.953 9132 | 479.83 | 5.139 4403 | 554.42 | 5.356 3045 | 656.01 | 5.617 0970 | 802.60 |
| 31 | .956 7954 | 480.91 | .142 7711 | 555.86 | .360 2466 | 658.02 | .621 9216 | 805.60 |
| 32 | .959 6841 | 481.99 | .146 1106 | 557.30 | .364 2007 | 660.04 | .626 7642 | 808.62 |
| 33 | .962 5793 | 483.08 | .149 4588 | 558.75 | .368 1671 | 662.07 | .631 6250 | 811.66 |
| 34 | .965 4811 | 484.18 | .152 8157 | 560.21 | .372 1456 | 664.11 | .636 5041 | 814.72 |
| 35 | 4.968 3894 | 485.28 | 5.156 1813 | 561.68 | 5.376 1364 | 666.17 | 5.641 4017 | 817.81 |
| 36 | .971 3044 | 486.38 | .159 5558 | 563.16 | .380 1396 | 668.24 | .646 3179 | 820.92 |
| 37 | .974 2260 | 487.49 | .162 9392 | 564.64 | .384 1553 | 670.32 | .651 2528 | 824.05 |
| 38 | .977 1543 | 488.61 | .166 3315 | 566.13 | .388 1834 | 672.41 | .656 2065 | 827.21 |
| 39 | .980 0893 | 489.73 | .169 7328 | 567.63 | .392 2242 | 674.52 | .661 1793 | 830.39 |
| 40 | 4.983 0311 | 490.85 | 5.173 1431 | 569.13 | 5.396 2777 | 676.64 | 5.666 1713 | 833.60 |
| 41 | .985 9795 | 491.98 | .176 5624 | 570.65 | .400 3439 | 678.77 | .671 1825 | 836.83 |
| 42 | .988 9348 | 493.12 | .179 9908 | 572.17 | .404 4229 | 680.92 | .676 2132 | 840.08 |
| 43 | .991 8970 | 494.26 | .183 4284 | 573.70 | .408 5149 | 683.08 | .681 2635 | 843.36 |
| 44 | .994 8659 | 495.40 | .186 8752 | 575.24 | .412 6199 | 685.25 | .686 3336 | 846.67 |
| 45 | 4.997 8418 | 496.55 | 5.190 3312 | 576.78 | 5.416 7379 | 687.44 | 5.691 4236 | 850.00 |
| 46 | 5.000 8246 | 497.71 | .193 7966 | 578.34 | .420 8692 | 689.64 | .696 5337 | 853.36 |
| 47 | .003 8143 | 498.87 | .197 2713 | 579.90 | .425 0136 | 691.85 | .701 6640 | 856.75 |
| 48 | .006 8111 | 500.04 | .200 7554 | 581.47 | .429 1714 | 694.08 | .706 8147 | 860.16 |
| 49 | .009 8148 | 501.21 | .204 2489 | 583.05 | .433 3427 | 696.33 | .711 9860 | 863.60 |
| 50 | 5.012 8256 | 502.39 | 5.207 7520 | 584.64 | 5.437 5274 | 698.59 | 5.717 1779 | 867.06 |
| 51 | .015 8435 | 503.57 | .211 2646 | 586.23 | .441 7258 | 700.86 | .722 3908 | 870.56 |
| 52 | .018 8665 | 504.76 | .214 7868 | 587.84 | .445 9378 | 703.15 | .727 6247 | 874.08 |
| 53 | .021 9008 | 505.95 | .218 3186 | 589.45 | .450 1636 | 705.45 | .732 8798 | 877.63 |
| 54 | .024 9399 | 507.15 | .221 8602 | 591.07 | .454 4032 | 707.77 | .738 1563 | 881.21 |
| 55 | 5.027 9864 | 508.36 | 5.225 4116 | 592.71 | 5.458 6568 | 710.10 | 5.743 4544 | 884.82 |
| 56 | .031 0402 | 509.57 | .228 9727 | 594.35 | .462 9244 | 712.45 | .748 7742 | 888.46 |
| 57 | .034 1013 | 510.79 | .232 5437 | 596.00 | .467 2062 | 714.81 | .754 1159 | 892.13 |
| 58 | .037 1697 | 512.01 | .236 1247 | 597.66 | .471 5022 | 717.19 | .759 4798 | 895.83 |
| 59 | .040 2454 | 513.24 | .239 7156 | 599.32 | .475 8125 | 719.59 | .764 8659 | 899.56 |
| 60 | 5.043 3285 | 514.47 | 5.243 3165 | 601.00 | 5.480 1373 | 722.00 | 5.770 2745 | 903.31 |

TABLE VI.

For finding the True Anomaly or the Time from the Perihelion in a Parabolic Orbit.

| v. | 176° | | 177° | | 178° | | 179° | |
|----|------------|-----------|------------|-----------|------------|-----------|-------------|-----------|
| | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". | log M. | Diff. 1". |
| 0' | 5.770 2745 | 903.3 | 6.144 6289 | 1205.3 | 6.672 5724 | 1808.8 | 7.575 4640 | 3619 |
| 1 | .775 7058 | 907.1 | .151 8807 | 1212.0 | .683 4709 | 1824.0 | .597 3596 | 3680 |
| 2 | .781 1599 | 910.9 | .159 1733 | 1218.8 | .694 4613 | 1839.5 | .619 6295 | 3744 |
| 3 | .786 6370 | 914.8 | .166 5070 | 1225.7 | .705 5454 | 1855.3 | .642 2868 | 3809 |
| 4 | .792 1374 | 918.7 | .173 8823 | 1232.7 | .716 7248 | 1871.3 | .665 3452 | 3877 |
| 5 | 5.797 6612 | 922.6 | 6.181 2997 | 1239.8 | 6.728 0010 | 1887.5 | 7.688 8192 | 3948 |
| 6 | .803 2086 | 926.6 | .188 7597 | 1246.9 | .739 3758 | 1904.1 | .712 7239 | 4021 |
| 7 | .808 7798 | 930.6 | .196 2628 | 1254.1 | .750 8509 | 1921.0 | .737 0756 | 4097 |
| 8 | .814 3751 | 934.6 | .203 8095 | 1261.4 | .762 4279 | 1938.2 | .761 8913 | 4176 |
| 9 | .819 9946 | 938.6 | .211 4002 | 1268.8 | .774 1090 | 1955.6 | .787 1889 | 4257 |
| 10 | 5.825 6386 | 942.7 | 6.219 0354 | 1276.3 | 6.785 8958 | 1973.4 | 7.812 9876 | 4343 |
| 11 | .831 3073 | 946.8 | .226 7158 | 1283.8 | .797 7904 | 1991.5 | .839 3075 | 4431 |
| 12 | .837 0008 | 951.0 | .234 4419 | 1291.5 | .809 7946 | 2010.0 | .866 1702 | 4524 |
| 13 | .842 7195 | 955.2 | .242 2142 | 1299.2 | .821 9106 | 2028.8 | .893 5986 | 4620 |
| 14 | .848 4634 | 959.5 | .250 0333 | 1307.1 | .834 1404 | 2048.0 | .921 6170 | 4720 |
| 15 | 5.854 2329 | 963.7 | 6.257 8997 | 1315.0 | 6.846 4863 | 2067.5 | 7.950 2513 | 4825 |
| 16 | .860 0282 | 968.0 | .265 8139 | 1323.0 | .858 9503 | 2087.3 | 7.979 5292 | 4935 |
| 17 | .865 8495 | 972.4 | .273 7766 | 1331.1 | .871 5348 | 2107.6 | 8.009 4802 | 5050 |
| 18 | .871 6970 | 976.8 | .281 7884 | 1339.4 | .884 2422 | 2128.3 | .040 1361 | 5170 |
| 19 | .877 5710 | 981.2 | .289 8499 | 1347.7 | .897 0749 | 2149.4 | .071 5309 | 5296 |
| 20 | 5.883 4717 | 985.7 | 6.297 9617 | 1356.2 | 6.910 0353 | 2170.9 | 8.103 7011 | 5428 |
| 21 | .889 3993 | 990.2 | .306 1244 | 1364.7 | .923 1261 | 2192.8 | .136 6857 | 5568 |
| 22 | .895 3542 | 994.8 | .314 3387 | 1373.3 | .936 3498 | 2215.2 | .170 5274 | 5714 |
| 23 | .901 3365 | 999.4 | .322 6052 | 1382.1 | .949 7093 | 2238.0 | .205 2717 | 5869 |
| 24 | .907 3465 | 1004.0 | .330 9247 | 1391.0 | .963 2073 | 2261.4 | .240 9679 | 6032 |
| 25 | 5.913 3845 | 1008.7 | 6.339 2977 | 1400.0 | 6.976 8466 | 2285.2 | 8.277 6700 | 6204 |
| 26 | .919 4507 | 1013.4 | .347 7249 | 1409.1 | 6.990 6304 | 2309.6 | .315 4361 | 6387 |
| 27 | .925 5454 | 1018.1 | .356 2072 | 1418.3 | 7.004 5616 | 2334.3 | .354 3298 | 6580 |
| 28 | .931 6688 | 1022.9 | .364 7451 | 1427.6 | .018 6437 | 2359.7 | .394 4205 | 6786 |
| 29 | .937 8213 | 1027.8 | .373 3395 | 1437.1 | .032 8796 | 2385.7 | .435 7842 | 7004 |
| 30 | 5.944 0030 | 1032.7 | 6.381 9910 | 1446.7 | 7.047 2729 | 2412.2 | 8.478 5044 | 7238 |
| 31 | .950 2144 | 1037.6 | .390 7005 | 1456.4 | .061 8271 | 2439.4 | .522 6731 | 7488 |
| 32 | .956 4556 | 1042.6 | .399 4687 | 1466.2 | .076 5458 | 2467.1 | .568 3920 | 7755 |
| 33 | .962 7269 | 1047.7 | .408 2965 | 1476.2 | .091 4329 | 2495.4 | .615 7739 | 8042 |
| 34 | .969 0287 | 1052.9 | .417 1846 | 1486.4 | .106 4921 | 2524.5 | .664 9442 | 8352 |
| 35 | 5.975 3613 | 1058.0 | 6.426 1337 | 1496.7 | 7.121 7276 | 2554.2 | 8.716 0431 | 8686 |
| 36 | .981 7249 | 1063.2 | .435 1449 | 1507.0 | .137 1434 | 2584.6 | .769 2286 | 9048 |
| 37 | .988 1198 | 1068.4 | .444 2191 | 1517.6 | .152 7440 | 2615.8 | .824 6779 | 9441 |
| 38 | 5.994 5464 | 1073.7 | .453 3569 | 1528.3 | .168 5336 | 2647.6 | .882 5925 | 9870 |
| 39 | 6.001 0050 | 1079.1 | .462 5594 | 1539.2 | .184 5171 | 2680.4 | .943 2018 | 10340 |
| 40 | 6.007 4958 | 1084.5 | 6.471 8275 | 1550.2 | 7.200 6993 | 2713.9 | 9.006 7690 | 10857 |
| 41 | .014 0192 | 1089.9 | .481 1620 | 1561.3 | .217 0850 | 2748.3 | .073 5974 | 11429 |
| 42 | .020 5756 | 1095.4 | .490 5641 | 1572.6 | .233 6796 | 2783.5 | .144 0401 | 12064 |
| 43 | .027 1652 | 1101.0 | .500 0346 | 1584.1 | .250 4884 | 2819.7 | .218 5102 | 12773 |
| 44 | .033 7885 | 1106.7 | .509 5746 | 1595.8 | .267 5170 | 2856.8 | .297 4963 | 13572 |
| 45 | 6.040 4457 | 1112.4 | 6.519 1850 | 1607.7 | 7.284 7712 | 2894.8 | 9.381 5820 | 14476 |
| 46 | .047 1372 | 1118.1 | .528 8669 | 1619.6 | .302 2571 | 2934.1 | .471 4711 | 15510 |
| 47 | .053 8634 | 1123.9 | .538 6216 | 1631.8 | .319 9810 | 2974.2 | .568 0247 | 16704 |
| 48 | .060 6246 | 1129.8 | .548 4499 | 1644.2 | .337 9494 | 3015.6 | .672 3106 | 18096 |
| 49 | .067 4212 | 1135.7 | .558 3530 | 1656.8 | .356 1692 | 3058.1 | .785 6758 | 19741 |
| 50 | 6.074 2535 | 1141.7 | 6.568 3320 | 1669.6 | 7.374 6475 | 3101.7 | 9.909 8535 | 21715 |
| 51 | .081 1219 | 1147.7 | .578 3881 | 1682.4 | .393 3918 | 3146.8 | 10.047 1256 | 24127 |
| 52 | .088 0267 | 1153.8 | .588 5227 | 1695.6 | .412 4099 | 3193.0 | .200 5829 | 27144 |
| 53 | .094 9689 | 1160.0 | .598 7368 | 1708.9 | .431 7097 | 3240.7 | .374 5584 | 31023 |
| 54 | .101 9479 | 1166.3 | .609 0317 | 1722.6 | .451 2999 | 3289.9 | .575 3986 | 36197 |
| 55 | 6.108 9647 | 1172.6 | 6.619 4086 | 1736.4 | 7.471 1892 | 3340.3 | 10.812 9421 | 43450 |
| 56 | .116 0196 | 1179.0 | .629 8689 | 1750.3 | .491 3870 | 3392.6 | 11.103 6719 | |
| 57 | .123 1131 | 1185.4 | .640 4141 | 1764.5 | .511 9029 | 3446.5 | 11.478 4880 | |
| 58 | .130 2455 | 1192.0 | .651 0455 | 1779.0 | .532 7472 | 3502.1 | 12.006 7617 | |
| 59 | .137 4173 | 1198.6 | .661 7645 | 1793.8 | .553 9305 | 3559.6 | 12.909 8516 | |
| 60 | 6.144 6289 | 1205.3 | 6.672 5724 | 1808.8 | 7.575 4640 | 3618.7 | | |

TABLE VII.

For finding the True Anomaly in a Parabolic Orbit when v is nearly 180° .

| w | | | Δ_0 | | | Diff. | | | w | | | Δ_0 | | | Diff. | | | w | | | Δ_0 | | | Diff. | | |
|----------|----|---|------------|-------|------|----------|---|---|----------|----|---|------------|-------|------|----------|----|---|----------|---|----|------------|-------|-------|----------|---|---|
| $^\circ$ | ' | " | ' | " | " | $^\circ$ | ' | " | $^\circ$ | ' | " | ' | " | " | $^\circ$ | ' | " | $^\circ$ | ' | " | ' | " | " | $^\circ$ | ' | " |
| 155 | 0 | | 3 | 23.09 | | | | | 160 | 0 | | 1 | 6.70 | | 165 | 0 | | 165 | 0 | | 1 | 15.85 | | | | |
| | 5 | | | 19.74 | 3.35 | | | | | 5 | | | 5.33 | 1.37 | | 10 | | | | 10 | | | 14.98 | 0.87 | | |
| | 10 | | | 16.43 | 3.31 | | | | | 10 | | | 3.97 | 1.36 | | 20 | | | | 20 | | | 14.16 | 0.82 | | |
| | 15 | | | 13.17 | 3.26 | | | | | 15 | | | 2.64 | 1.33 | | 30 | | | | 30 | | | 13.38 | 0.78 | | |
| | 20 | | | 9.95 | 3.22 | | | | | 20 | | | 1.33 | 1.31 | | 40 | | | | 40 | | | 12.63 | 0.75 | | |
| | 25 | | | 6.77 | 3.18 | | | | | 25 | | | 0.04 | 1.29 | | 50 | | | | 50 | | | 11.91 | 0.72 | | |
| | | | | | 3.14 | | | | | | | | | 1.26 | | | | | | | | | | 0.69 | | |
| 155 | 30 | | 3 | 3.63 | | | | | 160 | 30 | | 0 | 58.78 | | 166 | 0 | | 166 | 0 | | 0 | 11.22 | | | | |
| | 35 | | | 0.54 | 3.09 | | | | | 35 | | | 57.54 | 1.24 | | 10 | | | | 10 | | | 10.57 | 0.65 | | |
| | 40 | | 2 | 57.49 | 3.05 | | | | | 40 | | | 56.31 | 1.23 | | 20 | | | | 20 | | | 9.95 | 0.62 | | |
| | 45 | | | 54.48 | 3.01 | | | | | 45 | | | 55.11 | 1.20 | | 30 | | | | 30 | | | 9.36 | 0.59 | | |
| | 50 | | | 51.51 | 2.97 | | | | | 50 | | | 53.93 | 1.18 | | 40 | | | | 40 | | | 8.80 | 0.56 | | |
| | 55 | | | 48.58 | 2.93 | | | | | 55 | | | 52.77 | 1.16 | | 50 | | | | 50 | | | 8.26 | 0.54 | | |
| | | | | | 2.89 | | | | | | | | | 1.14 | | | | | | | | | | 0.51 | | |
| 156 | 0 | | 2 | 45.69 | | | | | 161 | 0 | | 0 | 51.63 | | 167 | 0 | | 167 | 0 | | 0 | 7.75 | | | | |
| | 5 | | | 42.84 | 2.85 | | | | | 5 | | | 50.50 | 1.13 | | 10 | | | | 10 | | | 7.27 | 0.48 | | |
| | 10 | | | 40.03 | 2.81 | | | | | 10 | | | 49.40 | 1.10 | | 20 | | | | 20 | | | 6.81 | 0.46 | | |
| | 15 | | | 37.26 | 2.77 | | | | | 15 | | | 48.32 | 1.08 | | 30 | | | | 30 | | | 6.37 | 0.44 | | |
| | 20 | | | 34.53 | 2.73 | | | | | 20 | | | 47.26 | 1.06 | | 40 | | | | 40 | | | 5.96 | 0.41 | | |
| | 25 | | | 31.83 | 2.70 | | | | | 25 | | | 46.21 | 1.05 | | 50 | | | | 50 | | | 5.57 | 0.39 | | |
| | | | | | 2.66 | | | | | | | | | 1.02 | | | | | | | | | | 0.37 | | |
| 156 | 30 | | 2 | 29.17 | | | | | 161 | 30 | | 0 | 45.19 | | 168 | 0 | | 168 | 0 | | 0 | 5.20 | | | | |
| | 35 | | | 26.55 | 2.62 | | | | | 35 | | | 44.18 | 1.01 | | 10 | | | | 10 | | | 4.84 | 0.36 | | |
| | 40 | | | 23.97 | 2.58 | | | | | 40 | | | 43.19 | 0.99 | | 20 | | | | 20 | | | 4.51 | 0.33 | | |
| | 45 | | | 21.43 | 2.54 | | | | | 45 | | | 42.22 | 0.97 | | 30 | | | | 30 | | | 4.20 | 0.31 | | |
| | 50 | | | 18.92 | 2.51 | | | | | 50 | | | 41.26 | 0.96 | | 40 | | | | 40 | | | 3.90 | 0.30 | | |
| | 55 | | | 16.44 | 2.48 | | | | | 55 | | | 40.33 | 0.93 | | 50 | | | | 50 | | | 3.62 | 0.28 | | |
| | | | | | 2.44 | | | | | | | | | 0.92 | | | | | | | | | | 0.26 | | |
| 157 | 0 | | 2 | 14.00 | | | | | 162 | 0 | | 0 | 39.41 | | 169 | 0 | | 169 | 0 | | 0 | 3.36 | | | | |
| | 5 | | | 11.59 | 2.41 | | | | | 5 | | | 38.51 | 0.90 | | 10 | | | | 10 | | | 3.11 | 0.25 | | |
| | 10 | | | 9.22 | 2.37 | | | | | 10 | | | 37.62 | 0.89 | | 20 | | | | 20 | | | 2.88 | 0.23 | | |
| | 15 | | | 6.89 | 2.33 | | | | | 15 | | | 36.75 | 0.87 | | 30 | | | | 30 | | | 2.66 | 0.22 | | |
| | 20 | | | 4.58 | 2.31 | | | | | 20 | | | 35.90 | 0.85 | | 40 | | | | 40 | | | 2.46 | 0.20 | | |
| | 25 | | | 2.31 | 2.27 | | | | | 25 | | | 35.06 | 0.84 | | 50 | | | | 50 | | | 2.27 | 0.19 | | |
| | | | | | 2.23 | | | | | | | | | 0.82 | | | | | | | | | | 0.18 | | |
| 157 | 30 | | 2 | 0.08 | | | | | 162 | 30 | | 0 | 34.24 | | 170 | 0 | | 170 | 0 | | 0 | 2.09 | | | | |
| | 35 | | 1 | 57.89 | 2.19 | | | | | 35 | | | 33.43 | 0.81 | | 10 | | | | 10 | | | 1.92 | 0.17 | | |
| | 40 | | | 55.72 | 2.17 | | | | | 40 | | | 32.64 | 0.79 | | 20 | | | | 20 | | | 1.76 | 0.16 | | |
| | 45 | | | 53.57 | 2.15 | | | | | 45 | | | 31.86 | 0.78 | | 30 | | | | 30 | | | 1.62 | 0.14 | | |
| | 50 | | | 51.46 | 2.11 | | | | | 50 | | | 31.10 | 0.76 | | 40 | | | | 40 | | | 1.48 | 0.14 | | |
| | 55 | | | 49.39 | 2.07 | | | | | 55 | | | 30.35 | 0.75 | | 50 | | | | 50 | | | 1.35 | 0.13 | | |
| | | | | | 2.04 | | | | | | | | | 0.73 | | | | | | | | | | 0.12 | | |
| 158 | 0 | | 1 | 47.35 | | | | | 163 | 0 | | 0 | 29.62 | | 171 | 0 | | 171 | 0 | | 0 | 1.23 | | | | |
| | 5 | | | 45.34 | 2.01 | | | | | 5 | | | 28.90 | 0.72 | | 10 | | | | 10 | | | 1.12 | 0.11 | | |
| | 10 | | | 43.35 | 1.99 | | | | | 10 | | | 28.20 | 0.70 | | 20 | | | | 20 | | | 1.02 | 0.10 | | |
| | 15 | | | 41.39 | 1.96 | | | | | 15 | | | 27.51 | 0.69 | | 30 | | | | 30 | | | 0.93 | 0.09 | | |
| | 20 | | | 39.47 | 1.92 | | | | | 20 | | | 26.83 | 0.68 | | 40 | | | | 40 | | | 0.84 | 0.09 | | |
| | 25 | | | 37.57 | 1.90 | | | | | 25 | | | 26.16 | 0.67 | | 50 | | | | 50 | | | 0.76 | 0.08 | | |
| | | | | | 1.87 | | | | | | | | | 0.65 | | | | | | | | | | 0.08 | | |
| 158 | 30 | | 1 | 35.70 | | | | | 163 | 30 | | 0 | 25.51 | | 172 | 0 | | 172 | 0 | | 0 | 0.68 | | | | |
| | 35 | | | 33.87 | 1.83 | | | | | 35 | | | 24.88 | 0.63 | | 10 | | | | 10 | | | 0.61 | 0.07 | | |
| | 40 | | | 32.06 | 1.81 | | | | | 40 | | | 24.25 | 0.63 | | 20 | | | | 20 | | | 0.55 | 0.06 | | |
| | 45 | | | 30.28 | 1.78 | | | | | 45 | | | 23.64 | 0.61 | | 30 | | | | 30 | | | 0.49 | 0.05 | | |
| | 50 | | | 28.52 | 1.76 | | | | | 50 | | | 23.04 | 0.60 | | 40 | | | | 40 | | | 0.44 | 0.05 | | |
| | 55 | | | 26.80 | 1.72 | | | | | 55 | | | 22.45 | 0.59 | | 50 | | | | 50 | | | 0.39 | 0.05 | | |
| | | | | | 1.70 | | | | | | | | | 0.57 | | | | | | | | | | 0.04 | | |
| 159 | 0 | | 1 | 25.10 | | | | | 164 | 0 | | 0 | 21.88 | | 173 | 0 | | 173 | 0 | | 0 | 0.35 | | | | |
| | 5 | | | 23.43 | 1.67 | | | | | 5 | | | 21.31 | 0.57 | | 10 | | | | 10 | | | 0.31 | 0.04 | | |
| | 10 | | | 21.78 | 1.65 | | | | | 10 | | | 20.76 | 0.55 | | 20 | | | | 20 | | | 0.27 | 0.04 | | |
| | 15 | | | 20.16 | 1.62 | | | | | 15 | | | 20.22 | 0.54 | | 30 | | | | 30 | | | 0.24 | 0.03 | | |
| | 20 | | | 18.57 | 1.59 | | | | | 20 | | | 19.69 | 0.53 | | 40 | | | | 40 | | | 0.21 | 0.03 | | |
| | 25 | | | 17.00 | 1.57 | | | | | 25 | | | 19.18 | 0.51 | | 50 | | | | 50 | | | 0.19 | 0.02 | | |
| | | | | | 1.55 | | | | | | | | | 0.51 | | | | | | | | | | 0.03 | | |
| 159 | 30 | | 1 | 15.45 | | | | | 164 | 30 | | 0 | 18.67 | | 174 | 0 | | 174 | 0 | | 0 | 0.16 | | | | |
| | 35 | | | 13.94 | 1.51 | | | | | 35 | | | 18.17 | 0.50 | | 10 | | | | 10 | | | 0.16 | 0.09 | | |
| | 40 | | | 12.44 | 1.50 | | | | | 40 | | | 17.69 | 0.48 | | 20 | | | | 20 | | | 0.15 | 0.09 | | |
| | 45 | | | 10.97 | 1.47 | | | | | 45 | | | 17.21 | 0.48 | | 30 | | | | 30 | | | 0.14 | 0.08 | | |
| | 50 | | | 9.53 | 1.44 | | | | | 50 | | | 16.75 | 0.46 | | 40 | | | | 40 | | | 0.13 | 0.08 | | |
| | 55 | | | 8.10 | 1.43 | | | | | 55 | | | 16.29 | 0.46 | | 50 | | | | 50 | | | 0.12 | 0.08 | | |
| | | | | | 1.40 | | | | | | | | | 0.44 | | | | | | | | | | 0.08 | | |
| 160 | 0 | | 1 | 6.70 | | | | | 165 | 0 | | 0 | 15.85 | | 180 | 0 | | 180 | 0 | | 0 | 0.00 | | | | |

TABLE VIII.

For finding the Time from the Perihelion in a Parabolic Orbit.

| v | log N | Diff. | v | log N | Diff. | v | log N | Diff. |
|------|------------|-------|-------|------------|-------|------|------------|-------|
| 0 0 | 0.025 5763 | | 30 0 | 0.020 7913 | | 60 0 | 0.008 8644 | |
| 30 | .025 5749 | 14 | 30 30 | .020 6368 | 1545 | 30 | .008 6458 | 2186 |
| 1 0 | .025 5707 | 42 | 31 0 | .020 4802 | 1566 | 61 0 | .008 4277 | 2181 |
| 30 | .025 5638 | 69 | 30 30 | .020 3215 | 1587 | 30 | .008 2103 | 2174 |
| 2 0 | .025 5542 | 96 | 32 0 | .020 1607 | 1608 | 62 0 | .007 9934 | 2169 |
| 30 | .025 5418 | 124 | 30 30 | .019 9979 | 1628 | 30 | .007 7774 | 2160 |
| | | 152 | | | 1649 | | | 2153 |
| 3 0 | 0.025 5266 | | 33 0 | 0.019 8330 | | 63 0 | 0.007 5621 | |
| 30 | .025 5087 | 179 | 30 30 | .019 6662 | 1668 | 30 | .007 3477 | 2144 |
| 4 0 | .025 4881 | 206 | 34 0 | .019 4974 | 1688 | 64 0 | .007 1343 | 2134 |
| 30 | .025 4647 | 234 | 30 30 | .019 3267 | 1707 | 30 | .006 9220 | 2123 |
| 5 0 | .025 4386 | 261 | 35 0 | .019 1540 | 1727 | 65 0 | .006 7108 | 2112 |
| 30 | .025 4097 | 289 | 30 30 | .018 9795 | 1745 | 30 | .006 5008 | 2100 |
| | | 316 | | | 1765 | | | 2086 |
| 6 0 | 0.025 3781 | | 36 0 | 0.018 8030 | | 66 0 | 0.006 2922 | |
| 30 | .025 3437 | 344 | 30 30 | .018 6248 | 1782 | 30 | .006 0849 | 2073 |
| 7 0 | .025 3066 | 371 | 37 0 | .018 4448 | 1800 | 67 0 | .005 8792 | 2057 |
| 30 | .025 2668 | 398 | 30 30 | .018 2629 | 1819 | 30 | .005 6750 | 2042 |
| 8 0 | .025 2243 | 425 | 38 0 | .018 0794 | 1835 | 68 0 | .005 4725 | 2025 |
| 30 | .025 1791 | 452 | 30 30 | .017 8941 | 1853 | 30 | .005 2717 | 2008 |
| | | 480 | | | 1869 | | | 1988 |
| 9 0 | 0.025 1311 | | 39 0 | 0.017 7072 | | 69 0 | 0.005 0729 | |
| 30 | .025 0805 | 506 | 30 30 | .017 5186 | 1886 | 30 | .004 8760 | 1969 |
| 10 0 | .025 0271 | 534 | 40 0 | .017 3283 | 1903 | 70 0 | .004 6811 | 1949 |
| 30 | .024 9711 | 560 | 30 30 | .017 1365 | 1918 | 30 | .004 4884 | 1927 |
| 11 0 | .024 9124 | 587 | 41 0 | .016 9432 | 1933 | 71 0 | .004 2980 | 1904 |
| 30 | .024 8510 | 614 | 30 30 | .016 7483 | 1949 | 30 | .004 1100 | 1880 |
| | | 641 | | | 1963 | | | 1855 |
| 12 0 | 0.024 7869 | | 42 0 | 0.016 5520 | | 72 0 | 0.003 9245 | |
| 30 | .024 7201 | 668 | 30 30 | .016 3542 | 1978 | 30 | .003 7416 | 1829 |
| 13 0 | .024 6507 | 694 | 43 0 | .016 1550 | 1992 | 73 0 | .003 5613 | 1803 |
| 30 | .024 5786 | 721 | 30 30 | .015 9545 | 2005 | 30 | .003 3839 | 1774 |
| 14 0 | .024 5039 | 747 | 44 0 | .015 7526 | 2019 | 74 0 | .003 2094 | 1745 |
| 30 | .024 4266 | 773 | 30 30 | .015 5495 | 2031 | 30 | .003 0380 | 1714 |
| | | 800 | | | 2045 | | | 1682 |
| 15 0 | 0.024 3466 | | 45 0 | 0.015 3450 | | 75 0 | 0.002 8698 | |
| 30 | .024 2641 | 825 | 30 30 | .015 1394 | 2056 | 30 | .002 7049 | 1649 |
| 16 0 | .024 1789 | 852 | 46 0 | .014 9326 | 2058 | 76 0 | .002 5433 | 1616 |
| 30 | .024 0911 | 878 | 30 30 | .014 7247 | 2079 | 30 | .002 3854 | 1579 |
| 17 0 | .024 0008 | 903 | 47 0 | .014 5157 | 2090 | 77 0 | .002 2311 | 1543 |
| 30 | .023 9079 | 929 | 30 30 | .014 3057 | 2100 | 30 | .002 0806 | 1505 |
| | | 954 | | | 2110 | | | 1465 |
| 18 0 | 0.023 8125 | | 48 0 | 0.014 0947 | | 78 0 | 0.001 9341 | |
| 30 | .023 7145 | 980 | 30 30 | .013 8827 | 2120 | 30 | .001 7917 | 1424 |
| 19 0 | .023 6140 | 1005 | 49 0 | .013 6698 | 2129 | 79 0 | .001 6535 | 1382 |
| 30 | .023 5109 | 1031 | 30 30 | .013 4561 | 2137 | 30 | .001 5196 | 1339 |
| 20 0 | .023 4054 | 1055 | 50 0 | .013 2416 | 2145 | 80 0 | .001 3903 | 1293 |
| 30 | .023 2973 | 1081 | 30 30 | .013 0263 | 2153 | 30 | .001 2656 | 1247 |
| | | 1105 | | | 2160 | | | 1198 |
| 21 0 | 0.023 1868 | | 51 0 | 0.012 8103 | | 81 0 | 0.001 1458 | |
| 30 | .023 0738 | 1130 | 30 30 | .012 5936 | 2167 | 30 | .001 0309 | 1149 |
| 22 0 | .022 9584 | 1154 | 52 0 | .012 3764 | 2172 | 82 0 | .000 9211 | 1098 |
| 30 | .022 8405 | 1179 | 30 30 | .012 1585 | 2179 | 30 | .000 8166 | 1045 |
| 23 0 | .022 7202 | 1203 | 53 0 | .011 9402 | 2183 | 83 0 | .000 7175 | 991 |
| 30 | .022 5975 | 1227 | 30 30 | .011 7215 | 2187 | 30 | .000 6240 | 935 |
| | | 1251 | | | 2191 | | | 876 |
| 24 0 | 0.022 4724 | | 54 0 | 0.011 5024 | | 84 0 | 0.000 5364 | |
| 30 | .022 3449 | 1275 | 30 30 | .011 2829 | 2195 | 30 | .000 4546 | 818 |
| 25 0 | .022 2151 | 1298 | 55 0 | .011 0632 | 2197 | 85 0 | .000 3790 | 756 |
| 30 | .022 0829 | 1322 | 30 30 | .010 8432 | 2200 | 30 | .000 3096 | 694 |
| 26 0 | .021 9484 | 1345 | 56 0 | .010 6231 | 2201 | 86 0 | .000 2468 | 628 |
| 30 | .021 8116 | 1368 | 30 30 | .010 4029 | 2202 | 30 | .000 1906 | 562 |
| | | 1390 | | | 2202 | | | 493 |
| 27 0 | 0.021 6726 | | 57 0 | 0.010 1827 | | 87 0 | 0.000 1413 | |
| 30 | .021 5312 | 1414 | 30 30 | .009 9625 | 2202 | 30 | .000 0990 | 423 |
| 28 0 | .021 3876 | 1436 | 58 0 | .009 7424 | 2201 | 88 0 | .000 0639 | 351 |
| 30 | .021 2418 | 1458 | 30 30 | .009 5225 | 2199 | 30 | .000 0363 | 276 |
| 29 0 | .021 0938 | 1480 | 59 0 | .009 3028 | 2197 | 89 0 | .000 0163 | 200 |
| 30 | .020 9436 | 1502 | 30 30 | .009 0834 | 2194 | 30 | .000 0041 | 122 |
| | | 1523 | | | 2190 | | | 41 |
| 30 0 | 0.020 7913 | | 60 0 | 0.008 8644 | | 90 0 | 0.000 0000 | |

TABLE VIII.

For finding the Time from the Perihelion in a Parabolic Orbit.

| ω | $\log N'$ | Diff. | ω | $\log N'$ | Diff. | ω | $\log N'$ | Diff. |
|----------|------------|-------|----------|------------|-------|----------|------------|-------|
| 90 0 | 0.000 0000 | | 120 0 | 9.963 1069 | 10995 | 150 0 | 9.889 0321 | 11583 |
| 30 | 9.999 9876 | 124 | 30 | .962 0074 | 11103 | 30 | .887 8738 | 11479 |
| 91 0 | 9.999 9507 | 309 | 121 0 | .960 8971 | 11207 | 151 0 | .886 7259 | 11372 |
| 30 | .999 8893 | 614 | 30 | .959 7764 | 11310 | 30 | .885 5887 | 11260 |
| 92 0 | 9.999 8039 | 854 | 122 0 | .958 6454 | 11408 | 152 0 | .884 4627 | 11146 |
| 30 | .999 6944 | 1095 | 30 | .957 5046 | 11504 | 30 | .883 3481 | 11026 |
| | | 1331 | | | | | | |
| 93 0 | 9.999 5613 | 1567 | 123 0 | 9.956 3542 | 11597 | 153 0 | 9.882 2455 | 10903 |
| 30 | .999 4046 | 1800 | 30 | .955 1945 | 11687 | 30 | .881 1552 | 10777 |
| 94 0 | 9.999 2246 | 2031 | 124 0 | .954 0258 | 11775 | 154 0 | .880 0775 | 10646 |
| 30 | .999 0215 | 2260 | 30 | .952 8483 | 11859 | 30 | .879 0129 | 10513 |
| 95 0 | 9.998 7955 | 2487 | 125 0 | .951 6624 | 11940 | 155 0 | .877 9616 | 10374 |
| 30 | .998 5468 | 2711 | 30 | .950 4684 | 12018 | 30 | .876 9242 | 10232 |
| | | | | | | | | |
| 96 0 | 9.998 2757 | 2933 | 126 0 | 9.949 2666 | 12093 | 156 0 | 9.875 9010 | 10088 |
| 30 | .997 9824 | 3155 | 30 | .948 0573 | 12165 | 30 | .874 8922 | 9938 |
| 97 0 | 9.997 6669 | 3372 | 127 0 | .946 8408 | 12234 | 157 0 | .873 8984 | 9786 |
| 30 | .997 3297 | 3589 | 30 | .945 6174 | 12299 | 30 | .872 9198 | 9629 |
| 98 0 | 9.996 9708 | 3802 | 128 0 | .944 3875 | 12362 | 158 0 | .871 9569 | 9470 |
| 30 | .996 5906 | 4015 | 30 | .943 1513 | 12421 | 30 | .871 0099 | 9307 |
| | | | | | | | | |
| 99 0 | 9.996 1891 | 4225 | 129 0 | 9.941 9092 | 12477 | 159 0 | 9.870 0792 | 9140 |
| 30 | .995 7666 | 4432 | 30 | .940 6615 | 12530 | 30 | .869 1652 | 8969 |
| 100 0 | 9.995 3234 | 4638 | 130 0 | .939 4085 | 12579 | 160 0 | .868 2683 | 8797 |
| 30 | .994 8596 | 4841 | 30 | .938 1506 | 12625 | 30 | .867 3886 | 8620 |
| 101 0 | 9.994 3755 | 5043 | 131 0 | .936 8881 | 12668 | 161 0 | .866 5266 | 8439 |
| 30 | .993 8712 | 5242 | 30 | .935 6213 | 12707 | 30 | .865 6827 | 8257 |
| | | | | | | | | |
| 102 0 | 9.993 3470 | 5439 | 132 0 | 9.934 3506 | 12743 | 162 0 | 9.864 8570 | 8070 |
| 30 | .992 8031 | 5634 | 30 | .933 0763 | 12776 | 30 | .864 0500 | 7880 |
| 103 0 | 9.992 2397 | 5827 | 133 0 | .931 7987 | 12804 | 163 0 | .863 2620 | 7688 |
| 30 | .991 6570 | 6017 | 30 | .930 5183 | 12830 | 30 | .862 4932 | 7493 |
| 104 0 | 9.991 0553 | 6206 | 134 0 | .929 2353 | 12852 | 164 0 | .861 7439 | 7294 |
| 30 | .990 4347 | 6391 | 30 | .927 9501 | 12871 | 30 | .861 0145 | 7092 |
| | | | | | | | | |
| 105 0 | 9.989 7956 | 6576 | 135 0 | 9.926 6630 | 12885 | 165 0 | 9.860 3053 | 6889 |
| 30 | .989 1380 | 6758 | 30 | .925 3745 | 12897 | 30 | .859 6164 | 6682 |
| 106 0 | 9.988 4622 | 6937 | 136 0 | .924 0848 | 12905 | 166 0 | .858 3010 | 6472 |
| 30 | .987 7685 | 7114 | 30 | .922 7943 | 12908 | 30 | .857 0750 | 6260 |
| 107 0 | 9.987 0571 | 7290 | 137 0 | .921 5035 | 12909 | 167 0 | .857 0704 | 6046 |
| 30 | .986 3281 | 7462 | 30 | .920 2126 | 12906 | 30 | .857 0704 | 5829 |
| | | | | | | | | |
| 108 0 | 9.985 5819 | 7633 | 138 0 | 9.918 9220 | 12899 | 168 0 | 9.856 4875 | 5609 |
| 30 | .984 8186 | 7801 | 30 | .917 6321 | 12888 | 30 | .855 9266 | 5388 |
| 109 0 | 9.984 0385 | 7967 | 139 0 | .916 3433 | 12874 | 169 0 | .855 3878 | 5164 |
| 30 | .983 2418 | 8130 | 30 | .915 0559 | 12856 | 30 | .854 8714 | 4939 |
| 110 0 | 9.982 4288 | 8292 | 140 0 | .913 7703 | 12833 | 170 0 | .854 3775 | 4710 |
| 30 | .981 5996 | 8451 | 30 | .912 4870 | 12808 | 30 | .853 9065 | 4481 |
| | | | | | | | | |
| 111 0 | 9.980 7545 | 8607 | 141 0 | 9.911 2062 | 12779 | 171 0 | 9.853 4584 | 4249 |
| 30 | .979 8938 | 8761 | 30 | .909 9283 | 12745 | 30 | .853 0335 | 4016 |
| 112 0 | 9.979 0177 | 8913 | 142 0 | .908 6538 | 12707 | 172 0 | .852 6319 | 3781 |
| 30 | .978 1264 | 9062 | 30 | .907 3831 | 12667 | 30 | .852 2538 | 3544 |
| 113 0 | 9.977 2202 | 9209 | 143 0 | .906 1164 | 12622 | 173 0 | .851 8994 | 3307 |
| 30 | .976 2993 | 9353 | 30 | .904 8542 | 12573 | 30 | .851 5687 | 3067 |
| | | | | | | | | |
| 114 0 | 9.975 3640 | 9495 | 144 0 | 9.903 5969 | 12520 | 174 0 | 9.851 2620 | 2826 |
| 30 | .974 4145 | 9635 | 30 | .902 3449 | 12464 | 30 | .850 9794 | 2585 |
| 115 0 | 9.973 4510 | 9771 | 145 0 | .901 0985 | 12403 | 175 0 | .850 7209 | 2341 |
| 30 | .972 4739 | 9906 | 30 | .899 8582 | 12339 | 30 | .850 4868 | 2098 |
| 116 0 | 9.971 4833 | 10037 | 146 0 | .898 6243 | 12271 | 176 0 | .850 2770 | 1853 |
| 30 | .970 4796 | 10167 | 30 | .897 3972 | 12198 | 30 | .850 0917 | 1608 |
| | | | | | | | | |
| 117 0 | 9.969 4629 | 10292 | 147 0 | 9.896 1774 | 12122 | 177 0 | 9.849 9309 | 1361 |
| 30 | .968 4337 | 10417 | 30 | .894 9652 | 12042 | 30 | .849 7948 | 1115 |
| 118 0 | 9.967 3920 | 10538 | 148 0 | .893 7610 | 11958 | 178 0 | .849 6833 | 867 |
| 30 | .966 3382 | 10656 | 30 | .892 5652 | 11870 | 30 | .849 5966 | 620 |
| 119 0 | 9.965 2726 | 10772 | 149 0 | .891 3782 | 11778 | 179 0 | .849 5346 | 372 |
| 30 | .964 1954 | 10885 | 30 | .890 2004 | 11683 | 30 | .849 4974 | 124 |
| | | | | | | | | |
| 120 0 | 9.963 1069 | | 150 0 | 9.889 0321 | | 180 0 | 9.849 4850 | |

TABLE IX.

For finding the True Anomaly or the Time from the Perihelion in Orbits of great eccentricity.

| ω | A | Diff. | B | Diff. | C | B' | Diff. | C' |
|----------|--------|-------|-------|-------|-------|-------|-------|-------|
| 0 | " | " | " | " | " | " | " | " |
| 0 | 0.00 | | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 1 | 0.00 | 0.00 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 2 | 0.01 | 0.01 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 3 | 0.05 | 0.04 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 4 | 0.12 | 0.07 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| | | 0.11 | | | | | | |
| 5 | 0.23 | | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 6 | 0.35 | 0.16 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 7 | 0.62 | 0.23 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 8 | 0.93 | 0.31 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 9 | 1.33 | 0.40 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| | | 0.49 | | | | | | |
| 10 | 1.82 | | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 11 | 2.42 | 0.60 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 12 | 3.14 | 0.72 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 13 | 3.99 | 0.85 | 0.000 | | 0.000 | 0.000 | | 0.000 |
| 14 | 4.99 | 1.00 | 0.001 | | 0.000 | 0.001 | | 0.000 |
| | | 1.14 | | | | | | |
| 15 | 6.13 | | 0.001 | | 0.000 | 0.001 | | 0.000 |
| 16 | 7.43 | 1.30 | 0.002 | .001 | 0.000 | 0.001 | .000 | 0.000 |
| 17 | 8.90 | 1.47 | 0.002 | .000 | 0.000 | 0.002 | .001 | 0.000 |
| 18 | 10.55 | 1.65 | 0.003 | .001 | 0.000 | 0.002 | .000 | 0.000 |
| 19 | 12.40 | 1.85 | 0.004 | .001 | 0.000 | 0.003 | .001 | 0.000 |
| | | 2.05 | | .001 | | | .001 | 0.000 |
| 20 | 14.45 | | 0.005 | | 0.000 | 0.004 | | 0.000 |
| 21 | 16.70 | 2.25 | 0.006 | .001 | 0.000 | 0.005 | .001 | 0.000 |
| 22 | 19.18 | 2.48 | 0.008 | .002 | 0.000 | 0.006 | .001 | 0.000 |
| 23 | 21.89 | 2.71 | 0.010 | .002 | 0.000 | 0.008 | .002 | 0.000 |
| 24 | 24.83 | 2.94 | 0.012 | .002 | 0.000 | 0.010 | .002 | 0.000 |
| | | 3.20 | | .002 | | | .002 | 0.000 |
| 25 | 28.03 | | 0.014 | | 0.000 | 0.012 | | 0.000 |
| 26 | 31.48 | 3.45 | 0.017 | .003 | 0.000 | 0.014 | .002 | 0.000 |
| 27 | 35.20 | 3.72 | 0.020 | .003 | 0.000 | 0.017 | .003 | 0.000 |
| 28 | 39.19 | 3.99 | 0.025 | .005 | 0.000 | 0.020 | .003 | 0.000 |
| 29 | 43.47 | 4.28 | 0.030 | .005 | 0.000 | 0.024 | .004 | 0.000 |
| | | 4.57 | | .005 | | | .004 | 0.000 |
| 30 | 48.04 | | 0.035 | | 0.000 | 0.028 | | 0.000 |
| 31 | 52.91 | 4.87 | 0.041 | .006 | 0.000 | 0.033 | .005 | 0.000 |
| 32 | 58.09 | 5.18 | 0.047 | .006 | 0.000 | 0.039 | .006 | 0.000 |
| 33 | 63.59 | 5.50 | 0.055 | .008 | 0.000 | 0.045 | .006 | 0.000 |
| 34 | 69.42 | 5.83 | 0.064 | .009 | 0.000 | 0.052 | .007 | 0.000 |
| | | 6.15 | | .009 | | | .008 | 0.000 |
| 35 | 75.57 | | 0.073 | | 0.000 | 0.060 | | 0.000 |
| 36 | 82.07 | 6.50 | 0.084 | .011 | 0.000 | 0.068 | .008 | 0.000 |
| 37 | 88.92 | 6.85 | 0.096 | .012 | 0.000 | 0.078 | .010 | 0.000 |
| 38 | 96.12 | 7.20 | 0.109 | .013 | 0.000 | 0.088 | .010 | 0.000 |
| 39 | 103.68 | 7.56 | 0.123 | .014 | 0.000 | 0.100 | .012 | 0.000 |
| | | 7.93 | | .016 | | | .013 | 0.000 |
| 40 | 111.61 | | 0.139 | | 0.000 | 0.113 | | 0.000 |
| 41 | 119.92 | 8.31 | 0.156 | .017 | 0.000 | 0.127 | .014 | 0.000 |
| 42 | 128.62 | 8.70 | 0.175 | .019 | 0.000 | 0.142 | .015 | 0.000 |
| 43 | 137.70 | 9.08 | 0.196 | .021 | 0.000 | 0.159 | .017 | 0.000 |
| 44 | 147.18 | 9.48 | 0.218 | .022 | 0.000 | 0.177 | .018 | 0.000 |
| | | 9.87 | | .025 | | | .020 | 0.000 |
| 45 | 157.05 | | 0.243 | | 0.000 | 0.197 | | 0.000 |
| 46 | 167.34 | 10.29 | 0.269 | .026 | 0.000 | 0.219 | .022 | 0.000 |
| 47 | 178.04 | 10.70 | 0.298 | .029 | 0.000 | 0.242 | .023 | 0.000 |
| 48 | 189.16 | 11.12 | 0.328 | .030 | 0.000 | 0.267 | .025 | 0.000 |
| 49 | 200.71 | 11.55 | 0.361 | .033 | 0.000 | 0.294 | .027 | 0.000 |
| | | 11.98 | | .036 | | | .029 | 0.000 |
| 50 | 212.69 | | 0.397 | | 0.000 | 0.323 | | 0.000 |
| 51 | 225.10 | 12.41 | 0.436 | .039 | 0.000 | 0.354 | .031 | 0.000 |
| 52 | 237.95 | 12.85 | 0.477 | .041 | 0.001 | 0.388 | .034 | 0.000 |
| 53 | 251.25 | 13.30 | 0.521 | .044 | 0.001 | 0.424 | .036 | 0.000 |
| 54 | 265.01 | 13.76 | 0.567 | .046 | 0.001 | 0.462 | .038 | 0.000 |
| | | 14.20 | | .050 | | | .040 | 0.000 |
| 55 | 279.21 | | 0.617 | | 0.001 | 0.502 | | 0.000 |
| 56 | 293.88 | 14.67 | 0.671 | .054 | 0.002 | 0.546 | .044 | 0.001 |
| 57 | 309.02 | 15.14 | 0.727 | .056 | 0.002 | 0.592 | .046 | 0.001 |
| 58 | 324.62 | 15.60 | 0.787 | .060 | 0.002 | 0.641 | .049 | 0.001 |
| 59 | 340.70 | 16.08 | 0.851 | .064 | 0.002 | 0.693 | .052 | 0.001 |
| | | 16.56 | | .068 | | | .056 | 0.001 |
| 60 | 357.26 | | 0.919 | | 0.003 | 0.749 | | 0.002 |

TABLE IX.

For finding the True Anomaly or the Time from the Perihelion in Orbits of great eccentricity.

| α | A | Diff. | B | Diff. | C | B' | Diff. | C' |
|----------|---------|-------|--------|-------|-------|--------|-------|-------|
| 0 | " | " | " | " | " | " | " | " |
| 60 | 357.26 | | 0.919 | | 0.003 | 0.749 | | 0.002 |
| 61 | 374.30 | 17.04 | 0.990 | .071 | 0.003 | 0.807 | .058 | 0.002 |
| 62 | 391.84 | 17.54 | 1.066 | .076 | 0.003 | 0.869 | .062 | 0.002 |
| 63 | 409.86 | 18.02 | 1.145 | .079 | 0.004 | 0.935 | .066 | 0.002 |
| 64 | 428.38 | 18.52 | 1.230 | .085 | 0.004 | 1.004 | .069 | 0.002 |
| | | 19.02 | | .088 | | | .073 | |
| 65 | 447.40 | | 1.318 | | 0.004 | 1.077 | | 0.003 |
| 66 | 466.92 | 19.52 | 1.411 | .093 | 0.005 | 1.154 | .077 | 0.003 |
| 67 | 486.96 | 20.04 | 1.510 | .099 | 0.005 | 1.235 | .081 | 0.003 |
| 68 | 507.51 | 20.55 | 1.613 | .103 | 0.006 | 1.321 | .086 | 0.004 |
| 69 | 528.58 | 21.07 | 1.721 | .108 | 0.006 | 1.411 | .090 | 0.004 |
| | | 21.59 | | .114 | | | .094 | |
| 70 | 550.17 | | 1.835 | | 0.007 | 1.505 | | 0.004 |
| 71 | 572.29 | 22.12 | 1.954 | .119 | 0.007 | 1.605 | .100 | 0.005 |
| 72 | 594.94 | 22.65 | 2.078 | .124 | 0.008 | 1.709 | .104 | 0.005 |
| 73 | 618.12 | 23.18 | 2.209 | .131 | 0.009 | 1.819 | .110 | 0.006 |
| 74 | 641.85 | 23.73 | 2.345 | .136 | 0.009 | 1.934 | .115 | 0.006 |
| | | 24.28 | | .143 | | | .121 | |
| 75 | 666.13 | | 2.488 | | 0.010 | 2.055 | | 0.007 |
| 76 | 690.96 | 24.83 | 2.637 | .149 | 0.011 | 2.181 | .126 | 0.007 |
| 77 | 717.34 | 25.38 | 2.793 | .156 | 0.012 | 2.314 | .133 | 0.008 |
| 78 | 742.29 | 25.95 | 2.956 | .163 | 0.013 | 2.453 | .139 | 0.008 |
| 79 | 768.81 | 26.52 | 3.125 | .169 | 0.014 | 2.599 | .146 | 0.009 |
| | | 27.09 | | .177 | | | .153 | |
| 80 | 795.90 | | 3.302 | | 0.015 | 2.752 | | 0.010 |
| 81 | 823.57 | 27.67 | 3.486 | .184 | 0.016 | 2.912 | .160 | 0.011 |
| 82 | 851.84 | 28.27 | 3.678 | .192 | 0.017 | 3.079 | .167 | 0.012 |
| 83 | 880.70 | 28.86 | 3.878 | .200 | 0.018 | 3.255 | .176 | 0.013 |
| 84 | 910.16 | 29.46 | 4.087 | .209 | 0.020 | 3.439 | .184 | 0.014 |
| | | 30.07 | | .216 | | | .192 | |
| 85 | 940.23 | | 4.303 | | 0.021 | 3.631 | | 0.015 |
| 86 | 970.92 | 30.69 | 4.529 | .226 | 0.023 | 3.833 | .202 | 0.016 |
| 87 | 1002.24 | 31.32 | 4.764 | .235 | 0.024 | 4.044 | .211 | 0.018 |
| 88 | 1034.20 | 31.96 | 5.008 | .244 | 0.026 | 4.266 | .222 | 0.019 |
| 89 | 1066.81 | 32.61 | 5.262 | .254 | 0.028 | 4.498 | .232 | 0.021 |
| | | 33.27 | | .265 | | | .243 | |
| 90 | 1100.08 | | 5.527 | | 0.030 | 4.741 | | 0.023 |
| 91 | 1134.02 | 33.94 | 5.801 | .274 | 0.032 | 4.996 | .255 | 0.025 |
| 92 | 1168.64 | 34.62 | 6.087 | .286 | 0.034 | 5.263 | .267 | 0.027 |
| 93 | 1203.95 | 35.31 | 6.385 | .298 | 0.036 | 5.544 | .281 | 0.029 |
| 94 | 1239.97 | 36.02 | 6.694 | .309 | 0.038 | 5.838 | .294 | 0.032 |
| | | 36.75 | | .322 | | | .309 | |
| 95 | 1276.72 | | 7.016 | | 0.041 | 6.147 | | 0.035 |
| 96 | 1314.21 | 37.49 | 7.350 | .334 | 0.044 | 6.471 | .324 | 0.038 |
| 97 | 1352.45 | 38.24 | 7.698 | .348 | 0.047 | 6.812 | .341 | 0.041 |
| 98 | 1391.46 | 39.01 | 8.060 | .362 | 0.050 | 7.171 | .359 | 0.045 |
| 99 | 1431.27 | 39.81 | 8.437 | .377 | 0.053 | 7.549 | .378 | 0.049 |
| | | 40.61 | | .392 | | | .397 | |
| 100 0 | 1471.88 | | 8.829 | | 0.056 | 7.946 | | 0.053 |
| 30 | 1492.50 | 20.62 | 9.032 | .203 | 0.058 | 8.152 | .206 | 0.055 |
| 101 0 | 1513.33 | 20.83 | 9.238 | .206 | 0.060 | 8.364 | .212 | 0.058 |
| 30 | 1534.38 | 21.05 | 9.449 | .211 | 0.062 | 8.582 | .218 | 0.060 |
| 102 0 | 1555.64 | 21.26 | 9.664 | .215 | 0.064 | 8.805 | .223 | 0.063 |
| 30 | 1577.12 | 21.48 | 9.883 | .219 | 0.066 | 9.035 | .230 | 0.066 |
| | | 21.70 | | .225 | | | .236 | |
| 103 0 | 1598.82 | | 10.108 | | 0.068 | 9.271 | | 0.069 |
| 30 | 1620.75 | 21.93 | 10.337 | .229 | 0.070 | 9.513 | .242 | 0.072 |
| 104 0 | 1642.91 | 22.16 | 10.570 | .233 | 0.072 | 9.761 | .248 | 0.075 |
| 30 | 1665.30 | 22.39 | 10.809 | .239 | 0.074 | 10.017 | .256 | 0.078 |
| 105 0 | 1687.93 | 22.63 | 11.053 | .244 | 0.077 | 10.280 | .263 | 0.082 |
| 30 | 1710.80 | 22.87 | 11.302 | .249 | 0.079 | 10.550 | .270 | 0.085 |
| | | 23.12 | | .255 | | | .278 | |
| 106 0 | 1733.92 | | 11.557 | | 0.082 | 10.828 | | 0.089 |
| 30 | 1757.28 | 23.36 | 11.817 | .260 | 0.084 | 11.114 | .286 | 0.093 |
| 107 0 | 1780.90 | 23.62 | 12.083 | .266 | 0.087 | 11.408 | .294 | 0.098 |
| 30 | 1804.77 | 23.87 | 12.354 | .271 | 0.090 | 11.711 | .303 | 0.102 |
| 108 0 | 1828.90 | 24.13 | 12.632 | .278 | 0.093 | 12.022 | .311 | 0.107 |
| 30 | 1853.30 | 24.40 | 12.916 | .284 | 0.096 | 12.343 | .321 | 0.112 |
| | | 24.67 | | .291 | | | .330 | |
| 109 0 | 1877.97 | | 13.207 | | 0.099 | 12.673 | | 0.117 |

TABLE IX.

For finding the True Anomaly or the Time from the Perihelion in Orbits of great eccentricity

| θ | A | Diff. | B | Diff. | C | Diff. | B' | Diff. | C' | Diff. |
|------------|---------|-------|--------|-------|-------|-------|--------|-------|-------|-------|
| $^{\circ}$ | $''$ | $''$ | $''$ | $''$ | $''$ | $''$ | $''$ | $''$ | $''$ | $''$ |
| 109 0 | 1877.97 | | 13.207 | | 0.099 | | 12.673 | | 0.117 | |
| 30 | 1902.91 | 24.94 | 13.504 | .297 | 0.102 | .003 | 13.013 | .340 | 0.122 | .005 |
| 110 0 | 1928.13 | 25.22 | 13.808 | .304 | 0.106 | .004 | 13.363 | .350 | 0.128 | .006 |
| 30 | 1953.64 | 25.51 | 14.119 | .311 | 0.109 | .003 | 13.724 | .361 | 0.134 | .006 |
| 111 0 | 1979.44 | 25.80 | 14.438 | .319 | 0.113 | .004 | 14.095 | .371 | 0.141 | .007 |
| 30 | 2005.54 | 26.10 | 14.764 | .326 | 0.116 | .003 | 14.478 | .383 | 0.148 | .007 |
| | | 26.40 | | .333 | | .004 | | .396 | | |
| 112 0 | 2031.94 | | 15.097 | | 0.120 | | 14.874 | | 0.155 | |
| 30 | 2058.64 | 26.70 | 15.439 | .342 | 0.124 | .004 | 15.282 | .408 | 0.162 | .007 |
| 113 0 | 2085.66 | 27.02 | 15.789 | .350 | 0.128 | .004 | 15.702 | .420 | 0.170 | .008 |
| 30 | 2113.00 | 27.34 | 16.148 | .359 | 0.132 | .004 | 16.135 | .433 | 0.178 | .008 |
| 114 0 | 2140.66 | 27.66 | 16.515 | .367 | 0.137 | .005 | 16.583 | .448 | 0.187 | .009 |
| 30 | 2168.66 | 28.00 | 16.892 | .377 | 0.142 | .005 | 17.045 | .462 | 0.196 | .009 |
| | | 28.34 | | .386 | | | | .477 | | .010 |
| 115 0 | 2197.00 | | 17.278 | | 0.147 | | 17.522 | | 0.206 | |
| 30 | 2225.69 | 28.69 | 17.674 | .396 | 0.152 | .005 | 18.015 | .493 | 0.216 | .010 |
| 116 0 | 2254.73 | 29.04 | 18.080 | .406 | 0.157 | .005 | 18.524 | .509 | 0.227 | .011 |
| 30 | 2284.13 | 29.40 | 18.496 | .416 | 0.162 | .005 | 19.050 | .526 | 0.239 | .012 |
| 117 0 | 2313.91 | 29.78 | 18.924 | .428 | 0.168 | .006 | 19.594 | .544 | 0.251 | .012 |
| 30 | 2344.06 | 30.15 | 19.363 | .439 | 0.174 | .006 | 20.156 | .562 | 0.264 | .013 |
| | | 30.54 | | .450 | | .006 | | .582 | | .013 |
| 118 0 | 2374.60 | | 19.813 | | 0.180 | | 20.738 | | 0.277 | |
| 30 | 2405.54 | 30.94 | 20.276 | .463 | 0.186 | .006 | 21.339 | .601 | 0.291 | .014 |
| 119 0 | 2436.88 | 31.34 | 20.751 | .475 | 0.193 | .007 | 21.962 | .623 | 0.306 | .015 |
| 30 | 2468.64 | 31.76 | 21.240 | .489 | 0.200 | .007 | 22.606 | .644 | 0.322 | .016 |
| 120 0 | 2500.83 | 32.19 | 21.742 | .502 | 0.207 | .007 | 23.273 | .667 | 0.339 | .017 |
| 30 | 2533.45 | 32.62 | 22.258 | .516 | 0.214 | .007 | 23.964 | .691 | 0.357 | .018 |
| | | 33.06 | | .531 | | .008 | | .716 | | .019 |
| 121 0 | 2566.51 | | 22.789 | | 0.222 | | 24.680 | | 0.376 | |
| 30 | 2600.03 | 33.52 | 23.336 | .547 | 0.230 | .008 | 25.422 | .742 | 0.396 | .020 |
| 122 0 | 2634.02 | 33.99 | 23.898 | .562 | 0.239 | .009 | 26.191 | .769 | 0.417 | .021 |
| 30 | 2668.49 | 34.47 | 24.477 | .579 | 0.248 | .009 | 26.988 | .797 | 0.439 | .022 |
| 123 0 | 2703.46 | 34.97 | 25.073 | .596 | 0.258 | .010 | 27.815 | .827 | 0.463 | .024 |
| 30 | 2738.93 | 35.47 | 25.687 | .614 | 0.268 | .010 | 28.673 | .858 | 0.488 | .025 |
| | | 35.98 | | .633 | | | | .891 | | .027 |
| 124 0 | 2774.91 | | 26.320 | | 0.278 | | 29.564 | | 0.515 | |
| 30 | 2811.43 | 36.52 | 26.973 | .653 | 0.289 | .011 | 30.489 | .925 | 0.544 | .029 |
| 125 0 | 2848.50 | 37.07 | 27.646 | .673 | 0.300 | .012 | 31.450 | .961 | 0.574 | .030 |
| 30 | 2886.13 | 37.63 | 28.341 | .695 | 0.312 | .012 | 32.448 | .998 | 0.606 | .032 |
| 126 0 | 2924.33 | 38.20 | 29.057 | .716 | 0.325 | .013 | 33.485 | 1.037 | 0.640 | .034 |
| 30 | 2963.12 | 38.79 | 29.797 | .740 | 0.338 | .013 | 34.563 | 1.078 | 0.676 | .036 |
| | | 39.41 | | .765 | | .014 | | 1.122 | | .039 |
| 127 0 | 3002.53 | | 30.562 | | 0.352 | | 35.685 | | 0.715 | |
| 30 | 3042.56 | 40.03 | 31.351 | .789 | 0.367 | .015 | 36.852 | 1.167 | 0.757 | .042 |
| 128 0 | 3083.23 | 40.67 | 32.167 | .816 | 0.382 | .015 | 38.067 | 1.215 | 0.800 | .043 |
| 30 | 3124.57 | 41.34 | 33.011 | .844 | 0.398 | .016 | 39.331 | 1.264 | 0.846 | .046 |
| 129 0 | 3166.59 | 42.02 | 33.885 | .874 | 0.415 | .017 | 40.649 | 1.318 | 0.896 | .050 |
| 30 | 3209.31 | 42.72 | 34.789 | .904 | 0.433 | .018 | 42.022 | 1.373 | 0.949 | .053 |
| | | 43.45 | | .936 | | .019 | | 1.430 | | .056 |
| 130 0 | 3252.76 | | 35.725 | | 0.452 | | 43.452 | | 1.005 | |
| 20 | 3282.13 | 29.37 | 36.367 | .642 | 0.465 | .013 | 44.439 | 0.987 | 1.045 | .040 |
| 40 | 3311.85 | 29.72 | 37.025 | .658 | 0.479 | .014 | 45.455 | 1.016 | 1.087 | .042 |
| 131 0 | 3341.90 | 30.05 | 37.699 | .674 | 0.493 | .014 | 46.500 | 1.045 | 1.130 | .043 |
| 20 | 3372.31 | 30.41 | 38.389 | .690 | 0.508 | .015 | 47.575 | 1.075 | 1.175 | .045 |
| 40 | 3403.09 | 30.78 | 39.097 | .708 | 0.523 | .015 | 48.682 | 1.107 | 1.223 | .048 |
| | | 31.14 | | .725 | | .016 | | 1.138 | | .050 |
| 132 0 | 3434.23 | | 39.822 | | 0.539 | | 49.820 | | 1.273 | |
| 20 | 3465.74 | 31.51 | 40.564 | .742 | 0.555 | .016 | 50.992 | 1.172 | 1.325 | .052 |
| 40 | 3497.63 | 31.89 | 41.326 | .762 | 0.572 | .017 | 52.199 | 1.207 | 1.379 | .054 |
| 133 0 | 3529.91 | 32.28 | 42.108 | .782 | 0.590 | .018 | 53.442 | 1.243 | 1.436 | .057 |
| 20 | 3562.60 | 32.69 | 42.910 | .802 | 0.609 | .019 | 54.723 | 1.281 | 1.495 | .059 |
| 40 | 3595.69 | 33.09 | 43.733 | .823 | 0.629 | .020 | 56.042 | 1.319 | 1.558 | .063 |
| | | 33.51 | | .843 | | .020 | | 1.359 | | .065 |
| 134 0 | 3629.20 | | 44.576 | | 0.649 | | 57.401 | | 1.623 | |
| 20 | 3663.13 | 33.93 | 45.442 | .866 | 0.669 | .020 | 58.802 | 1.401 | 1.692 | .069 |
| 40 | 3697.50 | 34.37 | 46.331 | .889 | 0.691 | .022 | 60.247 | 1.445 | 1.764 | .072 |
| 135 0 | 3732.31 | 34.81 | 47.245 | .914 | 0.714 | .023 | 61.736 | 1.489 | 1.839 | .075 |
| 20 | 3767.58 | 35.27 | 48.183 | .938 | 0.738 | .024 | 63.273 | 1.537 | 1.917 | .078 |
| 40 | 3803.31 | 35.73 | 49.147 | .964 | 0.763 | .025 | 64.857 | 1.584 | 2.000 | .083 |
| | | 36.21 | | .991 | | | | 1.634 | | .087 |
| 136 0 | 3839.52 | | 50.138 | | 0.788 | | 66.491 | | 2.087 | |

TABLE IX.

For finding the True Anomaly or the Time from the Perihelion in Orbits of great excentricity

| α | A | Diff. | B | Diff. | C | Diff. | B' | Diff. | C' | Diff. |
|--------------|---------|-------|---------|-------|-------|-------|---------|-------|--------|-------|
| " | " | " | " | " | " | " | " | " | " | " |
| 136 0 | 3839.52 | | 50.138 | | 0.788 | | 66.491 | | 2.087 | |
| 20 | 3876.21 | 36.69 | 51.156 | 1.018 | 0.815 | .027 | 68.178 | 1.687 | 2.178 | .091 |
| 40 | 3913.41 | 37.20 | 52.203 | 1.047 | 0.843 | .028 | 69.920 | 1.742 | 2.274 | .096 |
| 137 0 | 3951.12 | 37.71 | 53.280 | 1.077 | 0.873 | .030 | 71.718 | 1.798 | 2.375 | .101 |
| 20 | 3989.35 | 38.23 | 54.388 | 1.108 | 0.904 | .031 | 73.575 | 1.857 | 2.480 | .105 |
| 40 | 4028.11 | 38.76 | 55.528 | 1.140 | 0.936 | .032 | 75.493 | 1.918 | 2.591 | .111 |
| 138 0 | 4067.42 | 39.31 | 56.702 | 1.174 | 0.969 | .033 | 77.475 | 1.982 | 2.708 | .117 |
| 20 | 4107.28 | 39.86 | 57.910 | 1.208 | 1.004 | .035 | 79.523 | 2.048 | 2.831 | .123 |
| 40 | 4147.72 | 40.44 | 59.154 | 1.244 | 1.041 | .037 | 81.641 | 2.118 | 2.960 | .129 |
| 139 0 | 4188.75 | 41.03 | 60.436 | 1.282 | 1.079 | .038 | 83.830 | 2.189 | 3.096 | .136 |
| 20 | 4230.38 | 41.63 | 61.757 | 1.321 | 1.119 | .040 | 86.094 | 2.264 | 3.239 | .143 |
| 40 | 4272.63 | 42.25 | 63.119 | 1.362 | 1.161 | .042 | 88.436 | 2.342 | 3.390 | .151 |
| 140 0 | 4315.52 | 42.89 | 64.523 | 1.404 | 1.205 | .044 | 90.860 | 2.424 | 3.549 | .159 |
| 20 | 4359.06 | 43.54 | 65.971 | 1.448 | 1.251 | .046 | 93.369 | 2.509 | 3.717 | .168 |
| 40 | 4403.26 | 44.20 | 67.465 | 1.494 | 1.299 | .048 | 95.967 | 2.598 | 3.893 | .176 |
| 141 0 | 4448.15 | 44.89 | 69.007 | 1.542 | 1.350 | .051 | 98.657 | 2.690 | 4.080 | .187 |
| 20 | 4493.73 | 45.58 | 70.599 | 1.592 | 1.404 | .054 | 101.443 | 2.786 | 4.277 | .197 |
| 40 | 4540.03 | 46.30 | 72.243 | 1.644 | 1.460 | .056 | 104.331 | 2.888 | 4.484 | .207 |
| 142 0 | 4587.07 | 47.04 | 73.941 | 1.698 | 1.518 | .058 | 107.324 | 2.993 | 4.704 | .220 |
| 10 | 4610.88 | 23.81 | 74.811 | 0.870 | 1.549 | .031 | 108.861 | 1.537 | 4.819 | .115 |
| 20 | 4634.88 | 24.00 | 75.695 | 0.884 | 1.580 | .031 | 110.427 | 1.566 | 4.936 | .117 |
| 30 | 4659.07 | 24.19 | 76.595 | 0.900 | 1.612 | .032 | 112.022 | 1.595 | 5.057 | .121 |
| 40 | 4683.46 | 24.39 | 77.509 | 0.914 | 1.645 | .033 | 113.646 | 1.624 | 5.181 | .124 |
| 50 | 4708.05 | 24.59 | 78.439 | 0.930 | 1.679 | .034 | 115.301 | 1.655 | 5.309 | .128 |
| 143 0 | 4732.84 | 24.79 | 79.385 | 0.946 | 1.714 | .035 | 116.986 | 1.685 | 5.440 | .131 |
| 10 | 4757.84 | 25.00 | 80.347 | 0.962 | 1.749 | .035 | 118.704 | 1.718 | 5.575 | .135 |
| 20 | 4783.05 | 25.21 | 81.325 | 0.978 | 1.786 | .037 | 120.452 | 1.748 | 5.715 | .140 |
| 30 | 4808.48 | 25.41 | 82.321 | 0.996 | 1.823 | .037 | 122.233 | 1.781 | 5.858 | .143 |
| 40 | 4834.10 | 25.64 | 83.333 | 1.012 | 1.862 | .039 | 124.049 | 1.816 | 6.005 | .147 |
| 50 | 4859.95 | 25.85 | 84.363 | 1.030 | 1.901 | .039 | 125.899 | 1.850 | 6.157 | .152 |
| 144 0 | 4886.02 | 26.07 | 85.411 | 1.048 | 1.942 | .041 | 127.785 | 1.886 | 6.313 | .156 |
| 10 | 4912.31 | 26.29 | 86.478 | 1.067 | 1.984 | .042 | 129.707 | 1.922 | 6.473 | .160 |
| 20 | 4938.83 | 26.52 | 87.564 | 1.086 | 2.026 | .042 | 131.666 | 1.959 | 6.639 | .166 |
| 30 | 4965.58 | 26.75 | 88.668 | 1.104 | 2.070 | .044 | 133.663 | 1.997 | 6.809 | .170 |
| 40 | 4992.56 | 26.98 | 89.793 | 1.125 | 2.116 | .046 | 135.698 | 2.035 | 6.984 | .175 |
| 50 | 5019.78 | 27.22 | 90.938 | 1.145 | 2.162 | .046 | 137.774 | 2.076 | 7.165 | .181 |
| 145 0 | 5047.23 | 27.45 | 92.103 | 1.165 | 2.210 | .049 | 139.890 | 2.116 | 7.351 | .186 |
| 10 | 5074.93 | 27.70 | 93.290 | 1.187 | 2.259 | .049 | 142.048 | 2.158 | 7.543 | .192 |
| 20 | 5102.88 | 27.95 | 94.498 | 1.208 | 2.309 | .050 | 144.249 | 2.201 | 7.740 | .197 |
| 30 | 5131.08 | 28.20 | 95.729 | 1.231 | 2.361 | .052 | 146.494 | 2.245 | 7.943 | .203 |
| 40 | 5159.53 | 28.45 | 96.982 | 1.253 | 2.414 | .053 | 148.784 | 2.290 | 8.153 | .210 |
| 50 | 5188.24 | 28.71 | 98.259 | 1.277 | 2.469 | .055 | 151.120 | 2.336 | 8.369 | .216 |
| 146 0 | 5217.21 | 28.97 | 99.559 | 1.300 | 2.526 | .057 | 153.503 | 2.383 | 8.592 | .223 |
| 10 | 5246.45 | 29.24 | 100.884 | 1.325 | 2.584 | .058 | 155.934 | 2.431 | 8.822 | .230 |
| 20 | 5275.95 | 29.50 | 102.234 | 1.350 | 2.643 | .059 | 158.415 | 2.481 | 9.060 | .238 |
| 30 | 5305.73 | 29.78 | 103.610 | 1.376 | 2.704 | .061 | 160.947 | 2.532 | 9.304 | .244 |
| 40 | 5335.79 | 30.06 | 105.012 | 1.402 | 2.767 | .063 | 163.531 | 2.584 | 9.555 | .251 |
| 50 | 5366.13 | 30.34 | 106.441 | 1.429 | 2.833 | .066 | 166.168 | 2.637 | 9.815 | .260 |
| 147 0 | 5396.76 | 30.63 | 107.897 | 1.456 | 2.900 | .067 | 168.860 | 2.692 | 10.083 | .268 |
| 10 | 5427.67 | 30.91 | 109.382 | 1.485 | 2.969 | .069 | 171.608 | 2.748 | 10.359 | .276 |
| 20 | 5458.88 | 31.21 | 110.896 | 1.514 | 3.040 | .071 | 174.414 | 2.806 | 10.645 | .286 |
| 30 | 5490.39 | 31.51 | 112.439 | 1.543 | 3.113 | .073 | 177.280 | 2.866 | 10.940 | .295 |
| 40 | 5522.20 | 31.81 | 114.013 | 1.574 | 3.188 | .075 | 180.206 | 2.926 | 11.244 | .304 |
| 50 | 5554.33 | 32.13 | 115.619 | 1.606 | 3.266 | .078 | 183.194 | 2.988 | 11.558 | .314 |
| 148 0 | 5586.77 | 32.44 | 117.256 | 1.637 | 3.346 | .080 | 186.246 | 3.052 | 11.883 | .325 |
| 10 | 5619.52 | 32.75 | 118.926 | 1.670 | 3.428 | .082 | 189.364 | 3.118 | 12.218 | .335 |
| 20 | 5652.60 | 33.08 | 120.631 | 1.705 | 3.513 | .085 | 192.549 | 3.185 | 12.564 | .346 |
| 30 | 5686.01 | 33.41 | 122.370 | 1.739 | 3.601 | .088 | 195.804 | 3.255 | 12.921 | .357 |
| 40 | 5719.75 | 33.74 | 124.144 | 1.774 | 3.691 | .090 | 199.130 | 3.326 | 13.291 | .370 |
| 50 | 5753.83 | 34.08 | 125.955 | 1.811 | 3.784 | .093 | 202.528 | 3.398 | 13.673 | .382 |
| 149 0 | 5788.26 | 34.43 | 127.804 | 1.849 | 3.881 | .097 | 206.002 | 3.474 | 14.067 | .394 |

TABLE X.

For finding the True Anomaly or the Time from the Perihelion in Elliptic and Hyperbolic Orbits.

| A | Ellipse. | | | | | Hyperbola. | | | | |
|------|----------|-------|------------|--------------|--------------------|------------|-------|------------|----------------------|--------------------|
| | log B | Diff. | log C | log I. Diff. | log half II. Diff. | log B | Diff. | log C | log I. Diff. | log half II. Diff. |
| 0.00 | 0.0000 | | 0.0000 | 4.23990 | 1.778 | 0.0000 | | 0.0000 | 4.23982 _n | 1.771 |
| .01 | 0007 | 7 | .001 7432 | .24286 | .783 | 0007 | 7 | 0.998 2688 | .23686 | .767 |
| .02 | 0030 | 23 | .003 4985 | .24583 | .788 | 0030 | 23 | .996 5493 | .23392 | .762 |
| .03 | 0067 | 53 | .005 2659 | .24885 | .794 | 0067 | 51 | .994 8414 | .23098 | .758 |
| .04 | 0120 | 68 | .007 0457 | .25190 | .799 | 0118 | 66 | .993 1450 | .22807 | .753 |
| .05 | 0188 | 84 | 0.008 8381 | .25497 | 1.805 | 0184 | 81 | 9.991 4599 | 4.22518 _n | 1.748 |
| .06 | 0272 | 99 | .010 6432 | .25806 | .811 | 0265 | 94 | .989 7859 | .22230 | .743 |
| .07 | 0371 | 114 | .012 4613 | .26116 | .816 | 0359 | 109 | .988 1231 | .21943 | .739 |
| .08 | 0485 | 130 | .014 2924 | .26427 | .821 | 0468 | 123 | .986 4711 | .21659 | .734 |
| .09 | 0615 | 147 | .016 1367 | .26741 | .827 | 0591 | 137 | .984 8298 | .21376 | .730 |
| .10 | 0762 | 162 | 0.017 9945 | 4.27057 | 1.833 | 0728 | 152 | 9.983 1992 | 4.21094 _n | 1.725 |
| .11 | 0924 | 178 | .019 8659 | .27376 | .839 | 0880 | 165 | .981 5791 | .20815 | .720 |
| .12 | 1102 | 194 | .021 7511 | .27697 | .845 | 1045 | 178 | .979 9694 | .20537 | .716 |
| .13 | 1296 | 211 | .023 6503 | .28020 | .851 | 1223 | 193 | .978 3699 | .20260 | .711 |
| .14 | 1507 | 227 | .025 5637 | .28344 | .857 | 1416 | 206 | .976 7805 | .19986 | .706 |
| .15 | 1734 | 243 | 0.027 4916 | 4.28670 | 1.863 | 1622 | 220 | 9.975 2011 | 4.19712 _n | 1.700 |
| .16 | 1977 | 261 | .029 4340 | .28999 | .869 | 1842 | 233 | .973 6316 | .19440 | .695 |
| .17 | 2238 | 277 | .031 3913 | .29331 | .875 | 2075 | 246 | .972 0719 | .19170 | .690 |
| .18 | 2515 | 294 | .033 3636 | .29665 | .882 | 2321 | 260 | .970 5218 | .18901 | .685 |
| .19 | 2809 | 311 | .035 3511 | .30001 | .888 | 2581 | 273 | .968 9813 | .18633 | .679 |
| .20 | 3120 | 328 | 0.037 3542 | 4.30339 | 1.895 | 2854 | 286 | 9.967 4502 | 4.18367 _n | 1.672 |
| .21 | 3448 | 345 | .039 3730 | .30679 | .901 | 3140 | 299 | .965 9285 | .18102 | .666 |
| .22 | 3793 | 363 | .041 4077 | .31022 | .908 | 3439 | 312 | .964 4159 | .17840 | .661 |
| .23 | 4156 | 381 | .043 4585 | .31368 | .915 | 3751 | 325 | .962 9124 | .17579 | .655 |
| .24 | 4537 | 398 | .045 5259 | .31716 | .922 | 4076 | 338 | .961 4180 | .17319 | .649 |
| .25 | 4935 | 416 | 0.047 6099 | 4.32066 | 1.929 | 4414 | 351 | 9.959 9324 | 4.17061 _n | 1.643 |
| .26 | 5351 | 434 | .049 7109 | .32418 | .936 | 4765 | 363 | .958 4556 | .16803 | .637 |
| .27 | 5785 | 452 | .051 8200 | .32773 | .943 | 5128 | 376 | .956 9875 | .16547 | .631 |
| .28 | 6237 | 471 | .053 9646 | .33131 | .951 | 5504 | 389 | .955 5281 | .16292 | .625 |
| .29 | 6708 | 488 | .056 1179 | .33492 | .958 | 5893 | 401 | .954 0771 | .16038 | .618 |
| .30 | 7196 | | 0.058 2893 | 4.33856 | 1.966 | 6294 | | 9.952 6346 | 4.15785 _n | 1.613 |

TABLE X. Part II.

| τ | Ellipse. | | Hyperbola. | | τ | Ellipse. | | Hyperbola. | |
|------|----------|-------|------------|-------|------|----------|-------|------------|-------|
| | A | Diff. | A | Diff. | | A | Diff. | A | Diff. |
| 0.00 | 0.00000 | | 0.00000 | 1008 | 0.20 | 0.17266 | | 0.23867 | |
| .01 | .00992 | 992 | .01008 | 1025 | .21 | .18008 | 742 | .25309 | 1442 |
| .02 | .01969 | 977 | .02033 | 1041 | .22 | .18740 | 732 | .26779 | 1470 |
| .03 | .02930 | 961 | .03074 | 1058 | .23 | .19462 | 722 | .28280 | 1501 |
| .04 | .03877 | 947 | .04132 | 1077 | .24 | .20174 | 712 | .29813 | 1533 |
| | | 931 | | | | | 704 | | 1564 |
| .05 | 0.04808 | 918 | 0.05209 | 1094 | 0.25 | 0.20878 | 695 | 0.31377 | |
| .06 | .05726 | 904 | .06303 | 1114 | .26 | .21573 | 685 | | |
| .07 | .06630 | 891 | .07417 | 1133 | .27 | .22258 | 677 | | |
| .08 | .07521 | 877 | .08550 | 1152 | .28 | .22935 | 669 | | |
| .09 | .08398 | 865 | .09702 | 1173 | .29 | .23604 | 661 | | |
| .10 | 0.09263 | 853 | 0.10875 | 1194 | 0.30 | 0.24265 | 652 | | |
| .11 | .10116 | 840 | .12069 | 1216 | .31 | .24917 | 644 | | |
| .12 | .10956 | 827 | .13285 | 1237 | .32 | .25561 | 637 | | |
| .13 | .11783 | 816 | .14522 | 1260 | .33 | .26198 | 628 | | |
| .14 | .12599 | 805 | .15782 | 1285 | .34 | .26826 | 621 | | |
| .15 | 0.13404 | 794 | 0.17067 | 1308 | 0.35 | 0.27447 | 614 | | |
| .16 | .14198 | 783 | .18375 | 1334 | .36 | .28061 | 607 | | |
| .17 | .14981 | 772 | .19709 | 1359 | .37 | .28668 | 600 | | |
| .18 | .15753 | 762 | .21068 | 1386 | .38 | .29268 | 592 | | |
| .19 | .16515 | 751 | .22454 | 1413 | .39 | .29860 | 586 | | |
| .20 | 0.17266 | | 0.23867 | | .40 | 0.30446 | | | |

TABLE XI.
For the Motion in a Parabolic Orbit.

| η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.000 | 0.000 0000 | 0 | 0.060 | 0.000 0652 | 22 | 0.120 | 0.000 2617 | 44 |
| .001 | .000 0000 | 1 | .061 | .000 0674 | 23 | .121 | .000 2661 | 44 |
| .002 | .000 0001 | 1 | .062 | .000 0697 | 22 | .122 | .000 2705 | 45 |
| .003 | .000 0002 | 1 | .063 | .000 0719 | 23 | .123 | .000 2750 | 45 |
| .004 | .000 0003 | 1 | .064 | .000 0742 | 24 | .124 | .000 2795 | 46 |
| 0.005 | 0.000 0004 | 2 | 0.065 | 0.000 0766 | 24 | 0.125 | 0.000 2841 | 45 |
| .006 | .000 0006 | 3 | .066 | .000 0790 | 24 | .126 | .000 2886 | 47 |
| .007 | .000 0009 | 3 | .067 | .000 0814 | 24 | .127 | .000 2933 | 46 |
| .008 | .000 0012 | 3 | .068 | .000 0838 | 25 | .128 | .000 2979 | 47 |
| .009 | .000 0015 | 3 | .069 | .000 0863 | 25 | .129 | .000 3026 | 48 |
| 0.010 | 0.000 0018 | 4 | 0.070 | 0.000 0888 | 26 | 0.130 | 0.000 3074 | 47 |
| .011 | .000 0022 | 4 | .071 | .000 0914 | 26 | .131 | .000 3121 | 48 |
| .012 | .000 0026 | 5 | .072 | .000 0940 | 26 | .132 | .000 3169 | 49 |
| .013 | .000 0031 | 5 | .073 | .000 0966 | 27 | .133 | .000 3218 | 49 |
| .014 | .000 0035 | 6 | .074 | .000 0993 | 27 | .134 | .000 3267 | 49 |
| 0.015 | 0.000 0041 | 5 | 0.075 | 0.000 1020 | 27 | 0.135 | 0.000 3316 | 49 |
| .016 | .000 0046 | 6 | .076 | .000 1047 | 28 | .136 | .000 3365 | 50 |
| .017 | .000 0052 | 7 | .077 | .000 1075 | 28 | .137 | .000 3415 | 51 |
| .018 | .000 0059 | 6 | .078 | .000 1103 | 29 | .138 | .000 3466 | 50 |
| .019 | .000 0065 | 7 | .079 | .000 1132 | 29 | .139 | .000 3516 | 51 |
| 0.020 | 0.000 0072 | 8 | 0.080 | 0.000 1161 | 29 | 0.140 | 0.000 3567 | 52 |
| .021 | .000 0080 | 8 | .081 | .000 1190 | 29 | .141 | .000 3619 | 52 |
| .022 | .000 0088 | 8 | .082 | .000 1219 | 30 | .142 | .000 3671 | 52 |
| .023 | .000 0096 | 8 | .083 | .000 1249 | 31 | .143 | .000 3723 | 52 |
| .024 | .000 0104 | 9 | .084 | .000 1280 | 31 | .144 | .000 3775 | 53 |
| 0.025 | 0.000 0113 | 9 | 0.085 | 0.000 1311 | 31 | 0.145 | 0.000 3828 | 54 |
| .026 | .000 0122 | 10 | .086 | .000 1342 | 31 | .146 | .000 3882 | 53 |
| .027 | .000 0132 | 10 | .087 | .000 1373 | 32 | .147 | .000 3935 | 54 |
| .028 | .000 0142 | 10 | .088 | .000 1405 | 32 | .148 | .000 3989 | 55 |
| .029 | .000 0152 | 11 | .089 | .000 1437 | 33 | .149 | .000 4044 | 55 |
| 0.030 | 0.000 0163 | 11 | 0.090 | 0.000 1470 | 32 | 0.150 | 0.000 4099 | 55 |
| .031 | .000 0174 | 11 | .091 | .000 1502 | 34 | .151 | .000 4154 | 55 |
| .032 | .000 0185 | 12 | .092 | .000 1536 | 33 | .152 | .000 4209 | 56 |
| .033 | .000 0197 | 12 | .093 | .000 1569 | 34 | .153 | .000 4265 | 57 |
| .034 | .000 0209 | 13 | .094 | .000 1603 | 35 | .154 | .000 4322 | 56 |
| 0.035 | 0.000 0222 | 13 | 0.095 | 0.000 1638 | 35 | 0.155 | 0.000 4378 | 57 |
| .036 | .000 0235 | 13 | .096 | .000 1673 | 35 | .156 | .000 4435 | 58 |
| .037 | .000 0248 | 14 | .097 | .000 1708 | 35 | .157 | .000 4493 | 58 |
| .038 | .000 0262 | 13 | .098 | .000 1743 | 36 | .158 | .000 4551 | 58 |
| .039 | .000 0275 | 15 | .099 | .000 1779 | 36 | .159 | .000 4609 | 59 |
| 0.040 | 0.000 0290 | 14 | 0.100 | 0.000 1815 | 37 | 0.160 | 0.000 4668 | 58 |
| .041 | .000 0304 | 16 | .101 | .000 1852 | 37 | .161 | .000 4726 | 60 |
| .042 | .000 0320 | 15 | .102 | .000 1889 | 37 | .162 | .000 4786 | 60 |
| .043 | .000 0335 | 16 | .103 | .000 1926 | 38 | .163 | .000 4846 | 60 |
| .044 | .000 0351 | 16 | .104 | .000 1964 | 38 | .164 | .000 4906 | 60 |
| 0.045 | 0.000 0367 | 16 | 0.105 | 0.000 2002 | 38 | 0.165 | 0.000 4966 | 61 |
| .046 | .000 0383 | 17 | .106 | .000 2040 | 39 | .166 | .000 5027 | 61 |
| .047 | .000 0400 | 17 | .107 | .000 2079 | 39 | .167 | .000 5088 | 62 |
| .048 | .000 0417 | 18 | .108 | .000 2118 | 40 | .168 | .000 5150 | 62 |
| .049 | .000 0435 | 18 | .109 | .000 2158 | 40 | .169 | .000 5212 | 62 |
| 0.050 | 0.000 0453 | 18 | 0.110 | 0.000 2198 | 40 | 0.170 | 0.000 5274 | 63 |
| .051 | .000 0471 | 19 | .111 | .000 2238 | 41 | .171 | .000 5337 | 63 |
| .052 | .000 0490 | 19 | .112 | .000 2279 | 41 | .172 | .000 5400 | 64 |
| .053 | .000 0509 | 19 | .113 | .000 2320 | 41 | .173 | .000 5464 | 64 |
| .054 | .000 0528 | 20 | .114 | .000 2361 | 42 | .174 | .000 5528 | 64 |
| 0.055 | 0.000 0548 | 20 | 0.115 | 0.000 2403 | 42 | 0.175 | 0.000 5592 | 65 |
| .056 | .000 0568 | 21 | .116 | .000 2445 | 42 | .176 | .000 5657 | 65 |
| .057 | .000 0589 | 21 | .117 | .000 2487 | 43 | .177 | .000 5722 | 65 |
| .058 | .000 0610 | 21 | .118 | .000 2530 | 43 | .178 | .000 5787 | 66 |
| .059 | .000 0631 | 21 | .119 | .000 2573 | 44 | .179 | .000 5853 | 66 |
| 0.060 | 0.000 0652 | | 0.120 | 0.000 2617 | | 0.180 | 0.000 5919 | |

TABLE XI.

For the Motion in a Parabolic Orbit.

| η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.180 | 0.000 5919 | | 0.240 | 0.001 0603 | | 0.300 | 0.001 6733 | |
| .181 | .000 5986 | 67 | .241 | .001 0693 | 90 | .301 | .001 6848 | 115 |
| .182 | .000 6053 | 67 | .242 | .001 0784 | 91 | .302 | .001 6963 | 115 |
| .183 | .000 6120 | 67 | .243 | .001 0875 | 91 | .303 | .001 7079 | 116 |
| .184 | .000 6188 | 68 | .244 | .001 0966 | 91 | .304 | .001 7195 | 116 |
| | | | | | 92 | | | 117 |
| 0.185 | 0.000 6256 | | 0.245 | 0.001 1058 | | 0.305 | 0.001 7312 | |
| .186 | .000 6325 | 69 | .246 | .001 1150 | 92 | .306 | .001 7429 | 117 |
| .187 | .000 6393 | 68 | .247 | .001 1242 | 92 | .307 | .001 7546 | 117 |
| .188 | .000 6463 | 70 | .248 | .001 1335 | 93 | .308 | .001 7664 | 118 |
| .189 | .000 6532 | 69 | .249 | .001 1429 | 94 | .309 | .001 7783 | 119 |
| | | 70 | | | 93 | | | 118 |
| 0.190 | 0.000 6602 | | 0.250 | 0.001 1522 | | 0.310 | 0.001 7901 | |
| .191 | .000 6673 | 71 | .251 | .001 1617 | 95 | .311 | .001 8020 | 119 |
| .192 | .000 6744 | 71 | .252 | .001 1711 | 94 | .312 | .001 8140 | 120 |
| .193 | .000 6815 | 71 | .253 | .001 1806 | 95 | .313 | .001 8260 | 120 |
| .194 | .000 6887 | 72 | .254 | .001 1901 | 95 | .314 | .001 8381 | 121 |
| | | 72 | | | 96 | | | 121 |
| 0.195 | 0.000 6959 | | 0.255 | 0.001 1997 | | 0.315 | 0.001 8502 | |
| .196 | .000 7031 | 72 | .256 | .001 2093 | 96 | .316 | .001 8623 | 121 |
| .197 | .000 7104 | 73 | .257 | .001 2190 | 97 | .317 | .001 8745 | 122 |
| .198 | .000 7177 | 73 | .258 | .001 2287 | 97 | .318 | .001 8867 | 122 |
| .199 | .000 7250 | 73 | .259 | .001 2384 | 97 | .319 | .001 8989 | 122 |
| | | 74 | | | 98 | | | 124 |
| 0.200 | 0.000 7324 | | 0.260 | 0.001 2482 | | 0.320 | 0.001 9113 | |
| .201 | .000 7399 | 75 | .261 | .001 2580 | 98 | .321 | .001 9236 | 123 |
| .202 | .000 7473 | 74 | .262 | .001 2679 | 99 | .322 | .001 9360 | 124 |
| .203 | .000 7548 | 75 | .263 | .001 2778 | 99 | .323 | .001 9484 | 124 |
| .204 | .000 7624 | 76 | .264 | .001 2877 | 99 | .324 | .001 9609 | 125 |
| | | 76 | | | 100 | | | 125 |
| 0.205 | 0.000 7700 | | 0.265 | 0.001 2977 | | 0.325 | 0.001 9734 | |
| .206 | .000 7776 | 76 | .266 | .001 3077 | 100 | .326 | .001 9860 | 126 |
| .207 | .000 7853 | 77 | .267 | .001 3178 | 101 | .327 | .001 9986 | 126 |
| .208 | .000 7930 | 77 | .268 | .001 3279 | 101 | .328 | .002 0113 | 127 |
| .209 | .000 8007 | 77 | .269 | .001 3381 | 102 | .329 | .002 0240 | 127 |
| | | 78 | | | 101 | | | 127 |
| 0.210 | 0.000 8085 | | 0.270 | 0.001 3482 | | 0.330 | 0.002 0367 | |
| .211 | .000 8163 | 78 | .271 | .001 3585 | 103 | .331 | .002 0495 | 128 |
| .212 | .000 8242 | 79 | .272 | .001 3688 | 103 | .332 | .002 0624 | 129 |
| .213 | .000 8321 | 79 | .273 | .001 3791 | 103 | .333 | .002 0752 | 128 |
| .214 | .000 8400 | 79 | .274 | .001 3894 | 103 | .334 | .002 0882 | 130 |
| | | 80 | | | 104 | | | 129 |
| 0.215 | 0.000 8480 | | 0.275 | 0.001 3998 | | 0.335 | 0.002 1011 | |
| .216 | .000 8560 | 80 | .276 | .001 4103 | 105 | .336 | .002 1141 | 130 |
| .217 | .000 8641 | 81 | .277 | .001 4207 | 104 | .337 | .002 1272 | 131 |
| .218 | .000 8722 | 81 | .278 | .001 4313 | 106 | .338 | .002 1403 | 131 |
| .219 | .000 8803 | 81 | .279 | .001 4418 | 105 | .339 | .002 1534 | 131 |
| | | 82 | | | 106 | | | 132 |
| 0.220 | 0.000 8885 | | 0.280 | 0.001 4524 | | 0.340 | 0.002 1666 | |
| .221 | .000 8967 | 82 | .281 | .001 4631 | 107 | .341 | .002 1799 | 133 |
| .222 | .000 9050 | 83 | .282 | .001 4738 | 107 | .342 | .002 1931 | 132 |
| .223 | .000 9132 | 82 | .283 | .001 4845 | 107 | .343 | .002 2065 | 134 |
| .224 | .000 9216 | 84 | .284 | .001 4953 | 108 | .344 | .002 2198 | 133 |
| | | 84 | | | 108 | | | 135 |
| 0.225 | 0.000 9300 | | 0.285 | 0.001 5061 | | 0.345 | 0.002 2333 | |
| .226 | .000 9384 | 84 | .286 | .001 5169 | 108 | .346 | .002 2467 | 134 |
| .227 | .000 9468 | 84 | .287 | .001 5278 | 109 | .347 | .002 2602 | 135 |
| .228 | .000 9553 | 85 | .288 | .001 5388 | 110 | .348 | .002 2738 | 136 |
| .229 | .000 9638 | 85 | .289 | .001 5497 | 109 | .349 | .002 2874 | 136 |
| | | 86 | | | 111 | | | 136 |
| 0.230 | 0.000 9724 | | 0.290 | 0.001 5608 | | 0.350 | 0.002 3010 | |
| .231 | .000 9810 | 86 | .291 | .001 5718 | 110 | .351 | .002 3147 | 137 |
| .232 | .000 9897 | 87 | .292 | .001 5829 | 111 | .352 | .002 3284 | 137 |
| .233 | .000 9984 | 87 | .293 | .001 5941 | 112 | .353 | .002 3422 | 138 |
| .234 | .001 0071 | 87 | .294 | .001 6053 | 112 | .354 | .002 3560 | 138 |
| | | 88 | | | 112 | | | 139 |
| 0.235 | 0.001 0159 | | 0.295 | 0.001 6165 | | 0.355 | 0.002 3699 | |
| .236 | .001 0247 | 88 | .296 | .001 6278 | 113 | .356 | .002 3838 | 139 |
| .237 | .001 0335 | 88 | .297 | .001 6391 | 113 | .357 | .002 3977 | 139 |
| .238 | .001 0424 | 89 | .298 | .001 6505 | 114 | .358 | .002 4117 | 140 |
| .239 | .001 0513 | 89 | .299 | .001 6619 | 114 | .359 | .002 4258 | 141 |
| | | 90 | | | 114 | | | 141 |
| 0.240 | 0.001 0603 | | 0.300 | 0.001 6733 | | 0.360 | 0.002 4399 | |

TABLE XI.

For the Motion in a Parabolic Orbit.

| η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. | η | $\log \mu$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.360 | 0.002 4399 | 141 | 0.420 | 0.003 3720 | 170 | 0.480 | 0.004 4858 | 203 |
| .361 | .002 4540 | 142 | .421 | .003 3890 | 171 | .481 | .004 5061 | 202 |
| .362 | .002 4682 | 142 | .422 | .003 4061 | 171 | .482 | .004 5263 | 204 |
| .363 | .002 4824 | 143 | .423 | .003 4232 | 172 | .483 | .004 5467 | 203 |
| .364 | .002 4967 | 143 | .424 | .003 4404 | 172 | .484 | .004 5670 | 205 |
| 0.365 | 0.002 5110 | 144 | 0.425 | 0.003 4576 | 173 | 0.485 | 0.004 5875 | 205 |
| .366 | .002 5254 | 144 | .426 | .003 4749 | 174 | .486 | .004 6080 | 205 |
| .367 | .002 5398 | 145 | .427 | .003 4923 | 173 | .487 | .004 6285 | 207 |
| .368 | .002 5543 | 145 | .428 | .003 5096 | 175 | .488 | .004 6492 | 206 |
| .369 | .002 5688 | 146 | .429 | .003 5271 | 174 | .489 | .004 6698 | 208 |
| 0.370 | 0.002 5834 | 146 | 0.430 | 0.003 5445 | 176 | 0.490 | 0.004 6906 | 207 |
| .371 | .002 5980 | 146 | .431 | .003 5621 | 176 | .491 | .004 7113 | 209 |
| .372 | .002 6126 | 147 | .432 | .003 5797 | 176 | .492 | .004 7322 | 209 |
| .373 | .002 6273 | 148 | .433 | .003 5973 | 177 | .493 | .004 7531 | 209 |
| .374 | .002 6421 | 147 | .434 | .003 6150 | 177 | .494 | .004 7740 | 211 |
| 0.375 | 0.002 6568 | 149 | 0.435 | 0.003 6327 | 178 | 0.495 | 0.004 7951 | 210 |
| .376 | .002 6717 | 149 | .436 | .003 6505 | 178 | .496 | .004 8161 | 212 |
| .377 | .002 6866 | 149 | .437 | .003 6683 | 179 | .497 | .004 8373 | 212 |
| .378 | .002 7015 | 150 | .438 | .003 6862 | 180 | .498 | .004 8585 | 212 |
| .379 | .002 7165 | 150 | .439 | .003 7042 | 180 | .499 | .004 8797 | 213 |
| 0.380 | 0.002 7315 | 151 | 0.440 | 0.003 7222 | 180 | 0.500 | 0.004 9010 | 2163 |
| .381 | .002 7466 | 151 | .441 | .003 7402 | 181 | .51 | .005 1173 | 2224 |
| .382 | .002 7617 | 152 | .442 | .003 7583 | 182 | .52 | .005 3397 | 2284 |
| .383 | .002 7769 | 152 | .443 | .003 7765 | 182 | .53 | .005 5681 | 2348 |
| .384 | .002 7921 | 152 | .444 | .003 7947 | 183 | .54 | .005 8029 | 2412 |
| 0.385 | 0.002 8073 | 153 | 0.445 | 0.003 8130 | 183 | 0.55 | 0.006 0441 | 2478 |
| .386 | .002 8226 | 154 | .446 | .003 8313 | 183 | .56 | .006 2919 | 2545 |
| .387 | .002 8380 | 154 | .447 | .003 8496 | 184 | .57 | .006 5464 | 2615 |
| .388 | .002 8534 | 155 | .448 | .003 8680 | 185 | .58 | .006 8079 | 2686 |
| .389 | .002 8689 | 155 | .449 | .003 8865 | 185 | .59 | .007 0765 | 2760 |
| 0.390 | 0.002 8844 | 155 | 0.450 | 0.003 9050 | 186 | 0.60 | 0.007 3525 | 2836 |
| .391 | .002 8999 | 156 | .451 | .003 9236 | 186 | .61 | .007 6361 | 2913 |
| .392 | .002 9155 | 156 | .452 | .003 9422 | 187 | .62 | .007 9274 | 2994 |
| .393 | .002 9311 | 157 | .453 | .003 9609 | 188 | .63 | .008 2268 | 3077 |
| .394 | .002 9468 | 158 | .454 | .003 9797 | 187 | .64 | .008 5345 | 3163 |
| 0.395 | 0.002 9626 | 158 | 0.455 | 0.003 9984 | 189 | 0.65 | 0.008 8508 | 3251 |
| .396 | .002 9784 | 158 | .456 | .004 0173 | 189 | .66 | .009 1759 | 3344 |
| .397 | .002 9942 | 159 | .457 | .004 0362 | 189 | .67 | .009 5103 | 3439 |
| .398 | .003 0101 | 159 | .458 | .004 0551 | 190 | .68 | .009 8542 | 3539 |
| .399 | .003 0260 | 160 | .459 | .004 0741 | 191 | .69 | .010 2081 | 3642 |
| 0.400 | 0.003 0420 | 160 | 0.460 | 0.004 0932 | 191 | 0.70 | 0.010 5723 | 3750 |
| .401 | .003 0580 | 161 | .461 | .004 1123 | 192 | .71 | .010 9473 | 3863 |
| .402 | .003 0741 | 162 | .462 | .004 1315 | 192 | .72 | .011 3336 | 3980 |
| .403 | .003 0903 | 161 | .463 | .004 1507 | 193 | .73 | .011 7316 | 4103 |
| .404 | .003 1064 | 163 | .464 | .004 1700 | 193 | .74 | .012 1419 | 4233 |
| 0.405 | 0.003 1227 | 162 | 0.465 | 0.004 1893 | 194 | 0.75 | 0.012 5652 | 4370 |
| .406 | .003 1389 | 164 | .466 | .004 2087 | 194 | .76 | .013 0022 | 4514 |
| .407 | .003 1553 | 163 | .467 | .004 2281 | 195 | .77 | .013 4536 | 4666 |
| .408 | .003 1716 | 165 | .468 | .004 2476 | 196 | .78 | .013 9202 | 4829 |
| .409 | .003 1881 | 164 | .469 | .004 2672 | 196 | .79 | .014 4031 | 5002 |
| 0.410 | 0.003 2045 | 166 | 0.470 | 0.004 2868 | 196 | 0.80 | 0.014 9033 | 5186 |
| .411 | .003 2211 | 165 | .471 | .004 3064 | 197 | .81 | .015 4219 | 5384 |
| .412 | .003 2376 | 167 | .472 | .004 3261 | 198 | .82 | .015 9603 | 5599 |
| .413 | .003 2543 | 166 | .473 | .004 3459 | 198 | .83 | .016 5202 | 5831 |
| .414 | .003 2709 | 168 | .474 | .004 3657 | 199 | .84 | .017 1033 | 6087 |
| 0.415 | 0.003 2877 | 167 | 0.475 | 0.004 3856 | 199 | 0.85 | 0.017 7120 | 6366 |
| .416 | .003 3044 | 169 | .476 | .004 4055 | 200 | .86 | .018 3486 | 6679 |
| .417 | .003 3213 | 168 | .477 | .004 4255 | 201 | .87 | .019 0165 | 7030 |
| .418 | .003 3381 | 169 | .478 | .004 4456 | 201 | .88 | .019 7195 | 7434 |
| .419 | .003 3550 | 170 | .479 | .004 4657 | 201 | .89 | .020 4629 | 7900 |
| 0.420 | 0.003 3720 | | 0.480 | 0.004 4858 | | 0.90 | 0.021 2529 | |

TABLE XII.

| ζ | $\log m_1$ | $\log m_2$ | z_1' | | z_2' | | z_3' | | z_4' | |
|----------|------------|------------|--------|--------|--------|--------|--------|--------|--------|--------|
| | | | m_1 | m_2 | m_2 | m_1 | m_1 | m_2 | m_2 | m_1 |
| 0 0 | ∞ | 0.0000 | 0 0 | 90 0 | 90 0 | 180 0 | 180 0 | 180 0 | 0 0 | 0 0 |
| 1 | 4.2976 | 9.9999 | 2 23 | 90 20 | 90 20 | 178 40 | 178 40 | 179 0 | 359 0 | 359 5 |
| 2 | 3.3950 | 9.9996 | 4 46 | 90 40 | 90 40 | 177 20 | 177 20 | 178 0 | 358 0 | 358 9 |
| 3 | 2.8675 | 9.9992 | 7 8 | 91 0 | 91 0 | 176 0 | 176 0 | 177 0 | 357 0 | 357 14 |
| 4 | 2.4938 | 9.9986 | 9 32 | 91 20 | 91 20 | 174 40 | 174 40 | 176 0 | 356 0 | 356 18 |
| 5 | 2.2044 | 9.9978 | 11 55 | 91 41 | 91 41 | 173 19 | 173 19 | 175 0 | 355 0 | 355 23 |
| 6 | 1.9686 | 9.9968 | 14 19 | 92 1 | 92 1 | 171 59 | 171 59 | 174 0 | 354 0 | 354 28 |
| 7 | 1.7698 | 9.9957 | 16 42 | 92 22 | 92 22 | 170 38 | 170 38 | 172 59 | 353 1 | 353 32 |
| 8 | 1.5981 | 9.9943 | 19 7 | 92 42 | 92 42 | 169 18 | 169 18 | 171 59 | 352 1 | 352 37 |
| 9 | 1.4473 | 9.9928 | 21 32 | 93 3 | 93 3 | 167 57 | 167 57 | 170 58 | 351 2 | 351 42 |
| 10 | 1.3130 | 9.9911 | 23 57 | 93 25 | 93 25 | 166 35 | 166 35 | 169 57 | 350 3 | 350 47 |
| 11 | 1.1922 | 9.9892 | 26 23 | 93 46 | 93 46 | 165 14 | 165 14 | 168 55 | 349 4 | 349 52 |
| 12 | 1.0824 | 9.9871 | 28 50 | 94 8 | 94 8 | 163 52 | 163 52 | 167 54 | 348 6 | 348 56 |
| 13 | 0.9821 | 9.9848 | 31 17 | 94 31 | 94 31 | 162 29 | 162 29 | 166 51 | 347 8 | 348 1 |
| 14 | 0.8898 | 9.9823 | 33 46 | 94 53 | 94 53 | 161 7 | 161 7 | 165 48 | 346 11 | 347 6 |
| 15 | 0.8045 | 9.9796 | 36 15 | 95 17 | 95 17 | 159 43 | 159 43 | 164 44 | 345 14 | 346 11 |
| 16 | 0.7254 | 9.9767 | 38 46 | 95 40 | 95 40 | 158 20 | 158 20 | 163 40 | 344 17 | 345 16 |
| 17 | 0.6518 | 9.9736 | 41 18 | 96 5 | 96 5 | 156 55 | 156 55 | 162 34 | 343 21 | 344 21 |
| 18 | 0.5830 | 9.9702 | 43 51 | 96 30 | 96 30 | 155 30 | 155 30 | 161 27 | 342 27 | 343 27 |
| 19 | 0.5185 | 9.9667 | 46 26 | 96 56 | 96 56 | 154 4 | 154 4 | 160 19 | 341 32 | 342 32 |
| 20 | 0.4581 | 9.9629 | 49 2 | 97 23 | 97 23 | 152 37 | 152 37 | 159 9 | 340 38 | 341 37 |
| 21 | 0.4013 | 9.9588 | 51 41 | 97 50 | 97 50 | 151 10 | 151 10 | 157 58 | 339 45 | 340 43 |
| 22 | 0.3479 | 9.9545 | 54 22 | 98 19 | 98 19 | 149 41 | 149 41 | 156 45 | 338 53 | 339 49 |
| 23 | 0.2976 | 9.9499 | 57 5 | 98 49 | 98 49 | 148 11 | 148 11 | 155 29 | 338 0 | 338 54 |
| 24 | 0.2501 | 9.9451 | 59 51 | 99 20 | 99 20 | 146 40 | 146 40 | 154 11 | 337 9 | 338 0 |
| 25 | 0.2053 | 9.9400 | 62 40 | 99 53 | 99 53 | 145 7 | 145 7 | 152 50 | 336 19 | 337 6 |
| 26 | 0.1631 | 9.9345 | 65 33 | 100 28 | 100 28 | 143 32 | 143 32 | 151 25 | 335 28 | 336 13 |
| 27 | 0.1232 | 9.9287 | 68 30 | 101 5 | 101 5 | 141 55 | 141 55 | 149 55 | 334 38 | 335 19 |
| 28 | 0.0857 | 9.9226 | 71 33 | 101 45 | 101 45 | 140 15 | 140 15 | 148 22 | 333 49 | 334 25 |
| 29 | 0.0503 | 9.9161 | 74 41 | 102 27 | 102 27 | 138 33 | 138 33 | 146 42 | 333 1 | 333 32 |
| 30 | 0.0170 | 9.9092 | 77 58 | 103 13 | 103 13 | 136 46 | 136 46 | 144 55 | 332 12 | 332 39 |
| 31 | 9.9857 | 9.9019 | 81 23 | 104 4 | 104 4 | 134 56 | 134 56 | 142 59 | 331 24 | 331 46 |
| 32 | 9.9565 | 9.8940 | 85 0 | 105 1 | 105 1 | 132 59 | 132 59 | 140 51 | 330 37 | 330 54 |
| 33 | 9.9292 | 9.8856 | 88 54 | 106 6 | 106 6 | 130 54 | 130 54 | 138 27 | 329 49 | 330 2 |
| 34 | 9.9040 | 9.8765 | 93 11 | 107 22 | 107 22 | 128 38 | 128 38 | 135 39 | 329 2 | 329 10 |
| 35 | 9.8808 | 9.8665 | 98 7 | 108 58 | 108 58 | 126 2 | 126 2 | 132 13 | 328 14 | 328 19 |
| 36 | 9.8600 | 9.8555 | 104 20 | 111 13 | 111 13 | 122 47 | 122 47 | 127 29 | 327 27 | 327 28 |
| -36 52.2 | 9.8443 | 9.8443 | 116 34 | 116 34 | 116 34 | 116 34 | 116 34 | 116 34 | 326 45 | 326 45 |

This table exhibits the limits of the roots of the equation

$$\sin(\zeta' - \zeta) = m_0 \sin^4 \zeta',$$

when there are four real roots. The quantities m_1 and m_2 are the limiting values of m_0 , and the values of z_1' , z_2' , z_3' , and z_4' , corresponding to each of these, give the limits of the four real roots of the equation.

TABLE XII.

| ζ | $\log m_1$ | $\log m_2$ | z_1' | | z_2' | | z_3' | | z_4' | |
|----------|------------|------------|--------|-------|--------|-------|--------|--------|--------|--------|
| | | | m_2 | m_1 | m_1 | m_2 | m_2 | m_1 | m_1 | m_2 |
| 0 0 | ∞ | 0.0000 | 0 0 | 0 0 | 0 0 | 90 0 | 90 0 | 180 0 | 180 0 | 180 0 |
| 1 | 4.2976 | 9.9999 | 1 0 | 1 20 | 1 20 | 89 40 | 89 40 | 177 37 | 180 55 | 181 0 |
| 2 | 3.3950 | 9.9996 | 2 0 | 2 40 | 2 40 | 89 20 | 89 20 | 175 14 | 181 51 | 182 0 |
| 3 | 2.8675 | 9.9992 | 3 0 | 4 0 | 4 0 | 89 0 | 89 0 | 172 52 | 182 46 | 183 0 |
| 4 | 2.4938 | 9.9986 | 4 0 | 5 20 | 5 20 | 88 40 | 88 40 | 170 28 | 183 42 | 184 0 |
| 5 | 2.2044 | 9.9978 | 5 0 | 6 41 | 6 41 | 88 19 | 88 19 | 168 5 | 184 37 | 185 0 |
| 6 | 1.9686 | 9.9968 | 6 0 | 8 1 | 8 1 | 87 59 | 87 59 | 165 41 | 185 32 | 186 0 |
| 7 | 1.7698 | 9.9957 | 7 1 | 9 22 | 9 22 | 87 38 | 87 38 | 163 18 | 186 28 | 186 59 |
| 8 | 1.5981 | 9.9943 | 8 1 | 10 42 | 10 42 | 87 18 | 87 18 | 160 53 | 187 23 | 187 59 |
| 9 | 1.4473 | 9.9928 | 9 2 | 12 3 | 12 3 | 86 57 | 86 57 | 158 28 | 188 18 | 188 58 |
| 10 | 1.3130 | 9.9911 | 10 3 | 13 25 | 13 25 | 86 35 | 86 35 | 156 3 | 189 13 | 189 57 |
| 11 | 1.1922 | 9.9892 | 11 5 | 14 46 | 14 46 | 86 14 | 86 14 | 153 37 | 190 8 | 190 56 |
| 12 | 1.0824 | 9.9871 | 12 6 | 16 8 | 16 8 | 85 52 | 85 52 | 151 10 | 191 4 | 191 54 |
| 13 | 0.9821 | 9.9848 | 13 9 | 17 31 | 17 31 | 85 29 | 85 29 | 148 43 | 191 59 | 192 52 |
| 14 | 0.8898 | 9.9823 | 14 12 | 18 53 | 18 53 | 85 7 | 85 7 | 146 14 | 192 54 | 193 49 |
| 15 | 0.8045 | 9.9796 | 15 16 | 20 17 | 20 17 | 84 43 | 84 43 | 143 45 | 193 49 | 194 46 |
| 16 | 0.7254 | 9.9767 | 16 20 | 21 40 | 21 40 | 84 20 | 84 20 | 141 14 | 194 44 | 195 43 |
| 17 | 0.6518 | 9.9736 | 17 26 | 23 5 | 23 5 | 83 55 | 83 55 | 138 42 | 195 39 | 196 39 |
| 18 | 0.5830 | 9.9702 | 18 33 | 24 30 | 24 30 | 83 30 | 83 30 | 136 9 | 196 33 | 197 33 |
| 19 | 0.5185 | 9.9667 | 19 41 | 25 56 | 25 56 | 83 4 | 83 4 | 133 34 | 197 28 | 198 28 |
| 20 | 0.4581 | 9.9629 | 20 51 | 27 23 | 27 23 | 82 37 | 82 37 | 130 58 | 198 23 | 199 22 |
| 21 | 0.4013 | 9.9588 | 22 2 | 28 50 | 28 50 | 82 10 | 82 10 | 128 19 | 199 17 | 200 15 |
| 22 | 0.3479 | 9.9545 | 23 15 | 30 19 | 30 19 | 81 41 | 81 41 | 125 38 | 200 11 | 201 07 |
| 23 | 0.2976 | 9.9499 | 24 31 | 31 49 | 31 49 | 81 11 | 81 11 | 122 55 | 201 6 | 202 0 |
| 24 | 0.2501 | 9.9451 | 25 49 | 33 20 | 33 20 | 80 40 | 80 40 | 120 9 | 202 0 | 202 51 |
| 25 | 0.2053 | 9.9400 | 27 10 | 34 53 | 34 53 | 80 7 | 80 7 | 117 20 | 202 54 | 203 41 |
| 26 | 0.1631 | 9.9345 | 28 35 | 36 28 | 36 28 | 79 32 | 79 32 | 114 27 | 203 47 | 204 32 |
| 27 | 0.1232 | 9.9287 | 30 4 | 38 5 | 38 5 | 78 55 | 78 55 | 111 30 | 204 41 | 205 22 |
| 28 | 0.0857 | 9.9226 | 31 38 | 39 45 | 39 45 | 78 15 | 78 15 | 108 27 | 205 35 | 206 11 |
| 29 | 0.0503 | 9.9161 | 33 18 | 41 27 | 41 27 | 77 33 | 77 33 | 105 19 | 206 28 | 206 59 |
| 30 | 0.0170 | 9.9092 | 35 5 | 43 13 | 43 13 | 76 47 | 76 47 | 102 3 | 207 21 | 207 48 |
| 31 | 9.9857 | 9.9019 | 37 1 | 45 4 | 45 4 | 75 56 | 75 56 | 98 37 | 208 14 | 208 36 |
| 32 | 9.9565 | 9.8940 | 39 9 | 47 1 | 47 1 | 74 59 | 74 59 | 95 0 | 209 06 | 209 23 |
| 33 | 9.9292 | 9.8856 | 41 33 | 49 6 | 49 6 | 73 54 | 73 54 | 91 6 | 209 58 | 210 11 |
| 34 | 9.9040 | 9.8765 | 44 21 | 51 22 | 51 22 | 72 38 | 72 38 | 86 49 | 210 50 | 210 58 |
| 35 | 9.8808 | 9.8665 | 47 47 | 53 58 | 53 58 | 71 2 | 71 2 | 81 53 | 211 41 | 211 46 |
| 36 | 9.8600 | 9.8555 | 52 31 | 57 13 | 57 13 | 68 47 | 68 47 | 75 40 | 212 32 | 212 33 |
| +36 52.2 | 9.8443 | 9.8443 | 63 26 | 63 26 | 63 26 | 63 26 | 63 26 | 63 26 | 213 15 | 213 15 |

This table exhibits the limits of the roots of the equation

$$\sin(\zeta' - \zeta) = m_0 \sin^4 \zeta',$$

when there are four real roots. The quantities m_1 and m_2 are the limiting values of m_0 , and the values of z_1' , z_2' , z_3' , and z_4' , corresponding to each of these, give the limits of the four real roots of the equation.

TABLE XIII.

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.0000 | 0.000 0000 | 965 | 0.0060 | 0.005 7298 | 945 | 0.0120 | 0.011 3417 | 926 |
| .0001 | .000 0965 | 965 | .0061 | .005 8243 | 944 | .0121 | .011 4343 | 925 |
| .0002 | .000 1930 | 964 | .0062 | .005 9187 | 944 | .0122 | .011 5268 | 925 |
| .0003 | .000 2894 | 964 | .0063 | .006 0131 | 944 | .0123 | .011 6193 | 925 |
| .0004 | .000 3858 | 963 | .0064 | .006 1075 | 944 | .0124 | .011 7118 | 925 |
| 0.0005 | 0.000 4821 | 963 | 0.0065 | 0.006 2019 | 943 | 0.0125 | 0.011 8043 | 924 |
| .0006 | .000 5784 | 963 | .0066 | .006 2962 | 943 | .0126 | .011 8967 | 923 |
| .0007 | .000 6747 | 963 | .0067 | .006 3905 | 942 | .0127 | .011 9890 | 924 |
| .0008 | .000 7710 | 962 | .0068 | .006 4847 | 943 | .0128 | .012 0814 | 923 |
| .0009 | .000 8672 | 962 | .0069 | .006 5790 | 942 | .0129 | .012 1737 | 923 |
| 0.0010 | 0.000 9634 | 961 | 0.0070 | 0.006 6732 | 941 | 0.0130 | 0.012 2660 | 923 |
| .0011 | .001 0595 | 961 | .0071 | .006 7673 | 941 | .0131 | .012 3583 | 922 |
| .0012 | .001 1556 | 961 | .0072 | .006 8614 | 941 | .0132 | .012 4505 | 922 |
| .0013 | .001 2517 | 961 | .0073 | .006 9555 | 941 | .0133 | .012 5427 | 921 |
| .0014 | .001 3478 | 960 | .0074 | .007 0496 | 940 | .0134 | .012 6348 | 921 |
| 0.0015 | 0.001 4438 | 960 | 0.0075 | 0.007 1436 | 940 | 0.0135 | 0.012 7269 | 921 |
| .0016 | .001 5398 | 959 | .0076 | .007 2376 | 940 | .0136 | .012 8190 | 921 |
| .0017 | .001 6357 | 959 | .0077 | .007 3316 | 939 | .0137 | .012 9111 | 921 |
| .0018 | .001 7316 | 959 | .0078 | .007 4255 | 939 | .0138 | .013 0032 | 920 |
| .0019 | .001 8275 | 959 | .0079 | .007 5194 | 939 | .0139 | .013 0952 | 919 |
| 0.0020 | 0.001 9234 | 958 | 0.0080 | 0.007 6133 | 938 | 0.0140 | 0.013 1871 | 920 |
| .0021 | .002 0192 | 958 | .0081 | .007 7071 | 938 | .0141 | .013 2791 | 919 |
| .0022 | .002 1150 | 957 | .0082 | .007 8009 | 938 | .0142 | .013 3710 | 919 |
| .0023 | .002 2107 | 957 | .0083 | .007 8947 | 937 | .0143 | .013 4629 | 918 |
| .0024 | .002 3064 | 957 | .0084 | .007 9884 | 937 | .0144 | .013 5547 | 918 |
| 0.0025 | 0.002 4021 | 956 | 0.0085 | 0.008 0821 | 937 | 0.0145 | 0.013 6465 | 918 |
| .0026 | .002 4977 | 956 | .0086 | .008 1758 | 936 | .0146 | .013 7383 | 918 |
| .0027 | .002 5933 | 956 | .0087 | .008 2694 | 936 | .0147 | .013 8301 | 917 |
| .0028 | .002 6889 | 956 | .0088 | .008 3630 | 936 | .0148 | .013 9218 | 917 |
| .0029 | .002 7845 | 955 | .0089 | .008 4566 | 936 | .0149 | .014 0135 | 917 |
| 0.0030 | 0.002 8800 | 955 | 0.0090 | 0.008 5502 | 935 | 0.0150 | 0.014 1052 | 916 |
| .0031 | .002 9755 | 954 | .0091 | .008 6437 | 935 | .0151 | .014 1968 | 916 |
| .0032 | .003 0709 | 954 | .0092 | .008 7372 | 934 | .0152 | .014 2884 | 916 |
| .0033 | .003 1663 | 954 | .0093 | .008 8306 | 934 | .0153 | .014 3800 | 916 |
| .0034 | .003 2617 | 953 | .0094 | .008 9240 | 934 | .0154 | .014 4716 | 915 |
| 0.0035 | 0.003 3570 | 953 | 0.0095 | 0.009 0174 | 934 | 0.0155 | 0.014 5631 | 915 |
| .0036 | .003 4523 | 953 | .0096 | .009 1108 | 933 | .0156 | .014 6546 | 914 |
| .0037 | .003 5476 | 952 | .0097 | .009 2041 | 933 | .0157 | .014 7460 | 914 |
| .0038 | .003 6428 | 952 | .0098 | .009 2974 | 932 | .0158 | .014 8374 | 914 |
| .0039 | .003 7380 | 952 | .0099 | .009 3906 | 932 | .0159 | .014 9288 | 914 |
| 0.0040 | 0.003 8332 | 952 | 0.0100 | 0.009 4838 | 932 | 0.0160 | 0.015 0202 | 913 |
| .0041 | .003 9284 | 951 | .0101 | .009 5770 | 932 | .0161 | .015 1115 | 913 |
| .0042 | .004 0235 | 951 | .0102 | .009 6702 | 931 | .0162 | .015 2028 | 913 |
| .0043 | .004 1186 | 950 | .0103 | .009 7633 | 931 | .0163 | .015 2941 | 913 |
| .0044 | .004 2136 | 950 | .0104 | .009 8564 | 931 | .0164 | .015 3854 | 912 |
| 0.0045 | 0.004 3086 | 950 | 0.0105 | 0.009 9495 | 930 | 0.0165 | 0.015 4766 | 912 |
| .0046 | .004 4036 | 949 | .0106 | .010 0425 | 930 | .0166 | .015 5678 | 911 |
| .0047 | .004 4985 | 949 | .0107 | .010 1355 | 930 | .0167 | .015 6589 | 911 |
| .0048 | .004 5934 | 949 | .0108 | .010 2285 | 930 | .0168 | .015 7500 | 911 |
| .0049 | .004 6883 | 949 | .0109 | .010 3215 | 929 | .0169 | .015 8411 | 911 |
| 0.0050 | 0.004 7832 | 948 | 0.0110 | 0.010 4144 | 929 | 0.0170 | 0.015 9322 | 910 |
| .0051 | .004 8780 | 948 | .0111 | .010 5073 | 928 | .0171 | .016 0232 | 910 |
| .0052 | .004 9728 | 947 | .0112 | .010 6001 | 928 | .0172 | .016 1142 | 910 |
| .0053 | .005 0675 | 947 | .0113 | .010 6929 | 928 | .0173 | .016 2052 | 909 |
| .0054 | .005 1622 | 947 | .0114 | .010 7857 | 928 | .0174 | .016 2961 | 909 |
| 0.0055 | 0.005 2569 | 946 | 0.0115 | 0.010 8785 | 927 | 0.0175 | 0.016 3870 | 909 |
| .0056 | .005 3515 | 946 | .0116 | .010 9712 | 927 | .0176 | .016 4779 | 909 |
| .0057 | .005 4461 | 946 | .0117 | .011 0639 | 926 | .0177 | .016 5688 | 908 |
| .0058 | .005 5407 | 946 | .0118 | .011 1565 | 926 | .0178 | .016 6596 | 908 |
| .0059 | .005 6353 | 945 | .0119 | .011 2491 | 926 | .0179 | .016 7504 | 908 |
| 0.0060 | 0.005 7298 | | 0.0120 | 0.011 3417 | | 0.0180 | 0.016 8412 | |

TABLE XII

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.0180 | 0.016 8412 | | 0.0240 | 0.022 2330 | | 0.0300 | 0.027 5218 | |
| .0181 | .016 9319 | 907 | .0241 | .022 3220 | 890 | .0301 | .027 6091 | 873 |
| .0182 | .017 0226 | 907 | .0242 | .022 4109 | 889 | .0302 | .027 6964 | 873 |
| .0183 | .017 1133 | 907 | .0243 | .022 4998 | 889 | .0303 | .027 7836 | 872 |
| .0184 | .017 2039 | 906 | .0244 | .022 5887 | 889 | .0304 | .027 8708 | 872 |
| | | 906 | | | 889 | | | 872 |
| 0.0185 | 0.017 2945 | | 0.0245 | 0.022 6776 | | 0.0305 | 0.027 9580 | |
| .0186 | .017 3851 | 906 | .0246 | .022 7664 | 888 | .0306 | .028 0452 | 872 |
| .0187 | .017 4757 | 906 | .0247 | .022 8552 | 888 | .0307 | .028 1323 | 871 |
| .0188 | .017 5662 | 905 | .0248 | .022 9440 | 888 | .0308 | .028 2194 | 871 |
| .0189 | .017 6567 | 905 | .0249 | .023 0328 | 887 | .0309 | .028 3065 | 871 |
| | | 904 | | | 887 | | | 871 |
| 0.0190 | 0.017 7471 | | 0.0250 | 0.023 1215 | | 0.0310 | 0.028 3936 | |
| .0191 | .017 8376 | 905 | .0251 | .023 2102 | 887 | .0311 | .028 4806 | 870 |
| .0192 | .017 9280 | 904 | .0252 | .023 2988 | 886 | .0312 | .028 5676 | 870 |
| .0193 | .018 0183 | 903 | .0253 | .023 3875 | 887 | .0313 | .028 6546 | 870 |
| .0194 | .018 1087 | 904 | .0254 | .023 4761 | 886 | .0314 | .028 7415 | 869 |
| | | 903 | | | 886 | | | 869 |
| 0.0195 | 0.018 1990 | | 0.0255 | 0.023 5647 | | 0.0315 | 0.028 8284 | |
| .0196 | .018 2893 | 903 | .0256 | .023 6532 | 885 | .0316 | .028 9153 | 869 |
| .0197 | .018 3796 | 903 | .0257 | .023 7417 | 885 | .0317 | .029 0022 | 869 |
| .0198 | .018 4698 | 902 | .0258 | .023 8302 | 885 | .0318 | .029 0890 | 868 |
| .0199 | .018 5600 | 902 | .0259 | .023 9187 | 884 | .0319 | .029 1758 | 868 |
| | | 901 | | | 884 | | | 868 |
| 0.0200 | 0.018 6501 | | 0.0260 | 0.024 0071 | | 0.0320 | 0.029 2626 | |
| .0201 | .018 7403 | 902 | .0261 | .024 0956 | 885 | .0321 | .029 3494 | 868 |
| .0202 | .018 8304 | 901 | .0262 | .024 1839 | 883 | .0322 | .029 4361 | 867 |
| .0203 | .018 9205 | 901 | .0263 | .024 2723 | 884 | .0323 | .029 5228 | 867 |
| .0204 | .019 0105 | 900 | .0264 | .024 3606 | 883 | .0324 | .029 6095 | 866 |
| | | 900 | | | 883 | | | 866 |
| 0.0205 | 0.019 1005 | | 0.0265 | 0.024 4489 | | 0.0325 | 0.029 6961 | |
| .0206 | .019 1905 | 900 | .0266 | .024 5372 | 883 | .0326 | .029 7827 | 866 |
| .0207 | .019 2805 | 900 | .0267 | .024 6254 | 882 | .0327 | .029 8693 | 866 |
| .0208 | .019 3704 | 899 | .0268 | .024 7136 | 882 | .0328 | .029 9559 | 866 |
| .0209 | .019 4603 | 899 | .0269 | .024 8018 | 882 | .0329 | .030 0424 | 865 |
| | | 899 | | | 882 | | | 866 |
| 0.0210 | 0.019 5502 | | 0.0270 | 0.024 8900 | | 0.0330 | 0.030 1290 | |
| .0211 | .019 6401 | 899 | .0271 | .024 9781 | 881 | .0331 | .030 2154 | 864 |
| .0212 | .019 7299 | 898 | .0272 | .025 0662 | 881 | .0332 | .030 3019 | 865 |
| .0213 | .019 8197 | 898 | .0273 | .025 1543 | 881 | .0333 | .030 3883 | 864 |
| .0214 | .019 9094 | 897 | .0274 | .025 2423 | 880 | .0334 | .030 4747 | 864 |
| | | 898 | | | 880 | | | 864 |
| 0.0215 | 0.019 9992 | | 0.0275 | 0.025 3303 | | 0.0335 | 0.030 5611 | |
| .0216 | .020 0889 | 897 | .0276 | .025 4183 | 880 | .0336 | .030 6475 | 864 |
| .0217 | .020 1785 | 896 | .0277 | .025 5063 | 880 | .0337 | .030 7338 | 863 |
| .0218 | .020 2682 | 897 | .0278 | .025 5942 | 879 | .0338 | .030 8201 | 863 |
| .0219 | .020 3578 | 896 | .0279 | .025 6821 | 879 | .0339 | .030 9064 | 862 |
| | | 896 | | | 879 | | | 862 |
| 0.0220 | 0.020 4474 | | 0.0280 | 0.025 7700 | | 0.0340 | 0.030 9926 | |
| .0221 | .020 5369 | 895 | .0281 | .025 8579 | 879 | .0341 | .031 0788 | 862 |
| .0222 | .020 6264 | 895 | .0282 | .025 9457 | 878 | .0342 | .031 1650 | 862 |
| .0223 | .020 7159 | 895 | .0283 | .026 0335 | 878 | .0343 | .031 2512 | 861 |
| .0224 | .020 8054 | 894 | .0284 | .026 1213 | 877 | .0344 | .031 3373 | 861 |
| | | 894 | | | 877 | | | 861 |
| 0.0225 | 0.020 8948 | | 0.0285 | 0.026 2090 | | 0.0345 | 0.031 4234 | |
| .0226 | .020 9842 | 894 | .0286 | .026 2967 | 877 | .0346 | .031 5095 | 861 |
| .0227 | .021 0736 | 894 | .0287 | .026 3844 | 877 | .0347 | .031 5956 | 860 |
| .0228 | .021 1630 | 894 | .0288 | .026 4721 | 876 | .0348 | .031 6816 | 860 |
| .0229 | .021 2523 | 893 | .0289 | .026 5597 | 876 | .0349 | .031 7676 | 860 |
| | | 893 | | | 876 | | | 860 |
| 0.0230 | 0.021 3416 | | 0.0290 | 0.026 6473 | | 0.0350 | 0.031 8536 | |
| .0231 | .021 4309 | 893 | .0291 | .026 7349 | 876 | .0351 | .031 9396 | 859 |
| .0232 | .021 5201 | 892 | .0292 | .026 8224 | 875 | .0352 | .032 0255 | 859 |
| .0233 | .021 6093 | 892 | .0293 | .026 9099 | 875 | .0353 | .032 1114 | 859 |
| .0234 | .021 6985 | 891 | .0294 | .026 9974 | 875 | .0354 | .032 1973 | 858 |
| | | 891 | | | 875 | | | 858 |
| 0.0235 | 0.021 7876 | | 0.0295 | 0.027 0849 | | 0.0355 | 0.032 2831 | |
| .0236 | .021 8768 | 892 | .0296 | .027 1723 | 874 | .0356 | .032 3689 | 858 |
| .0237 | .021 9659 | 891 | .0297 | .027 2597 | 874 | .0357 | .032 4547 | 858 |
| .0238 | .022 0549 | 890 | .0298 | .027 3471 | 874 | .0358 | .032 5405 | 857 |
| .0239 | .022 1440 | 891 | .0299 | .027 4345 | 873 | .0359 | .032 6262 | 857 |
| | | 890 | | | 873 | | | 858 |
| 0.0240 | 0.022 2330 | | 0.0300 | 0.027 5218 | | 0.0360 | 0.032 7120 | |

TABLE XIII.

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.0360 | 0.032 7120 | 856 | 0.060 | 0.052 5626 | 7976 | 0.120 | 0.096 8849 | 6843 |
| 0.0361 | 0.032 7976 | 857 | 0.061 | 0.053 3602 | 7954 | 0.121 | 0.097 5692 | 6828 |
| 0.0362 | 0.032 8833 | 856 | 0.062 | 0.054 1556 | 7932 | 0.122 | 0.098 2520 | 6811 |
| 0.0363 | 0.032 9689 | 857 | 0.063 | 0.054 9488 | 7909 | 0.123 | 0.098 9331 | 6796 |
| 0.0364 | 0.033 0546 | 855 | 0.064 | 0.055 7397 | 7888 | 0.124 | 0.099 6127 | 6780 |
| 0.0365 | 0.033 1401 | 856 | 0.065 | 0.056 5285 | 7865 | 0.125 | 0.100 2907 | 6765 |
| 0.0366 | 0.033 2257 | 855 | 0.066 | 0.057 3150 | 7844 | 0.126 | 0.100 9672 | 6749 |
| 0.0367 | 0.033 3112 | 855 | 0.067 | 0.058 0994 | 7823 | 0.127 | 0.101 6421 | 6733 |
| 0.0368 | 0.033 3967 | 855 | 0.068 | 0.058 8817 | 7801 | 0.128 | 0.102 3154 | 6719 |
| 0.0369 | 0.033 4822 | 855 | 0.069 | 0.059 6618 | 7780 | 0.129 | 0.102 9873 | 6703 |
| 0.0370 | 0.033 5677 | 854 | 0.070 | 0.060 4398 | 7759 | 0.130 | 0.103 6576 | 6688 |
| 0.0371 | 0.033 6531 | 854 | 0.071 | 0.061 2157 | 7738 | 0.131 | 0.104 3264 | 6672 |
| 0.0372 | 0.033 7385 | 854 | 0.072 | 0.061 9895 | 7717 | 0.132 | 0.104 9936 | 6658 |
| 0.0373 | 0.033 8239 | 853 | 0.073 | 0.062 7612 | 7696 | 0.133 | 0.105 6594 | 6643 |
| 0.0374 | 0.033 9092 | 854 | 0.074 | 0.063 5308 | 7676 | 0.134 | 0.106 3237 | 6628 |
| 0.0375 | 0.033 9946 | 853 | 0.075 | 0.064 2984 | 7655 | 0.135 | 0.106 9865 | 6613 |
| 0.0376 | 0.034 0799 | 852 | 0.076 | 0.065 0639 | 7635 | 0.136 | 0.107 6478 | 6598 |
| 0.0377 | 0.034 1651 | 853 | 0.077 | 0.065 8274 | 7614 | 0.137 | 0.108 3076 | 6584 |
| 0.0378 | 0.034 2504 | 852 | 0.078 | 0.066 5888 | 7594 | 0.138 | 0.108 9660 | 6569 |
| 0.0379 | 0.034 3356 | 852 | 0.079 | 0.067 3482 | 7575 | 0.139 | 0.109 6229 | 6554 |
| 0.0380 | 0.034 4208 | 851 | 0.080 | 0.068 1057 | 7555 | 0.140 | 0.110 2783 | 6540 |
| 0.0381 | 0.034 5059 | 852 | 0.081 | 0.068 8612 | 7534 | 0.141 | 0.110 9323 | 6526 |
| 0.0382 | 0.034 5911 | 851 | 0.082 | 0.069 6146 | 7515 | 0.142 | 0.111 5849 | 6511 |
| 0.0383 | 0.034 6762 | 851 | 0.083 | 0.070 3661 | 7496 | 0.143 | 0.112 2360 | 6497 |
| 0.0384 | 0.034 7613 | 851 | 0.084 | 0.071 1157 | 7476 | 0.144 | 0.112 8857 | 6483 |
| 0.0385 | 0.034 8464 | 850 | 0.085 | 0.071 8633 | 7457 | 0.145 | 0.113 5340 | 6469 |
| 0.0386 | 0.034 9314 | 850 | 0.086 | 0.072 6090 | 7437 | 0.146 | 0.114 1809 | 6455 |
| 0.0387 | 0.035 0164 | 850 | 0.087 | 0.073 3527 | 7418 | 0.147 | 0.114 8264 | 6440 |
| 0.0388 | 0.035 1014 | 850 | 0.088 | 0.074 0945 | 7400 | 0.148 | 0.115 4704 | 6427 |
| 0.0389 | 0.035 1864 | 849 | 0.089 | 0.074 8345 | 7380 | 0.149 | 0.116 1131 | 6413 |
| 0.0390 | 0.035 2713 | 849 | 0.090 | 0.075 5725 | 7362 | 0.150 | 0.116 7544 | 6399 |
| 0.0391 | 0.035 3562 | 849 | 0.091 | 0.076 3087 | 7343 | 0.151 | 0.117 3943 | 6386 |
| 0.0392 | 0.035 4411 | 848 | 0.092 | 0.077 0430 | 7324 | 0.152 | 0.118 0329 | 6372 |
| 0.0393 | 0.035 5259 | 849 | 0.093 | 0.077 7754 | 7306 | 0.153 | 0.118 6701 | 6358 |
| 0.0394 | 0.035 6108 | 848 | 0.094 | 0.078 5060 | 7288 | 0.154 | 0.119 3059 | 6345 |
| 0.0395 | 0.035 6956 | 848 | 0.095 | 0.079 2348 | 7269 | 0.155 | 0.119 9404 | 6331 |
| 0.0396 | 0.035 7804 | 847 | 0.096 | 0.079 9617 | 7251 | 0.156 | 0.120 5735 | 6318 |
| 0.0397 | 0.035 8651 | 848 | 0.097 | 0.080 6868 | 7233 | 0.157 | 0.121 2053 | 6304 |
| 0.0398 | 0.035 9499 | 847 | 0.098 | 0.081 4101 | 7215 | 0.158 | 0.121 8357 | 6292 |
| 0.0399 | 0.036 0346 | 846 | 0.099 | 0.082 1316 | 7197 | 0.159 | 0.122 4649 | 6278 |
| 0.040 | 0.036 1192 | 8454 | 0.100 | 0.082 8513 | 7180 | 0.160 | 0.123 0927 | 6265 |
| 0.041 | 0.036 2036 | 8429 | 0.101 | 0.083 5693 | 7161 | 0.161 | 0.123 7192 | 6252 |
| 0.042 | 0.037 8075 | 8403 | 0.102 | 0.084 2854 | 7145 | 0.162 | 0.124 3444 | 6238 |
| 0.043 | 0.038 6478 | 8378 | 0.103 | 0.084 9999 | 7126 | 0.163 | 0.124 9682 | 6226 |
| 0.044 | 0.039 4856 | 8353 | 0.104 | 0.085 7125 | 7110 | 0.164 | 0.125 5908 | 6213 |
| 0.045 | 0.040 3209 | 8328 | 0.105 | 0.086 4235 | 7092 | 0.165 | 0.126 2121 | 6200 |
| 0.046 | 0.041 1537 | 8304 | 0.106 | 0.087 1327 | 7074 | 0.166 | 0.126 8321 | 6187 |
| 0.047 | 0.041 9841 | 8280 | 0.107 | 0.087 8401 | 7058 | 0.167 | 0.127 4508 | 6175 |
| 0.048 | 0.042 8121 | 8255 | 0.108 | 0.088 5459 | 7041 | 0.168 | 0.128 0683 | 6162 |
| 0.049 | 0.043 6376 | 8231 | 0.109 | 0.089 2500 | 7023 | 0.169 | 0.128 6845 | 6149 |
| 0.050 | 0.044 4607 | 8207 | 0.110 | 0.089 9523 | 7007 | 0.170 | 0.129 2994 | 6137 |
| 0.051 | 0.045 2814 | 8183 | 0.111 | 0.090 6530 | 6990 | 0.171 | 0.129 9131 | 6124 |
| 0.052 | 0.046 0997 | 8160 | 0.112 | 0.091 3520 | 6974 | 0.172 | 0.130 5255 | 6112 |
| 0.053 | 0.046 9157 | 8137 | 0.113 | 0.092 0494 | 6957 | 0.173 | 0.131 1367 | 6099 |
| 0.054 | 0.047 7294 | 8113 | 0.114 | 0.092 7451 | 6940 | 0.174 | 0.131 7466 | 6087 |
| 0.055 | 0.048 5407 | 8089 | 0.115 | 0.093 4391 | 6924 | 0.175 | 0.132 3553 | 6075 |
| 0.056 | 0.049 3496 | 8067 | 0.116 | 0.094 1315 | 6908 | 0.176 | 0.132 9628 | 6062 |
| 0.057 | 0.050 1563 | 8044 | 0.117 | 0.094 8223 | 6891 | 0.177 | 0.133 5690 | 6050 |
| 0.058 | 0.050 9607 | 8021 | 0.118 | 0.095 5114 | 6876 | 0.178 | 0.134 1740 | 6038 |
| 0.059 | 0.051 7628 | 7998 | 0.119 | 0.096 1990 | 6859 | 0.179 | 0.134 7778 | 6026 |
| 0.060 | 0.052 5626 | | 0.120 | 0.096 8849 | | 0.180 | 0.135 3804 | |

TABLE XIII.

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.180 | 0.135 3804 | 6014 | 0.240 | 0.169 5092 | 5378 | 0.300 | 0.200 2285 | 4872 |
| .181 | .135 9818 | 6003 | .241 | .170 0470 | 5368 | .301 | .200 7157 | 4864 |
| .182 | .136 5821 | 5990 | .242 | .170 5838 | 5359 | .302 | .201 2021 | 4857 |
| .183 | .137 1811 | 5978 | .243 | .171 1197 | 5350 | .303 | .201 6878 | 4849 |
| .184 | .137 7789 | 5966 | .244 | .171 6547 | 5340 | .304 | .202 1727 | 4842 |
| 0.185 | 0.138 3755 | 5955 | 0.245 | 0.172 1887 | 5331 | 0.305 | 0.202 6569 | 4834 |
| .186 | .138 9710 | 5943 | .246 | .172 7218 | 5322 | .306 | .203 1403 | 4827 |
| .187 | .139 5653 | 5932 | .247 | .173 2540 | 5313 | .307 | .203 6230 | 4820 |
| .188 | .140 1585 | 5919 | .248 | .173 7853 | 5303 | .308 | .204 1050 | 4812 |
| .189 | .140 7504 | 5908 | .249 | .174 3156 | 5295 | .309 | .204 5862 | 4805 |
| 0.190 | 0.141 3412 | 5897 | 0.250 | 0.174 8451 | 5285 | 0.310 | 0.205 0667 | 4797 |
| .191 | .141 9309 | 5885 | .251 | .175 3736 | 5277 | .311 | .205 5464 | 4790 |
| .192 | .142 5194 | 5874 | .252 | .175 9013 | 5267 | .312 | .206 0254 | 4783 |
| .193 | .143 1068 | 5863 | .253 | .176 4280 | 5258 | .313 | .206 5037 | 4776 |
| .194 | .143 6931 | 5851 | .254 | .176 9538 | 5250 | .314 | .206 9813 | 4768 |
| 0.195 | 0.144 2782 | 5840 | 0.255 | 0.177 4788 | 5241 | 0.315 | 0.207 4581 | 4761 |
| .196 | .144 8622 | 5828 | .256 | .178 0029 | 5232 | .316 | .207 9342 | 4754 |
| .197 | .145 4450 | 5818 | .257 | .178 5261 | 5223 | .317 | .208 4096 | 4747 |
| .198 | .146 0268 | 5806 | .258 | .179 0484 | 5214 | .318 | .208 8843 | 4739 |
| .199 | .146 6074 | 5795 | .259 | .179 5698 | 5205 | .319 | .209 3582 | 4733 |
| 0.200 | 0.147 1869 | 5784 | 0.260 | 0.180 0903 | 5197 | 0.320 | 0.209 8315 | 4725 |
| .201 | .147 7653 | 5774 | .261 | .180 6100 | 5188 | .321 | .210 3040 | 4719 |
| .202 | .148 3427 | 5762 | .262 | .181 1288 | 5179 | .322 | .210 7759 | 4711 |
| .203 | .148 9189 | 5751 | .263 | .181 6467 | 5171 | .323 | .211 2470 | 4704 |
| .204 | .149 4940 | 5741 | .264 | .182 1638 | 5162 | .324 | .211 7174 | 4697 |
| 0.205 | 0.150 0681 | 5730 | 0.265 | 0.182 6800 | 5153 | 0.325 | 0.212 1871 | 4691 |
| .206 | .150 6411 | 5719 | .266 | .183 1953 | 5145 | .326 | .212 6562 | 4683 |
| .207 | .151 2130 | 5708 | .267 | .183 7098 | 5137 | .327 | .213 1245 | 4676 |
| .208 | .151 7838 | 5697 | .268 | .184 2235 | 5128 | .328 | .213 5921 | 4670 |
| .209 | .152 3535 | 5687 | .269 | .184 7363 | 5120 | .329 | .214 0591 | 4662 |
| 0.210 | 0.152 9222 | 5677 | 0.270 | 0.185 2483 | 5111 | 0.330 | 0.214 5253 | 4656 |
| .211 | .153 4899 | 5666 | .271 | .185 7594 | 5102 | .331 | .214 9909 | 4649 |
| .212 | .154 0565 | 5655 | .272 | .186 2696 | 5095 | .332 | .215 4558 | 4642 |
| .213 | .154 6220 | 5645 | .273 | .186 7791 | 5086 | .333 | .215 9200 | 4635 |
| .214 | .155 1865 | 5634 | .274 | .187 2877 | 5078 | .334 | .216 3835 | 4629 |
| 0.215 | 0.155 7499 | 5624 | 0.275 | 0.187 7955 | 5069 | 0.335 | 0.216 8464 | 4621 |
| .216 | .156 3123 | 5614 | .276 | .188 3024 | 5061 | .336 | .217 3085 | 4615 |
| .217 | .156 8737 | 5603 | .277 | .188 8085 | 5053 | .337 | .217 7700 | 4608 |
| .218 | .157 4340 | 5593 | .278 | .189 3138 | 5045 | .338 | .218 2308 | 4602 |
| .219 | .157 9933 | 5583 | .279 | .189 8183 | 5037 | .339 | .218 6910 | 4595 |
| 0.220 | 0.158 5516 | 5573 | 0.280 | 0.190 3220 | 5029 | 0.340 | 0.219 1505 | 4588 |
| .221 | .159 1089 | 5563 | .281 | .190 8249 | 5020 | .341 | .219 6093 | 4582 |
| .222 | .159 6652 | 5552 | .282 | .191 3269 | 5012 | .342 | .220 0675 | 4575 |
| .223 | .160 2204 | 5543 | .283 | .191 8281 | 5005 | .343 | .220 5250 | 4568 |
| .224 | .160 7747 | 5532 | .284 | .192 3286 | 4996 | .344 | .220 9818 | 4562 |
| 0.225 | 0.161 3279 | 5523 | 0.285 | 0.192 8282 | 4989 | 0.345 | 0.221 4380 | 4555 |
| .226 | .161 8802 | 5513 | .286 | .193 3271 | 4980 | .346 | .221 8935 | 4548 |
| .227 | .162 4315 | 5502 | .287 | .193 8251 | 4973 | .347 | .222 3483 | 4542 |
| .228 | .162 9817 | 5493 | .288 | .194 3224 | 4964 | .348 | .222 8025 | 4536 |
| .229 | .163 5310 | 5483 | .289 | .194 8188 | 4957 | .349 | .223 2561 | 4529 |
| 0.230 | 0.164 0793 | 5474 | 0.290 | 0.195 3145 | 4949 | 0.350 | 0.223 7090 | 4523 |
| .231 | .164 6267 | 5463 | .291 | .195 8094 | 4941 | .351 | .224 1613 | 4517 |
| .232 | .165 1730 | 5454 | .292 | .196 3035 | 4933 | .352 | .224 6130 | 4510 |
| .233 | .165 7184 | 5444 | .293 | .196 7968 | 4926 | .353 | .225 0640 | 4503 |
| .234 | .166 2628 | 5435 | .294 | .197 2894 | 4917 | .354 | .225 5143 | 4497 |
| 0.235 | 0.166 8063 | 5425 | 0.295 | 0.197 7811 | 4910 | 0.355 | 0.225 9640 | 4491 |
| .236 | .167 3488 | 5415 | .296 | .198 2721 | 4903 | .356 | .226 4131 | 4484 |
| .237 | .167 8903 | 5406 | .297 | .198 7624 | 4894 | .357 | .226 8615 | 4478 |
| .238 | .168 4309 | 5396 | .298 | .199 2518 | 4888 | .358 | .227 3093 | 4472 |
| .239 | .168 9705 | 5387 | .299 | .199 7406 | 4879 | .359 | .227 7565 | 4466 |
| 0.240 | 0.169 5092 | | 0.300 | 0.200 2285 | | 0.360 | 0.228 2031 | |

TABLE XIII.

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.360 | 0.228 2031 | 4459 | 0.420 | 0.253 9153 | 4116 | 0.480 | 0.277 7272 | 3824 |
| .361 | .228 6490 | 4453 | .421 | .254 3269 | 4110 | .481 | .278 1096 | 3820 |
| .362 | .229 0943 | 4447 | .422 | .254 7379 | 4105 | .482 | .278 4916 | 3816 |
| .363 | .229 5390 | 4441 | .423 | .255 1484 | 4100 | .483 | .278 8732 | 3811 |
| .364 | .229 9831 | 4434 | .424 | .255 5584 | 4095 | .484 | .279 2543 | 3806 |
| 0.365 | 0.230 4265 | 4429 | 0.425 | 0.255 9679 | 4090 | 0.485 | 0.279 6349 | 3802 |
| .366 | .230 8694 | 4422 | .426 | .256 3769 | 4084 | .486 | .280 0151 | 3798 |
| .367 | .231 3116 | 4416 | .427 | .256 7853 | 4079 | .487 | .280 3949 | 3794 |
| .368 | .231 7532 | 4410 | .428 | .257 1932 | 4074 | .488 | .280 7743 | 3789 |
| .369 | .232 1942 | 4404 | .429 | .257 6006 | 4069 | .489 | .281 1532 | 3784 |
| 0.370 | 0.232 6346 | 4397 | 0.430 | 0.258 0075 | 4064 | 0.490 | 0.281 5316 | 3780 |
| .371 | .233 0743 | 4392 | .431 | .258 4139 | 4059 | .491 | .281 9096 | 3776 |
| .372 | .233 5135 | 4386 | .432 | .258 8198 | 4054 | .492 | .282 2872 | 3772 |
| .373 | .233 9521 | 4379 | .433 | .259 2252 | 4048 | .493 | .282 6644 | 3767 |
| .374 | .234 3900 | 4374 | .434 | .259 6300 | 4044 | .494 | .283 0411 | 3762 |
| 0.375 | 0.234 8274 | 4368 | 0.435 | 0.260 0344 | 4038 | 0.495 | 0.283 4173 | 3759 |
| .376 | .235 2642 | 4361 | .436 | .260 4382 | 4033 | .496 | .283 7932 | 3754 |
| .377 | .235 7003 | 4356 | .437 | .260 8415 | 4029 | .497 | .284 1686 | 3750 |
| .378 | .236 1359 | 4350 | .438 | .261 2444 | 4023 | .498 | .284 5436 | 3745 |
| .379 | .236 5709 | 4344 | .439 | .261 6467 | 4019 | .499 | .284 9181 | 3742 |
| 0.380 | 0.237 0053 | 4338 | 0.440 | 0.262 0486 | 4013 | 0.500 | 0.285 2923 | 3737 |
| .381 | .237 4391 | 4332 | .441 | .262 4499 | 4008 | .501 | .285 6660 | 3732 |
| .382 | .237 8723 | 4327 | .442 | .262 8507 | 4004 | .502 | .286 0392 | 3729 |
| .383 | .238 3050 | 4320 | .443 | .263 2511 | 3998 | .503 | .286 4121 | 3724 |
| .384 | .238 7370 | 4315 | .444 | .263 6509 | 3994 | .504 | .286 7845 | 3720 |
| 0.385 | 0.239 1685 | 4308 | 0.445 | 0.264 0503 | 3989 | 0.505 | 0.287 1565 | 3716 |
| .386 | .239 5993 | 4303 | .446 | .264 4492 | 3983 | .506 | .287 5281 | 3711 |
| .387 | .240 0296 | 4298 | .447 | .264 8475 | 3979 | .507 | .287 8992 | 3708 |
| .388 | .240 4594 | 4291 | .448 | .265 2454 | 3974 | .508 | .288 2700 | 3703 |
| .389 | .240 8885 | 4286 | .449 | .265 6428 | 3969 | .509 | .288 6403 | 3699 |
| 0.390 | 0.241 3171 | 4280 | 0.450 | 0.266 0397 | 3965 | 0.510 | 0.289 0102 | 3695 |
| .391 | .241 7451 | 4274 | .451 | .266 4362 | 3959 | .511 | .289 3797 | 3690 |
| .392 | .242 1725 | 4269 | .452 | .266 8321 | 3955 | .512 | .289 7487 | 3687 |
| .393 | .242 5994 | 4263 | .453 | .267 2276 | 3950 | .513 | .290 1174 | 3682 |
| .394 | .243 0257 | 4257 | .454 | .267 6226 | 3945 | .514 | .290 4856 | 3679 |
| 0.395 | 0.243 4514 | 4252 | 0.455 | 0.268 0171 | 3940 | 0.515 | 0.290 8535 | 3674 |
| .396 | .243 8766 | 4246 | .456 | .268 4111 | 3935 | .516 | .291 2209 | 3670 |
| .397 | .244 3012 | 4240 | .457 | .268 8046 | 3931 | .517 | .291 5879 | 3666 |
| .398 | .244 7252 | 4235 | .458 | .269 1977 | 3926 | .518 | .291 9545 | 3662 |
| .399 | .245 1487 | 4229 | .459 | .269 5903 | 3921 | .519 | .292 3207 | 3657 |
| 0.400 | 0.245 5716 | 4224 | 0.460 | 0.269 9824 | 3917 | 0.520 | 0.292 6864 | 3654 |
| .401 | .245 9940 | 4218 | .461 | .270 3741 | 3911 | .521 | .293 0518 | 3650 |
| .402 | .246 4158 | 4213 | .462 | .270 7652 | 3907 | .522 | .293 4168 | 3645 |
| .403 | .246 8371 | 4207 | .463 | .271 1559 | 3903 | .523 | .293 7813 | 3642 |
| .404 | .247 2578 | 4201 | .464 | .271 5462 | 3898 | .524 | .294 1455 | 3637 |
| 0.405 | 0.247 6779 | 4196 | 0.465 | 0.271 9360 | 3893 | 0.525 | 0.294 5092 | 3634 |
| .406 | .248 0975 | 4191 | .466 | .272 3253 | 3888 | .526 | .294 8726 | 3629 |
| .407 | .248 5166 | 4185 | .467 | .272 7141 | 3884 | .527 | .295 2355 | 3626 |
| .408 | .248 9351 | 4180 | .468 | .273 1025 | 3879 | .528 | .295 5981 | 3621 |
| .409 | .249 3531 | 4174 | .469 | .273 4904 | 3874 | .529 | .295 9602 | 3618 |
| 0.410 | 0.249 7705 | 4169 | 0.470 | 0.273 8778 | 3870 | 0.530 | 0.296 3220 | 3613 |
| .411 | .250 1874 | 4164 | .471 | .274 2648 | 3865 | .531 | .296 6833 | 3610 |
| .412 | .250 6038 | 4158 | .472 | .274 6513 | 3861 | .532 | .297 0443 | 3606 |
| .413 | .251 0196 | 4153 | .473 | .275 0374 | 3856 | .533 | .297 4049 | 3601 |
| .414 | .251 4349 | 4147 | .474 | .275 4230 | 3852 | .534 | .297 7650 | 3598 |
| 0.415 | 0.251 8496 | 4142 | 0.475 | 0.275 8082 | 3847 | 0.535 | 0.298 1248 | 3594 |
| .416 | .252 2638 | 4137 | .476 | .276 1929 | 3842 | .536 | .298 4842 | 3590 |
| .417 | .252 6775 | 4131 | .477 | .276 5771 | 3838 | .537 | .298 8432 | 3586 |
| .418 | .253 0906 | 4126 | .478 | .276 9609 | 3834 | .538 | .299 2018 | 3582 |
| .419 | .253 5032 | 4121 | .479 | .277 3443 | 3829 | .539 | .299 5600 | 3578 |
| 0.420 | 0.253 9153 | | 0.480 | 0.277 7272 | | 0.540 | 0.299 9178 | |

TABLE XIII.

For finding the Ratio of the Sector to the Triangle.

| η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. | η | $\log s^2$ | Diff. |
|--------|------------|-------|--------|------------|-------|--------|------------|-------|
| 0.540 | 0.299 9178 | | 0.560 | 0.306 9938 | | 0.580 | 0.313 9215 | |
| .541 | .300 2752 | 3574 | .561 | .307 3437 | 3499 | .581 | .314 2641 | 3426 |
| .542 | .300 6323 | 3571 | .562 | .307 6931 | 3494 | .582 | .314 6064 | 3423 |
| .543 | .300 9890 | 3567 | .563 | .308 0422 | 3491 | .583 | .314 9483 | 3419 |
| .544 | .301 3452 | 3562 | .564 | .308 3910 | 3488 | .584 | .315 2898 | 3415 |
| | | 3559 | | | 3484 | | | 3412 |
| 0.545 | 0.301 7011 | | 0.565 | 0.308 7394 | | 0.585 | 0.315 6310 | |
| .546 | .302 0566 | 3555 | .566 | .309 0874 | 3480 | .586 | .315 9719 | 3409 |
| .547 | .302 4117 | 3551 | .567 | .309 4350 | 3476 | .587 | .316 3124 | 3405 |
| .548 | .302 7664 | 3547 | .568 | .309 7823 | 3473 | .588 | .316 6525 | 3401 |
| .549 | .303 1208 | 3544 | .569 | .310 1292 | 3469 | .589 | .316 9923 | 3398 |
| | | 3540 | | | 3466 | | | 3395 |
| 0.550 | 0.303 4748 | | 0.570 | 0.310 4758 | | 0.590 | 0.317 3318 | |
| .551 | .303 8284 | 3536 | .571 | .310 8220 | 3462 | .591 | .317 6709 | 3391 |
| .552 | .304 1816 | 3532 | .572 | .311 1678 | 3458 | .592 | .318 0096 | 3387 |
| .553 | .304 5344 | 3528 | .573 | .311 5133 | 3455 | .593 | .318 3480 | 3384 |
| .554 | .304 8869 | 3525 | .574 | .311 8584 | 3451 | .594 | .318 6861 | 3381 |
| | | 3521 | | | 3447 | | | 3377 |
| 0.555 | 0.305 2390 | | 0.575 | 0.312 2031 | | 0.595 | 0.319 0238 | |
| .556 | .305 5907 | 3517 | .576 | .312 5475 | 3444 | .596 | .319 3612 | 3374 |
| .557 | .305 9420 | 3513 | .577 | .312 8915 | 3440 | .597 | .319 6983 | 3371 |
| .558 | .306 2930 | 3510 | .578 | .313 2352 | 3437 | .598 | .320 0350 | 3367 |
| .559 | .306 6436 | 3506 | .579 | .313 5785 | 3433 | .599 | .320 3714 | 3364 |
| | | 3502 | | | 3430 | | | 3360 |
| 0.560 | 0.306 9938 | | 0.580 | 0.313 9215 | | 0.600 | 0.320 7074 | |

TABLE XIV.

For finding the Ratio of the Sector to the Triangle.

| x | ζ | | | | x | ζ | | | |
|-------|------------|-------|------------|-------|-------|------------|-------|------------|-------|
| | Ellipse. | Diff. | Hyperbola. | Diff. | | Ellipse. | Diff. | Hyperbola. | Diff. |
| 0.000 | 0.000 0000 | | 0.000 0000 | | 0.030 | 0.000 0523 | | 0.000 0506 | |
| .001 | .000 0001 | 1 | .000 0001 | 1 | .031 | .000 0559 | 36 | .000 0539 | 33 |
| .002 | .000 0002 | 1 | .000 0002 | 1 | .032 | .000 0596 | 37 | .000 0575 | 36 |
| .003 | .000 0005 | 3 | .000 0005 | 3 | .033 | .000 0634 | 38 | .000 0611 | 36 |
| .004 | .000 0009 | 4 | .000 0009 | 4 | .034 | .000 0674 | 40 | .000 0648 | 37 |
| | | 5 | | 5 | | | 40 | | 38 |
| 0.005 | 0.000 0014 | | 0.000 0014 | | 0.035 | 0.000 0714 | | 0.000 0686 | |
| .006 | .000 0021 | 7 | .000 0020 | 8 | .036 | .000 0756 | 42 | .000 0726 | 40 |
| .007 | .000 0028 | 7 | .000 0028 | 8 | .037 | .000 0799 | 43 | .000 0766 | 40 |
| .008 | .000 0037 | 9 | .000 0036 | 8 | .038 | .000 0844 | 45 | .000 0807 | 41 |
| .009 | .000 0047 | 10 | .000 0046 | 10 | .039 | .000 0889 | 45 | .000 0850 | 43 |
| | | 11 | | 11 | | | 47 | | 44 |
| 0.010 | 0.000 0058 | | 0.000 0057 | | 0.040 | 0.000 0936 | | 0.000 0894 | |
| .011 | .000 0070 | 12 | .000 0069 | 12 | .041 | .000 0984 | 48 | .000 0938 | 44 |
| .012 | .000 0083 | 13 | .000 0082 | 13 | .042 | .000 1033 | 49 | .000 0984 | 46 |
| .013 | .000 0097 | 14 | .000 0096 | 14 | .043 | .000 1084 | 51 | .000 1031 | 47 |
| .014 | .000 0113 | 16 | .000 0111 | 15 | .044 | .000 1135 | 51 | .000 1079 | 48 |
| | | 17 | | 16 | | | 53 | | 49 |
| 0.015 | 0.000 0130 | | 0.000 0127 | | 0.045 | 0.000 1188 | | 0.000 1128 | |
| .016 | .000 0148 | 18 | .000 0145 | 18 | .046 | .000 1242 | 54 | .000 1178 | 50 |
| .017 | .000 0167 | 19 | .000 0164 | 19 | .047 | .000 1298 | 56 | .000 1229 | 51 |
| .018 | .000 0187 | 20 | .000 0183 | 19 | .048 | .000 1354 | 56 | .000 1281 | 52 |
| .019 | .000 0209 | 22 | .000 0204 | 21 | .049 | .000 1412 | 58 | .000 1334 | 53 |
| | | 22 | | 22 | | | 59 | | 55 |
| 0.020 | 0.000 0231 | | 0.000 0226 | | 0.050 | 0.000 1471 | | 0.000 1389 | |
| .021 | .000 0255 | 24 | .000 0249 | 23 | .051 | .000 1532 | 61 | .000 1444 | 55 |
| .022 | .000 0280 | 25 | .000 0273 | 24 | .052 | .000 1593 | 61 | .000 1500 | 56 |
| .023 | .000 0306 | 26 | .000 0298 | 25 | .053 | .000 1656 | 63 | .000 1558 | 58 |
| .024 | .000 0334 | 28 | .000 0325 | 27 | .054 | .000 1720 | 64 | .000 1616 | 58 |
| | | 28 | | 27 | | | 65 | | 59 |
| 0.025 | 0.000 0362 | | 0.000 0352 | | 0.055 | 0.000 1785 | | 0.000 1675 | |
| .026 | .000 0392 | 30 | .000 0381 | 29 | .056 | .000 1852 | 67 | .000 1736 | 61 |
| .027 | .000 0423 | 31 | .000 0410 | 29 | .057 | .000 1920 | 68 | .000 1798 | 62 |
| .028 | .000 0455 | 32 | .000 0441 | 31 | .058 | .000 1989 | 69 | .000 1860 | 62 |
| .029 | .000 0489 | 34 | .000 0473 | 32 | .059 | .000 2060 | 71 | .000 1924 | 64 |
| | | 34 | | 33 | | | 71 | | 64 |
| 0.030 | 0.000 0523 | | 0.000 0506 | | 0.060 | 0.000 2131 | | 0.000 1988 | |

TABLE XIV.

For finding the Ratio of the Sector to the Triangle.

| z | ξ | | | | z | ξ | | | |
|-------|------------|-------|------------|-------|-------|------------|-------|------------|-------|
| | Ellipse. | Diff. | Hyperbola. | Diff. | | Ellipse. | Diff. | Hyperbola. | Diff. |
| 0.060 | 0.000 2131 | | 0.000 1988 | | 0.120 | 0.000 8845 | | 0.000 7698 | |
| .061 | .000 2204 | 73 | .000 2054 | 66 | .121 | .000 8999 | 154 | .000 7822 | 124 |
| .062 | .000 2278 | 74 | .000 2121 | 67 | .122 | .000 9154 | 155 | .000 7948 | 126 |
| .063 | .000 2354 | 76 | .000 2189 | 68 | .123 | .000 9311 | 157 | .000 8074 | 126 |
| .064 | .000 2431 | 77 | .000 2257 | 68 | .124 | .000 9469 | 158 | .000 8202 | 128 |
| | | 78 | | 70 | | | 159 | | 128 |
| 0.065 | 0.000 2509 | | 0.000 2327 | | 0.125 | 0.000 9628 | | 0.000 8330 | |
| .066 | .000 2588 | 79 | .000 2398 | 71 | .126 | .000 9789 | 161 | .000 8459 | 129 |
| .067 | .000 2669 | 81 | .000 2470 | 72 | .127 | .000 9951 | 162 | .000 8590 | 131 |
| .068 | .000 2751 | 82 | .000 2543 | 73 | .128 | .001 0115 | 164 | .000 8721 | 131 |
| .069 | .000 2834 | 83 | .000 2617 | 74 | .129 | .001 0280 | 165 | .000 8853 | 132 |
| | | 84 | | 74 | | | 167 | | 133 |
| 0.070 | 0.000 2918 | | 0.000 2691 | | 0.130 | 0.001 0447 | | 0.000 8986 | |
| .071 | .000 3004 | 86 | .000 2767 | 76 | .131 | .001 0615 | 168 | .000 9120 | 134 |
| .072 | .000 3091 | 87 | .000 2844 | 77 | .132 | .001 0784 | 169 | .000 9255 | 135 |
| .073 | .000 3180 | 89 | .000 2922 | 78 | .133 | .001 0955 | 171 | .000 9390 | 135 |
| .074 | .000 3269 | 89 | .000 3001 | 79 | .134 | .001 1128 | 173 | .000 9527 | 137 |
| | | 91 | | 80 | | | 173 | | 138 |
| 0.075 | 0.000 3360 | | 0.000 3081 | | 0.135 | 0.001 1301 | | 0.000 9665 | |
| .076 | .000 3453 | 93 | .000 3162 | 81 | .136 | .001 1477 | 176 | .000 9803 | 138 |
| .077 | .000 3546 | 93 | .000 3244 | 82 | .137 | .001 1654 | 177 | .000 9943 | 140 |
| .078 | .000 3641 | 95 | .000 3327 | 83 | .138 | .001 1832 | 178 | .001 0083 | 140 |
| .079 | .000 3738 | 97 | .000 3411 | 84 | .139 | .001 2012 | 180 | .001 0224 | 141 |
| | | 97 | | 85 | | | 181 | | 142 |
| 0.080 | 0.000 3835 | | 0.000 3496 | | 0.140 | 0.001 2193 | | 0.001 0366 | |
| .081 | .000 3934 | 99 | .000 3582 | 86 | .141 | .001 2376 | 183 | .001 0509 | 143 |
| .082 | .000 4034 | 100 | .000 3669 | 87 | .142 | .001 2560 | 184 | .001 0653 | 144 |
| .083 | .000 4136 | 102 | .000 3757 | 88 | .143 | .001 2745 | 185 | .001 0798 | 145 |
| .084 | .000 4239 | 103 | .000 3846 | 89 | .144 | .001 2933 | 188 | .001 0944 | 146 |
| | | 104 | | 90 | | | 188 | | 147 |
| 0.085 | 0.000 4343 | | 0.000 3936 | | 0.145 | 0.001 3121 | | 0.001 1091 | |
| .086 | .000 4448 | 105 | .000 4027 | 91 | .146 | .001 3311 | 190 | .001 1238 | 147 |
| .087 | .000 4555 | 107 | .000 4119 | 92 | .147 | .001 3503 | 192 | .001 1387 | 149 |
| .088 | .000 4663 | 108 | .000 4212 | 93 | .148 | .001 3696 | 193 | .001 1536 | 149 |
| .089 | .000 4773 | 110 | .000 4306 | 94 | .149 | .001 3891 | 195 | .001 1686 | 150 |
| | | 111 | | 95 | | | 196 | | 152 |
| 0.090 | 0.000 4884 | | 0.000 4401 | | 0.150 | 0.001 4087 | | 0.001 1838 | |
| .091 | .000 4996 | 112 | .000 4496 | 95 | .151 | .001 4285 | 198 | .001 1990 | 152 |
| .092 | .000 5109 | 113 | .000 4593 | 97 | .152 | .001 4484 | 199 | .001 2143 | 153 |
| .093 | .000 5224 | 115 | .000 4691 | 98 | .153 | .001 4684 | 200 | .001 2296 | 153 |
| .094 | .000 5341 | 117 | .000 4790 | 99 | .154 | .001 4886 | 202 | .001 2451 | 155 |
| | | 117 | | 100 | | | 204 | | 156 |
| 0.095 | 0.000 5458 | | 0.000 4890 | | 0.155 | 0.001 5090 | | 0.001 2607 | |
| .096 | .000 5577 | 119 | .000 4991 | 101 | .156 | .001 5295 | 205 | .001 2763 | 156 |
| .097 | .000 5697 | 120 | .000 5092 | 101 | .157 | .001 5502 | 207 | .001 2921 | 158 |
| .098 | .000 5819 | 122 | .000 5195 | 103 | .158 | .001 5710 | 208 | .001 3079 | 158 |
| .099 | .000 5942 | 123 | .000 5299 | 104 | .159 | .001 5920 | 210 | .001 3238 | 159 |
| | | 124 | | 104 | | | 211 | | 160 |
| 0.100 | 0.000 6066 | | 0.000 5403 | | 0.160 | 0.001 6131 | | 0.001 3398 | |
| .101 | .000 6192 | 126 | .000 5509 | 106 | .161 | .001 6344 | 213 | .001 3559 | 161 |
| .102 | .000 6319 | 127 | .000 5616 | 107 | .162 | .001 6559 | 215 | .001 3721 | 162 |
| .103 | .000 6448 | 129 | .000 5723 | 107 | .163 | .001 6775 | 216 | .001 3883 | 162 |
| .104 | .000 6578 | 130 | .000 5832 | 109 | .164 | .001 6992 | 217 | .001 4047 | 164 |
| | | 131 | | 109 | | | 219 | | 164 |
| 0.105 | 0.000 6709 | | 0.000 5941 | | 0.165 | 0.001 7211 | | 0.001 4211 | |
| .106 | .000 6842 | 133 | .000 6052 | 111 | .166 | .001 7432 | 221 | .001 4377 | 166 |
| .107 | .000 6976 | 134 | .000 6163 | 111 | .167 | .001 7654 | 222 | .001 4543 | 166 |
| .108 | .000 7111 | 135 | .000 6275 | 112 | .168 | .001 7878 | 224 | .001 4710 | 167 |
| .109 | .000 7248 | 137 | .000 6389 | 114 | .169 | .001 8103 | 225 | .001 4878 | 168 |
| | | 138 | | 114 | | | 227 | | 169 |
| 0.110 | 0.000 7386 | | 0.000 6503 | | 0.170 | 0.001 8330 | | 0.001 5047 | |
| .111 | .000 7526 | 140 | .000 6618 | 115 | .171 | .001 8558 | 228 | .001 5216 | 169 |
| .112 | .000 7667 | 141 | .000 6734 | 116 | .172 | .001 8788 | 230 | .001 5387 | 171 |
| .113 | .000 7809 | 142 | .000 6851 | 117 | .173 | .001 9020 | 232 | .001 5558 | 171 |
| .114 | .000 7953 | 144 | .000 6969 | 118 | .174 | .001 9253 | 233 | .001 5730 | 172 |
| | | 145 | | 119 | | | 234 | | 173 |
| 0.115 | 0.000 8098 | | 0.000 7088 | | 0.175 | 0.001 9487 | | 0.001 5903 | |
| .116 | .000 8245 | 147 | .000 7208 | 120 | .176 | .001 9724 | 237 | .001 6077 | 174 |
| .117 | .000 8393 | 148 | .000 7329 | 121 | .177 | .001 9961 | 237 | .001 6252 | 175 |
| .118 | .000 8542 | 149 | .000 7451 | 122 | .178 | .002 0201 | 240 | .001 6428 | 176 |
| .119 | .000 8693 | 151 | .000 7574 | 123 | .179 | .002 0442 | 241 | .001 6604 | 176 |
| | | 152 | | 124 | | | 243 | | 178 |
| 0.120 | 0.000 8845 | | 0.000 7698 | | 0.180 | 0.002 0685 | | 0.001 6782 | |

TABLE XIV.

For finding the Ratio of the Sector to the Triangle.

| x | ξ | | | | x | ξ | | | |
|-------|------------|-------|------------|-------|-------|------------|-------|------------|-------|
| | Ellipse. | Diff. | Hyperbola. | Diff. | | Ellipse. | Diff. | Hyperbola. | Diff. |
| 0.180 | 0.002 0685 | 244 | 0.001 6782 | 178 | 0.240 | 0.003 8289 | 346 | 0.002 8939 | 227 |
| .181 | .002 0929 | 246 | .001 6960 | 179 | .241 | .003 8635 | 348 | .002 9166 | 228 |
| .182 | .002 1175 | 247 | .001 7139 | 180 | .242 | .003 8983 | 350 | .002 9394 | 229 |
| .183 | .002 1422 | 249 | .001 7319 | 181 | .243 | .003 9333 | 352 | .002 9623 | 229 |
| .184 | .002 1671 | 251 | .001 7500 | 181 | .244 | .003 9685 | 354 | .002 9852 | 231 |
| 0.185 | 0.002 1922 | 252 | 0.001 7681 | 183 | 0.245 | 0.004 0039 | 355 | 0.003 0083 | 231 |
| .186 | .002 2174 | 254 | .001 7864 | 183 | .246 | .004 0394 | 358 | .003 0314 | 231 |
| .187 | .002 2428 | 255 | .001 8047 | 184 | .247 | .004 0752 | 359 | .003 0545 | 233 |
| .188 | .002 2683 | 258 | .001 8231 | 185 | .248 | .004 1111 | 361 | .003 0778 | 233 |
| .189 | .002 2941 | 258 | .001 8416 | 186 | .249 | .004 1472 | 363 | .003 1011 | 234 |
| 0.190 | 0.002 3199 | 261 | 0.001 8602 | 187 | 0.250 | 0.004 1835 | 364 | 0.003 1245 | 235 |
| .191 | .002 3460 | 262 | .001 8789 | 187 | .251 | .004 2199 | 367 | .003 1480 | 236 |
| .192 | .002 3722 | 263 | .001 8976 | 189 | .252 | .004 2566 | 368 | .003 1716 | 236 |
| .193 | .002 3985 | 266 | .001 9165 | 189 | .253 | .004 2934 | 371 | .003 1952 | 237 |
| .194 | .002 4251 | 267 | .001 9354 | 190 | .254 | .004 3305 | 372 | .003 2189 | 238 |
| 0.195 | 0.002 4518 | 268 | 0.001 9544 | 191 | 0.255 | 0.004 3677 | 374 | 0.003 2427 | 239 |
| .196 | .002 4786 | 270 | .001 9735 | 191 | .256 | .004 4051 | 376 | .003 2666 | 239 |
| .197 | .002 5056 | 272 | .001 9926 | 193 | .257 | .004 4427 | 377 | .003 2905 | 241 |
| .198 | .002 5328 | 274 | .002 0119 | 193 | .258 | .004 4804 | 380 | .003 3146 | 241 |
| .199 | .002 5602 | 275 | .002 0312 | 195 | .259 | .004 5184 | 382 | .003 3387 | 241 |
| 0.200 | 0.002 5877 | 277 | 0.002 0507 | 195 | 0.260 | 0.004 5566 | 383 | 0.003 3628 | 243 |
| .201 | .002 6154 | 279 | .002 0702 | 195 | .261 | .004 5949 | 385 | .003 3871 | 243 |
| .202 | .002 6433 | 280 | .002 0897 | 197 | .262 | .004 6334 | 387 | .003 4114 | 244 |
| .203 | .002 6713 | 282 | .002 1094 | 198 | .263 | .004 6721 | 390 | .003 4358 | 245 |
| .204 | .002 6995 | 283 | .002 1292 | 198 | .264 | .004 7111 | 391 | .003 4603 | 245 |
| 0.205 | 0.002 7278 | 286 | 0.002 1490 | 199 | 0.265 | 0.004 7502 | 392 | 0.003 4848 | 246 |
| .206 | .002 7564 | 287 | .002 1689 | 200 | .266 | .004 7894 | 395 | .003 5094 | 247 |
| .207 | .002 7851 | 288 | .002 1889 | 201 | .267 | .004 8289 | 397 | .003 5341 | 248 |
| .208 | .002 8139 | 290 | .002 2090 | 201 | .268 | .004 8686 | 399 | .003 5589 | 249 |
| .209 | .002 8429 | 293 | .002 2291 | 203 | .269 | .004 9085 | 400 | .003 5838 | 249 |
| 0.210 | 0.002 8722 | 293 | 0.002 2494 | 203 | 0.270 | 0.004 9485 | 403 | 0.003 6087 | 250 |
| .211 | .002 9015 | 296 | .002 2697 | 204 | .271 | .004 9888 | 404 | .003 6337 | 250 |
| .212 | .002 9311 | 297 | .002 2901 | 205 | .272 | .005 0292 | 407 | .003 6587 | 252 |
| .213 | .002 9608 | 299 | .002 3106 | 205 | .273 | .005 0699 | 408 | .003 6839 | 252 |
| .214 | .002 9907 | 300 | .002 3311 | 207 | .274 | .005 1107 | 410 | .003 7091 | 253 |
| 0.215 | 0.003 0207 | 302 | 0.002 3518 | 207 | 0.275 | 0.005 1517 | 413 | 0.003 7344 | 254 |
| .216 | .003 0509 | 305 | .002 3725 | 207 | .276 | .005 1930 | 414 | .003 7598 | 254 |
| .217 | .003 0814 | 305 | .002 3932 | 210 | .277 | .005 2344 | 416 | .003 7852 | 255 |
| .218 | .003 1119 | 308 | .002 4142 | 210 | .278 | .005 2760 | 418 | .003 8107 | 256 |
| .219 | .003 1427 | 309 | .002 4352 | 210 | .279 | .005 3178 | 420 | .003 8363 | 257 |
| 0.220 | 0.003 1736 | 311 | 0.002 4562 | 212 | 0.280 | 0.005 3598 | 422 | 0.003 8620 | 257 |
| .221 | .003 2047 | 312 | .002 4774 | 212 | .281 | .005 4020 | 424 | .003 8877 | 258 |
| .222 | .003 2359 | 315 | .002 4986 | 213 | .282 | .005 4444 | 426 | .003 9135 | 259 |
| .223 | .003 2674 | 316 | .002 5199 | 213 | .283 | .005 4870 | 428 | .003 9394 | 260 |
| .224 | .003 2990 | 318 | .002 5412 | 215 | .284 | .005 5298 | 430 | .003 9654 | 260 |
| 0.225 | 0.003 3308 | 319 | 0.002 5627 | 215 | 0.285 | 0.005 5728 | 432 | 0.003 9914 | 261 |
| .226 | .003 3627 | 322 | .002 5842 | 216 | .286 | .005 6160 | 434 | .004 0175 | 262 |
| .227 | .003 3949 | 323 | .002 6058 | 217 | .287 | .005 6594 | 436 | .004 0437 | 263 |
| .228 | .003 4272 | 325 | .002 6275 | 218 | .288 | .005 7030 | 438 | .004 0700 | 263 |
| .229 | .003 4597 | 327 | .002 6493 | 218 | .289 | .005 7468 | 440 | .004 0963 | 264 |
| 0.230 | 0.003 4924 | 328 | 0.002 6711 | 220 | 0.290 | 0.005 7908 | 442 | 0.004 1227 | 264 |
| .231 | .003 5252 | 330 | .002 6931 | 220 | .291 | .005 8350 | 445 | .004 1491 | 266 |
| .232 | .003 5582 | 332 | .002 7151 | 220 | .292 | .005 8795 | 446 | .004 1757 | 266 |
| .233 | .003 5914 | 334 | .002 7371 | 222 | .293 | .005 9241 | 448 | .004 2023 | 267 |
| .234 | .003 6248 | 336 | .002 7593 | 223 | .294 | .005 9689 | 450 | .004 2290 | 267 |
| 0.235 | 0.003 6584 | 337 | 0.002 7816 | 223 | 0.295 | 0.006 0139 | 452 | 0.004 2557 | 269 |
| .236 | .003 6921 | 339 | .002 8039 | 224 | .296 | .006 0591 | 454 | .004 2826 | 269 |
| .237 | .003 7260 | 341 | .002 8263 | 224 | .297 | .006 1045 | 457 | .004 3095 | 269 |
| .238 | .003 7601 | 343 | .002 8487 | 226 | .298 | .006 1502 | 458 | .004 3364 | 271 |
| .239 | .003 7944 | 345 | .002 8713 | 226 | .299 | .006 1960 | 461 | .004 3635 | 271 |
| 0.240 | 0.003 8289 | | 0.002 8939 | | 0.300 | 0.006 2421 | | 0.004 3906 | |

TABLE XV.

For Elliptic Orbits of great eccentricity.

| ϵ or δ | $\log B_0$ or $\log B'_0$ | Diff. | $\log N$ | Diff. | ϵ or δ | $\log B_0$ or $\log B'_0$ | Diff. | $\log N$ | Diff. |
|------------------------|---------------------------|-------|------------|-------|------------------------|---------------------------|-------|------------|-------|
| 0 | 0.000 0000 | 0 | 0.000 0000 | 7 | 30 | 0.000 0066 | 9 | 0.000 6400 | 436 |
| 1 | .000 0000 | 0 | .000 0007 | 21 | 31 | .000 0075 | 11 | .000 6836 | 450 |
| 2 | .000 0000 | 0 | .000 0028 | 36 | 32 | .000 0086 | 11 | .000 7286 | 464 |
| 3 | .000 0000 | 0 | .000 0064 | 49 | 33 | .000 0097 | 12 | .000 7750 | 479 |
| 4 | .000 0000 | 0 | .000 0113 | 64 | 34 | .000 0109 | 13 | .000 8229 | 493 |
| 5 | 0.000 0000 | 0 | 0.000 0177 | 78 | 35 | 0.000 0122 | 15 | 0.000 8722 | 508 |
| 6 | .000 0000 | 0 | .000 0255 | 92 | 36 | .000 0137 | 16 | .000 9230 | 523 |
| 7 | .000 0000 | 0 | .000 0347 | 107 | 37 | .000 0153 | 18 | .000 9753 | 537 |
| 8 | .000 0000 | 1 | .000 0454 | 120 | 38 | .000 0171 | 19 | .001 0290 | 552 |
| 9 | .000 0001 | 0 | .000 0574 | 135 | 39 | .000 0190 | 20 | .001 0842 | 567 |
| 10 | 0.000 0001 | 0 | 0.000 0709 | 149 | 40 | 0.000 0210 | 22 | 0.001 1409 | 581 |
| 11 | .000 0001 | 1 | .000 0858 | 163 | 41 | .000 0232 | 23 | .001 1990 | 596 |
| 12 | .000 0002 | 0 | .000 1021 | 178 | 42 | .000 0255 | 26 | .001 2586 | 611 |
| 13 | .000 0002 | 1 | .000 1199 | 191 | 43 | .000 0281 | 27 | .001 3197 | 626 |
| 14 | .000 0003 | 1 | .000 1390 | 206 | 44 | .000 0308 | 29 | .001 3823 | 640 |
| 15 | 0.000 0004 | 1 | 0.000 1596 | 220 | 45 | 0.000 0337 | 31 | 0.001 4463 | 655 |
| 16 | .000 0005 | 2 | .000 1816 | 235 | 46 | .000 0368 | 33 | .001 5118 | 670 |
| 17 | .000 0007 | 2 | .000 2051 | 248 | 47 | .000 0401 | 36 | .001 5788 | 685 |
| 18 | .000 0009 | 2 | .000 2299 | 263 | 48 | .000 0437 | 38 | .001 6473 | 700 |
| 19 | .000 0011 | 2 | .000 2562 | 277 | 49 | .000 0475 | 40 | .001 7173 | 715 |
| 20 | 0.000 0013 | 3 | 0.000 2839 | 292 | 50 | 0.000 0515 | 43 | 0.001 7888 | 730 |
| 21 | .000 0016 | 3 | .000 3131 | 306 | 51 | .000 0558 | 46 | .001 8618 | 744 |
| 22 | .000 0019 | 3 | .000 3437 | 320 | 52 | .000 0604 | 48 | .001 9362 | 760 |
| 23 | .000 0023 | 4 | .000 3757 | 334 | 53 | .000 0652 | 51 | .002 0122 | 775 |
| 24 | .000 0027 | 5 | .000 4091 | 349 | 54 | .000 0703 | 54 | .002 0897 | 790 |
| 25 | 0.000 0032 | 5 | 0.000 4440 | 363 | 55 | 0.000 0757 | 58 | 0.002 1687 | 806 |
| 26 | .000 0037 | 6 | .000 4803 | 378 | 56 | .000 0815 | 60 | .002 2493 | 820 |
| 27 | .000 0043 | 7 | .000 5181 | 392 | 57 | .000 0875 | 64 | .002 3313 | 836 |
| 28 | .000 0050 | 7 | .000 5573 | 407 | 58 | .000 0939 | 68 | .002 4149 | 851 |
| 29 | .000 0057 | 9 | .000 5980 | 420 | 59 | .000 1007 | 71 | .002 5000 | 866 |
| 30 | 0.000 0066 | | 0.000 6400 | | 60 | 0.000 1078 | | 0.002 5866 | |

TABLE XVI.

For Hyperbolic Orbits.

| m or n | $\log Q$ or $\log Q'$ | $\log I.$ Diff. | \log half II. Diff. | m or n | $\log Q$ or $\log Q'$ | $\log I.$ Diff. | \log half II. Diff. |
|------------|-----------------------|----------------------|-----------------------|------------|-----------------------|----------------------|-----------------------|
| 0.00 | 0.000 0000 | — | 2.1149 _n | 0.10 | 9.998 7021 | 3.41256 _n | 2.1046 _n |
| .01 | 9.999 9870 | 2.41597 _n | 2.1146 _n | .11 | .998 4308 | 3.45326 _n | 2.1025 _n |
| .02 | .999 9479 | 2.71675 _n | 2.1142 _n | .12 | .998 1342 | 3.49028 _n | 2.1003 _n |
| .03 | .999 8828 | 2.89259 _n | 2.1137 _n | .13 | .997 8123 | 3.52423 _n | 2.0978 _n |
| .04 | .999 7917 | 3.01741 _n | 2.1130 _n | .14 | .997 4654 | 3.55547 _n | 2.0952 _n |
| 0.05 | 9.999 6746 | 3.11411 _n | 2.1121 _n | 0.15 | 9.997 0936 | 3.58453 _n | 2.0923 _n |
| .06 | .999 5316 | 3.19290 _n | 2.1110 _n | .16 | .996 6971 | 3.61154 _n | 2.0892 _n |
| .07 | .999 3628 | 3.25940 _n | 2.1097 _n | .17 | .996 2760 | 3.63679 _n | 2.0860 _n |
| .08 | .999 1682 | 3.31687 _n | 2.1082 _n | .18 | .995 8305 | 3.66048 _n | 2.0826 _n |
| .09 | .998 9479 | 3.36745 _n | 2.1065 _n | .19 | .995 3608 | 3.68276 _n | 2.0790 _n |
| 0.10 | 9.998 7021 | 3.41256 _n | 2.1046 _n | 0.20 | 9.994 8671 | 3.70378 _n | 2.0752 _n |

TABLE XVII.
For special Perturbations.

| q, q', q'' | For positive values of the Argument. | | | | For negative values of the Argument. | | | |
|--------------|--------------------------------------|-------|----------------------|-------|--------------------------------------|-------|----------------------|-------|
| | log f | Diff. | log f' , log f'' | Diff. | log f | Diff. | log f' , log f'' | Diff. |
| 0.0000 | 0.477 1213 | 1086 | 0.301 0300 | 869 | 0.477 1213 | 1086 | 0.301 0300 | 869 |
| .0001 | .477 0127 | 1085 | .300 9431 | 868 | .477 2299 | 1086 | .301 1169 | 868 |
| .0002 | .476 9042 | 1085 | .300 8563 | 868 | .477 3385 | 1086 | .301 2037 | 869 |
| .0003 | .476 7957 | 1085 | .300 7695 | 868 | .477 4471 | 1087 | .301 2906 | 870 |
| .0004 | .476 6872 | 1085 | .300 6827 | 868 | .477 5558 | 1087 | .301 3776 | 869 |
| 0.0005 | 0.476 5787 | 1085 | 0.300 5959 | 867 | 0.477 6645 | 1087 | 0.301 4645 | 870 |
| .0006 | .476 4702 | 1084 | .300 5092 | 868 | .477 7732 | 1087 | .301 5515 | 869 |
| .0007 | .476 3618 | 1084 | .300 4224 | 867 | .477 8819 | 1087 | .301 6384 | 870 |
| .0008 | .476 2534 | 1084 | .300 3357 | 867 | .477 9906 | 1088 | .301 7254 | 870 |
| .0009 | .476 1450 | 1083 | .300 2490 | 867 | .478 0994 | 1088 | .301 8124 | 871 |
| 0.0010 | 0.476 0367 | 1083 | 0.300 1623 | 867 | 0.478 2082 | 1088 | 0.301 8995 | 870 |
| .0011 | .475 9284 | 1083 | .300 0756 | 867 | .478 3170 | 1089 | .301 9865 | 870 |
| .0012 | .475 8201 | 1083 | .299 9889 | 866 | .478 4259 | 1089 | .302 0736 | 870 |
| .0013 | .475 7118 | 1083 | .299 9023 | 866 | .478 5348 | 1089 | .302 1606 | 871 |
| .0014 | .475 6035 | 1082 | .299 8157 | 866 | .478 6437 | 1089 | .302 2477 | 871 |
| 0.0015 | 0.475 4953 | 1082 | 0.299 7291 | 866 | 0.478 7526 | 1089 | 0.302 3348 | 872 |
| .0016 | .475 3871 | 1082 | .299 6425 | 866 | .478 8615 | 1090 | .302 4220 | 871 |
| .0017 | .475 2789 | 1082 | .299 5559 | 866 | .478 9705 | 1090 | .302 5091 | 872 |
| .0018 | .475 1707 | 1081 | .299 4693 | 865 | .479 0795 | 1090 | .302 5963 | 872 |
| .0019 | .475 0626 | 1081 | .299 3828 | 865 | .479 1885 | 1090 | .302 6835 | 872 |
| 0.0020 | 0.474 9545 | 1081 | 0.299 2963 | 865 | 0.479 2975 | 1090 | 0.302 7707 | 872 |
| .0021 | .474 8464 | 1081 | .299 2098 | 865 | .479 4065 | 1091 | .302 8579 | 872 |
| .0022 | .474 7383 | 1080 | .299 1233 | 865 | .479 5156 | 1091 | .302 9451 | 873 |
| .0023 | .474 6303 | 1080 | .299 0368 | 864 | .479 6247 | 1091 | .303 0324 | 872 |
| .0024 | .474 5223 | 1080 | .298 9504 | 865 | .479 7338 | 1092 | .303 1196 | 873 |
| 0.0025 | 0.474 4143 | 1080 | 0.298 8639 | 864 | 0.479 8430 | 1092 | 0.303 2069 | 873 |
| .0026 | .474 3063 | 1080 | .298 7775 | 864 | .479 9522 | 1092 | .303 2942 | 873 |
| .0027 | .474 1983 | 1079 | .298 6911 | 864 | .480 0614 | 1092 | .303 3815 | 874 |
| .0028 | .474 0904 | 1079 | .298 6047 | 863 | .480 1706 | 1092 | .303 4689 | 873 |
| .0029 | .473 9825 | 1079 | .298 5184 | 864 | .480 2798 | 1093 | .303 5562 | 874 |
| 0.0030 | 0.473 8746 | 1079 | 0.298 4320 | 863 | 0.480 3891 | 1093 | 0.303 6436 | 874 |
| .0031 | .473 7667 | 1078 | .298 3457 | 863 | .480 4984 | 1093 | .303 7310 | 874 |
| .0032 | .473 6589 | 1078 | .298 2594 | 863 | .480 6077 | 1093 | .303 8184 | 874 |
| .0033 | .473 5511 | 1078 | .298 1731 | 863 | .480 7170 | 1094 | .303 9058 | 875 |
| .0034 | .473 4433 | 1078 | .298 0868 | 863 | .480 8264 | 1094 | .303 9933 | 874 |
| 0.0035 | 0.473 3355 | 1077 | 0.298 0005 | 862 | 0.480 9358 | 1094 | 0.304 0807 | 875 |
| .0036 | .473 2278 | 1077 | .297 9143 | 863 | .481 0452 | 1095 | .304 1682 | 875 |
| .0037 | .473 1201 | 1077 | .297 8280 | 862 | .481 1547 | 1094 | .304 2557 | 875 |
| .0038 | .473 0124 | 1077 | .297 7418 | 862 | .481 2641 | 1095 | .304 3432 | 876 |
| .0039 | .472 9047 | 1077 | .297 6556 | 861 | .481 3736 | 1095 | .304 4308 | 875 |
| 0.0040 | 0.472 7970 | 1076 | 0.297 5695 | 862 | 0.481 4831 | 1095 | 0.304 5183 | 876 |
| .0041 | .472 6894 | 1076 | .297 4833 | 861 | .481 5926 | 1096 | .304 6059 | 876 |
| .0042 | .472 5818 | 1076 | .297 3972 | 862 | .481 7022 | 1096 | .304 6935 | 876 |
| .0043 | .472 4742 | 1076 | .297 3110 | 861 | .481 8118 | 1096 | .304 7811 | 876 |
| .0044 | .472 3666 | 1075 | .297 2249 | 861 | .481 9214 | 1096 | .304 8687 | 876 |
| 0.0045 | 0.472 2591 | 1075 | 0.297 1388 | 860 | 0.482 0310 | 1097 | 0.304 9563 | 877 |
| .0046 | .472 1516 | 1075 | .297 0528 | 861 | .482 1407 | 1097 | .305 0440 | 877 |
| .0047 | .472 0441 | 1075 | .296 9667 | 860 | .482 2504 | 1097 | .305 1317 | 877 |
| .0048 | .471 9366 | 1074 | .296 8807 | 861 | .482 3601 | 1097 | .305 2194 | 877 |
| .0049 | .471 8292 | 1074 | .296 7946 | 860 | .482 4698 | 1098 | .305 3071 | 877 |
| 0.0050 | 0.471 7218 | 1074 | 0.296 7086 | 860 | 0.482 5796 | 1098 | 0.305 3948 | 877 |
| .0051 | .471 6144 | 1074 | .296 6226 | 859 | .482 6894 | 1098 | .305 4825 | 878 |
| .0052 | .471 5070 | 1074 | .296 5367 | 860 | .482 7992 | 1098 | .305 5703 | 878 |
| .0053 | .471 3996 | 1073 | .296 4507 | 859 | .482 9090 | 1098 | .305 6581 | 878 |
| .0054 | .471 2923 | 1073 | .296 3648 | 860 | .483 0188 | 1099 | .305 7459 | 878 |
| 0.0055 | 0.471 1850 | 1073 | 0.296 2788 | 859 | 0.483 1287 | 1099 | 0.305 8337 | 878 |
| .0056 | .471 0777 | 1073 | .296 1929 | 859 | .483 2386 | 1099 | .305 9215 | 879 |
| .0057 | .470 9704 | 1072 | .296 1070 | 858 | .483 3485 | 1099 | .306 0094 | 879 |
| .0058 | .470 8632 | 1072 | .296 0212 | 859 | .483 4584 | 1100 | .306 0973 | 878 |
| .0059 | .470 7560 | 1072 | .295 9353 | 858 | .483 5684 | 1100 | .306 1851 | 879 |
| .0060 | .470 6488 | 1072 | .295 8495 | 858 | .483 6784 | | .306 2730 | |

TABLE XVII.
For special Perturbations.

| q, q', q'' | For positive values of the Argument. | | | | For negative values of the Argument. | | | |
|--------------|--------------------------------------|-------|----------------------|-------|--------------------------------------|-------|----------------------|-------|
| | log f | Diff. | log f' , log f'' | Diff. | log f | Diff. | log f' , log f'' | Diff. |
| .0.0060 | 0.470 6488 | | 0.295 8495 | 858 | 0.483 6784 | | 0.306 2730 | 880 |
| .0061 | .470 5416 | 1072 | .295 7637 | 858 | .483 7884 | 1100 | .306 3610 | 880 |
| .0062 | .470 4345 | 1071 | .295 6779 | 858 | .483 8984 | 1100 | .306 4489 | 879 |
| .0063 | .470 3274 | 1071 | .295 5921 | 858 | .484 0085 | 1101 | .306 5369 | 880 |
| .0064 | .470 2203 | 1071 | .295 5063 | 858 | .484 1186 | 1101 | .306 6248 | 879 |
| .0.0065 | 0.470 1132 | | 0.295 4205 | 857 | 0.484 2287 | | 0.306 7128 | 881 |
| .0066 | .470 0062 | 1070 | .295 3348 | 857 | .484 3388 | 1101 | .306 8009 | 880 |
| .0067 | .469 8992 | 1070 | .295 3491 | 857 | .484 4490 | 1102 | .306 8889 | 880 |
| .0068 | .469 7922 | 1070 | .295 1634 | 857 | .484 5592 | 1102 | .306 9769 | 881 |
| .0069 | .469 6852 | 1070 | .295 0777 | 857 | .484 6694 | 1102 | .307 0650 | 881 |
| .0.0070 | 0.469 5782 | | 0.294 9920 | 856 | 0.484 7796 | | 0.307 1531 | 881 |
| .0071 | .469 4713 | 1069 | .294 9064 | 856 | .484 8898 | 1102 | .307 2412 | 881 |
| .0072 | .469 3644 | 1069 | .294 8208 | 856 | .485 0001 | 1103 | .307 3293 | 881 |
| .0073 | .469 2575 | 1069 | .294 7351 | 856 | .485 1104 | 1103 | .307 4174 | 882 |
| .0074 | .469 1506 | 1069 | .294 6495 | 855 | .485 2207 | 1104 | .307 5056 | 882 |
| .0.0075 | 0.469 0437 | | 0.294 5640 | 856 | 0.485 3311 | | 0.307 5938 | 882 |
| .0076 | .468 9369 | 1068 | .294 4784 | 856 | .485 4415 | 1104 | .307 6820 | 882 |
| .0077 | .468 8301 | 1068 | .294 3928 | 856 | .485 5519 | 1104 | .307 7702 | 882 |
| .0078 | .468 7233 | 1068 | .294 3073 | 855 | .485 6623 | 1104 | .307 8584 | 882 |
| .0079 | .468 6165 | 1067 | .294 2218 | 855 | .485 7728 | 1105 | .307 9466 | 883 |
| .0.0080 | 0.468 5098 | | 0.294 1363 | 855 | 0.485 8833 | | 0.308 0349 | 883 |
| .0081 | .468 4031 | 1067 | .294 0508 | 855 | .485 9938 | 1105 | .308 1232 | 883 |
| .0082 | .468 2964 | 1067 | .293 9653 | 854 | .486 1043 | 1106 | .308 2115 | 883 |
| .0083 | .468 1897 | 1066 | .293 8799 | 854 | .486 2149 | 1106 | .308 2998 | 883 |
| .0084 | .468 0831 | 1066 | .293 7945 | 855 | .486 3255 | 1106 | .308 3881 | 884 |
| .0.0085 | 0.467 9765 | | 0.293 7090 | 854 | 0.486 4361 | | 0.308 4765 | 883 |
| .0086 | .467 8699 | 1066 | .293 6236 | 854 | .486 5467 | 1106 | .308 5648 | 884 |
| .0087 | .467 7633 | 1066 | .293 5383 | 853 | .486 6573 | 1106 | .308 6532 | 884 |
| .0088 | .467 6567 | 1066 | .293 4529 | 854 | .486 7680 | 1107 | .308 7416 | 884 |
| .0089 | .467 5502 | 1065 | .293 3675 | 854 | .486 8787 | 1107 | .308 8301 | 885 |
| .0.0090 | 0.467 4437 | | 0.293 2822 | 853 | 0.486 9894 | | 0.308 9185 | 885 |
| .0091 | .467 3372 | 1065 | .293 1969 | 853 | .487 1001 | 1107 | .309 0070 | 885 |
| .0092 | .467 2307 | 1064 | .293 1116 | 853 | .487 2109 | 1108 | .309 0954 | 885 |
| .0093 | .467 1243 | 1064 | .293 0263 | 852 | .487 3217 | 1108 | .309 1839 | 886 |
| .0094 | .467 0179 | 1064 | .292 9411 | 853 | .487 4325 | 1108 | .309 2725 | 885 |
| .0.0095 | 0.466 9115 | | 0.292 8558 | 852 | 0.487 5433 | | 0.309 3610 | 885 |
| .0096 | .466 8051 | 1064 | .292 7706 | 852 | .487 6542 | 1109 | .309 4495 | 886 |
| .0097 | .466 6988 | 1063 | .292 6854 | 852 | .487 7651 | 1109 | .309 5381 | 886 |
| .0098 | .466 5925 | 1063 | .292 6002 | 852 | .487 8760 | 1109 | .309 6267 | 886 |
| .0099 | .466 4862 | 1063 | .292 5150 | 852 | .487 9869 | 1110 | .309 7153 | 886 |
| .0.0100 | 0.466 3799 | | 0.292 4298 | 851 | 0.488 0979 | | 0.309 8039 | 887 |
| .0101 | .466 2736 | 1062 | .292 3447 | 852 | .488 2089 | 1110 | .309 8926 | 886 |
| .0102 | .466 1674 | 1062 | .292 2595 | 851 | .488 3199 | 1110 | .309 9812 | 887 |
| .0103 | .466 0612 | 1062 | .292 1744 | 851 | .488 4309 | 1111 | .310 0699 | 887 |
| .0104 | .465 9550 | 1062 | .292 0893 | 850 | .488 5420 | 1111 | .310 1586 | 887 |
| .0.0105 | 0.465 8488 | | 0.292 0043 | 851 | 0.488 6531 | | 0.310 2473 | 887 |
| .0106 | .465 7427 | 1061 | .291 9192 | 851 | .488 7642 | 1111 | .310 3360 | 888 |
| .0107 | .465 6366 | 1061 | .291 8341 | 850 | .488 8753 | 1112 | .310 4248 | 888 |
| .0108 | .465 5305 | 1061 | .291 7491 | 850 | .488 9865 | 1112 | .310 5136 | 887 |
| .0109 | .465 4244 | 1061 | .291 6641 | 850 | .489 0977 | 1112 | .310 6023 | 888 |
| .0.0110 | 0.465 3183 | | 0.291 5791 | 850 | 0.489 2089 | | 0.310 6911 | 889 |
| .0111 | .465 2123 | 1060 | .291 4941 | 849 | .489 3201 | 1112 | .310 7800 | 888 |
| .0112 | .465 1063 | 1060 | .291 4092 | 850 | .489 4314 | 1113 | .310 8688 | 889 |
| .0113 | .465 0003 | 1060 | .291 3242 | 849 | .489 5427 | 1113 | .310 9577 | 888 |
| .0114 | .464 8943 | 1059 | .291 2393 | 849 | .489 6540 | 1113 | .311 0465 | 889 |
| .0.0115 | 0.464 7884 | | 0.291 1544 | 849 | 0.489 7653 | | 0.311 1354 | 889 |
| .0116 | .464 6825 | 1059 | .291 0695 | 849 | .489 8767 | 1114 | .311 2243 | 890 |
| .0117 | .464 5766 | 1059 | .290 9846 | 849 | .489 9881 | 1114 | .311 3133 | 889 |
| .0118 | .464 4707 | 1059 | .290 8997 | 848 | .490 0995 | 1114 | .311 4022 | 890 |
| .0119 | .464 3648 | 1058 | .290 8149 | 849 | .490 2109 | 1114 | .311 4912 | 890 |
| .0120 | .464 2590 | | .290 7300 | | .490 3223 | | .311 5802 | 890 |

TABLE XVII.
For special Perturbations.

| ϕ, ϕ', ϕ'' | For positive values of the Argument. | | | | For negative values of the Argument. | | | |
|-----------------------|--------------------------------------|-------|---------------------|-------|--------------------------------------|-------|---------------------|-------|
| | $\log f$ | Diff. | $\log f', \log f''$ | Diff. | $\log f$ | Diff. | $\log f', \log f''$ | Diff. |
| .0.0120 | .464 2590 | 1058 | .290 7300 | 848 | .490 3223 | 1115 | .311 5802 | 890 |
| .0121 | .464 1532 | 1058 | .290 6452 | 848 | .490 4338 | 1115 | .311 6692 | 890 |
| .0122 | .464 0474 | 1058 | .290 5604 | 848 | .490 5453 | 1115 | .311 7582 | 890 |
| .0123 | .463 9416 | 1057 | .290 4756 | 847 | .490 6568 | 1116 | .311 8472 | 890 |
| .0124 | .463 8359 | 1057 | .290 3909 | 848 | .490 7684 | 1116 | .311 9363 | 891 |
| .0.0125 | .463 7302 | 1057 | .290 3061 | 847 | .490 8800 | 1116 | .312 0254 | 891 |
| .0126 | .463 6245 | 1057 | .290 2214 | 847 | .490 9916 | 1116 | .312 1145 | 891 |
| .0127 | .463 5188 | 1057 | .290 1367 | 847 | .491 1032 | 1117 | .312 2036 | 891 |
| .0128 | .463 4132 | 1056 | .290 0520 | 847 | .491 2149 | 1117 | .312 2927 | 892 |
| .0129 | .463 3076 | 1056 | .289 9673 | 847 | .491 3266 | 1117 | .312 3819 | 891 |
| .0.0130 | .463 2020 | 1056 | .289 8826 | 846 | .491 4383 | 1117 | .312 4710 | 892 |
| .0131 | .463 0964 | 1056 | .289 7980 | 846 | .491 5500 | 1118 | .312 5602 | 892 |
| .0132 | .462 9908 | 1055 | .289 7134 | 847 | .491 6618 | 1118 | .312 6494 | 893 |
| .0133 | .462 8853 | 1055 | .289 6287 | 846 | .491 7736 | 1118 | .312 7387 | 892 |
| .0134 | .462 7798 | 1055 | .289 5441 | 845 | .491 8854 | 1118 | .312 8279 | 893 |
| .0.0135 | .462 6743 | 1055 | .289 4596 | 846 | .491 9972 | 1119 | .313 9172 | 892 |
| .0136 | .462 5688 | 1055 | .289 3750 | 846 | .492 1091 | 1119 | .313 0064 | 893 |
| .0137 | .462 4633 | 1054 | .289 2904 | 845 | .492 2210 | 1119 | .313 0957 | 893 |
| .0138 | .462 3579 | 1054 | .289 2059 | 845 | .492 3329 | 1119 | .313 1850 | 894 |
| .0139 | .462 2525 | 1054 | .289 1214 | 845 | .492 4448 | 1119 | .313 2744 | 893 |
| .0.0140 | .462 1471 | 1054 | .289 0369 | 845 | .492 5567 | 1120 | .313 3637 | 894 |
| .0141 | .462 0417 | 1053 | .288 9524 | 845 | .492 6687 | 1120 | .313 4531 | 894 |
| .0142 | .461 9364 | 1053 | .288 8679 | 844 | .492 7807 | 1120 | .313 5425 | 894 |
| .0143 | .461 8311 | 1053 | .288 7835 | 845 | .492 8927 | 1120 | .313 6319 | 894 |
| .0144 | .461 7258 | 1053 | .288 6990 | 844 | .493 0047 | 1121 | .313 7213 | 895 |
| .0.0145 | .461 6205 | 1052 | .288 6146 | 844 | .493 1168 | 1121 | .313 8108 | 894 |
| .0146 | .461 5153 | 1052 | .288 5302 | 844 | .493 2289 | 1121 | .313 9002 | 895 |
| .0147 | .461 4101 | 1052 | .288 4458 | 843 | .493 3410 | 1122 | .313 9897 | 895 |
| .0148 | .461 3049 | 1052 | .288 3615 | 844 | .493 4532 | 1122 | .314 0792 | 895 |
| .0149 | .461 1997 | 1052 | .288 2771 | 843 | .493 5654 | 1122 | .314 1687 | 896 |
| .0.0150 | .461 0945 | 1051 | .288 1928 | 843 | .493 6776 | 1122 | .314 2583 | 895 |
| .0151 | .460 9894 | 1051 | .288 1085 | 844 | .493 7898 | 1123 | .314 3478 | 896 |
| .0152 | .460 8843 | 1051 | .288 0241 | 842 | .493 9021 | 1123 | .314 4374 | 896 |
| .0153 | .460 7792 | 1051 | .287 9399 | 843 | .494 0144 | 1123 | .314 5270 | 896 |
| .0154 | .460 6741 | 1051 | .287 8556 | 843 | .494 1267 | 1123 | .314 6166 | 896 |
| .0.0155 | .460 5690 | 1050 | .287 7713 | 842 | .494 2390 | 1124 | .314 7062 | 897 |
| .0156 | .460 4640 | 1050 | .287 6871 | 842 | .494 3514 | 1124 | .314 7959 | 896 |
| .0157 | .460 3590 | 1050 | .287 6029 | 842 | .494 4638 | 1124 | .314 8855 | 897 |
| .0158 | .460 2540 | 1050 | .287 5187 | 842 | .494 5762 | 1124 | .314 9752 | 897 |
| .0159 | .460 1490 | 1049 | .287 4345 | 842 | .494 6886 | 1124 | .315 0649 | 897 |
| .0.0160 | .460 0441 | 1049 | .287 3503 | 842 | .494 8010 | 1125 | .315 1546 | 898 |
| .0161 | .459 9392 | 1049 | .287 2661 | 841 | .494 9135 | 1125 | .315 2444 | 897 |
| .0162 | .459 8343 | 1049 | .287 1820 | 841 | .495 0260 | 1125 | .315 3341 | 898 |
| .0163 | .459 7294 | 1049 | .287 0979 | 841 | .495 1385 | 1125 | .315 4239 | 898 |
| .0164 | .459 6245 | 1048 | .287 0138 | 841 | .495 2510 | 1126 | .315 5137 | 898 |
| .0.0165 | .459 5197 | 1048 | .286 9297 | 841 | .495 3636 | 1126 | .315 6035 | 899 |
| .0166 | .459 4149 | 1048 | .286 8456 | 841 | .495 4762 | 1126 | .315 6934 | 898 |
| .0167 | .459 3101 | 1048 | .286 7615 | 840 | .495 5888 | 1127 | .315 7832 | 899 |
| .0168 | .459 2053 | 1047 | .286 6775 | 840 | .495 7015 | 1127 | .315 8731 | 899 |
| .0169 | .459 1006 | 1047 | .286 5935 | 840 | .495 8142 | 1127 | .315 9630 | 899 |
| .0.0170 | .458 9959 | 1047 | .286 5095 | 840 | .495 9269 | 1127 | .316 0529 | 899 |
| .0171 | .458 8912 | 1047 | .286 4255 | 840 | .496 0396 | 1128 | .316 1428 | 899 |
| .0172 | .458 7865 | 1047 | .286 3415 | 840 | .496 1524 | 1128 | .316 2327 | 900 |
| .0173 | .458 6818 | 1046 | .286 2575 | 839 | .496 2652 | 1128 | .316 3227 | 900 |
| .0174 | .458 5772 | 1046 | .286 1736 | 840 | .496 3780 | 1128 | .316 4127 | 900 |
| .0.0175 | .458 4726 | 1046 | .286 0896 | 839 | .496 4908 | 1129 | .316 5027 | 900 |
| .0176 | .458 3680 | 1046 | .286 0057 | 839 | .496 6037 | 1129 | .316 5927 | 900 |
| .0177 | .458 2634 | 1045 | .285 9218 | 838 | .496 7166 | 1129 | .316 6827 | 901 |
| .0178 | .458 1589 | 1045 | .285 8380 | 839 | .496 8295 | 1129 | .316 7728 | 901 |
| .0179 | .458 0544 | 1045 | .285 7541 | 839 | .496 9424 | 1130 | .316 8629 | 901 |
| .0180 | .457 9499 | 1045 | .285 6702 | 839 | .497 0554 | | .316 9530 | 901 |

TABLE XVII.
For special Perturbations.

| ϕ, ϕ', ϕ'' | For positive values of the Argument. | | | | For negative values of the Argument. | | | |
|-----------------------|--------------------------------------|-------|----------------------|-------|--------------------------------------|-------|----------------------|-------|
| | log f | Diff. | log f' , log f'' | Diff. | log f | Diff. | log f' , log f'' | Diff. |
| 0.0180 | 0.457 9499 | 1045 | 0.285 6702 | 838 | 0.497 0554 | 1130 | 0.316 9530 | 901 |
| 0.0181 | 457 8454 | 1045 | .285 5864 | 838 | .497 1684 | 1130 | .317 0431 | 901 |
| 0.0182 | 457 7409 | 1044 | .285 5026 | 838 | .497 2814 | 1130 | .317 1332 | 902 |
| 0.0183 | 457 6365 | 1044 | .285 4188 | 838 | .497 3944 | 1131 | .317 2234 | 901 |
| 0.0184 | 457 5321 | 1044 | .285 3350 | 838 | .497 5075 | 1131 | .317 3135 | 902 |
| 0.0185 | 0.457 4277 | 1044 | 0.285 2512 | 837 | 0.497 6206 | 1131 | 0.317 4037 | 902 |
| 0.0186 | 457 3233 | 1044 | .285 1675 | 837 | .497 7337 | 1131 | .317 4939 | 902 |
| 0.0187 | 457 2189 | 1043 | .285 0838 | 838 | .497 8468 | 1132 | .317 5841 | 903 |
| 0.0188 | 457 1146 | 1043 | .285 0000 | 837 | .497 9600 | 1132 | .317 6744 | 902 |
| 0.0189 | 457 0103 | 1043 | .284 9163 | 837 | .498 0732 | 1132 | .317 7646 | 903 |
| 0.0190 | 0.456 9060 | 1043 | 0.284 8326 | 836 | 0.498 1864 | 1132 | 0.317 8549 | 903 |
| 0.0191 | 456 8017 | 1042 | .284 7490 | 837 | .498 2996 | 1133 | .317 9452 | 903 |
| 0.0192 | 456 6975 | 1042 | .284 6653 | 836 | .498 4129 | 1133 | .318 0355 | 904 |
| 0.0193 | 456 5933 | 1042 | .284 5817 | 836 | .498 5262 | 1133 | .318 1259 | 903 |
| 0.0194 | 456 4891 | 1042 | .284 4981 | 836 | .498 6395 | 1133 | .318 2162 | 904 |
| 0.0195 | 0.456 3849 | 1041 | 0.284 4145 | 836 | 0.498 7528 | 1134 | 0.318 3066 | 904 |
| 0.0196 | 456 2808 | 1041 | .284 3309 | 836 | .498 8662 | 1134 | .318 3970 | 904 |
| 0.0197 | 456 1767 | 1041 | .284 2473 | 836 | .498 9796 | 1134 | .318 4874 | 904 |
| 0.0198 | 456 0726 | 1041 | .284 1637 | 835 | .499 0930 | 1134 | .318 5778 | 905 |
| 0.0199 | 455 9685 | 1041 | .284 0802 | 835 | .499 2064 | 1135 | .318 5683 | 905 |
| 0.0200 | 0.455 8644 | 1040 | 0.283 9967 | 835 | 0.499 3199 | 1135 | 0.318 7588 | 904 |
| 0.0201 | 455 7604 | 1040 | .283 9132 | 835 | .499 4334 | 1135 | .318 8492 | 906 |
| 0.0202 | 455 6564 | 1040 | .283 8297 | 835 | .499 5469 | 1135 | .318 9398 | 905 |
| 0.0203 | 455 5524 | 1040 | .283 7462 | 835 | .499 6604 | 1136 | .319 0303 | 905 |
| 0.0204 | 455 4484 | 1040 | .283 6627 | 834 | .499 7740 | 1136 | .319 1208 | 906 |
| 0.0205 | 0.455 3444 | 1039 | 0.283 5793 | 835 | 0.499 8876 | 1136 | 0.319 2114 | 906 |
| 0.0206 | 455 2405 | 1039 | .283 4958 | 834 | .500 0012 | 1137 | .319 3020 | 906 |
| 0.0207 | 455 1366 | 1039 | .283 4124 | 834 | .500 1149 | 1137 | .319 3926 | 906 |
| 0.0208 | 455 0327 | 1039 | .283 3290 | 834 | .500 2286 | 1137 | .319 4832 | 906 |
| 0.0209 | 454 9288 | 1039 | .283 2456 | 833 | .500 3423 | 1137 | .319 5738 | 907 |
| 0.0210 | 0.454 8249 | 1038 | 0.283 1623 | 834 | 0.500 4560 | 1137 | 0.319 6645 | 907 |
| 0.0211 | 454 7211 | 1038 | .283 0789 | 833 | .500 5697 | 1138 | .319 7552 | 907 |
| 0.0212 | 454 6173 | 1038 | .282 9956 | 833 | .500 6835 | 1138 | .319 8459 | 907 |
| 0.0213 | 454 5135 | 1038 | .282 9123 | 833 | .500 7973 | 1138 | .319 9366 | 907 |
| 0.0214 | 454 4097 | 1037 | .282 8290 | 833 | .500 9111 | 1139 | .320 0273 | 908 |
| 0.0215 | 0.454 3060 | 1037 | 0.282 7457 | 833 | 0.501 0250 | 1139 | 0.320 1181 | 907 |
| 0.0216 | 454 2023 | 1037 | .282 6624 | 832 | .501 1389 | 1139 | .320 2088 | 908 |
| 0.0217 | 454 0986 | 1037 | .282 5792 | 833 | .501 2528 | 1139 | .320 2996 | 908 |
| 0.0218 | 453 9949 | 1037 | .282 4959 | 832 | .501 3667 | 1140 | .320 3904 | 909 |
| 0.0219 | 453 8912 | 1036 | .282 4127 | 832 | .501 4807 | 1140 | .320 4813 | 908 |
| 0.0220 | 0.453 7876 | 1036 | 0.282 3295 | 832 | 0.501 5947 | 1140 | 0.320 5721 | 909 |
| 0.0221 | 453 6840 | 1036 | .282 2463 | 832 | .501 7087 | 1140 | .320 6630 | 909 |
| 0.0222 | 453 5804 | 1036 | .282 1631 | 831 | .501 8227 | 1141 | .320 7539 | 909 |
| 0.0223 | 453 4768 | 1035 | .282 0800 | 832 | .501 9368 | 1141 | .320 8448 | 909 |
| 0.0224 | 453 3733 | 1035 | .281 9968 | 831 | .502 0509 | 1141 | .320 9357 | 909 |
| 0.0225 | 0.453 2698 | 1035 | 0.281 9137 | 831 | 0.502 1650 | 1141 | 0.321 0266 | 910 |
| 0.0226 | 453 1663 | 1035 | .281 8306 | 831 | .502 2791 | 1142 | .321 1176 | 910 |
| 0.0227 | 453 0628 | 1035 | .281 7475 | 831 | .502 3933 | 1142 | .321 2086 | 910 |
| 0.0228 | 452 9593 | 1035 | .281 6644 | 830 | .502 5075 | 1142 | .321 2996 | 910 |
| 0.0229 | 452 8558 | 1034 | .281 5814 | 831 | .502 6217 | 1143 | .321 3906 | 910 |
| 0.0230 | 0.452 7524 | 1034 | 0.281 4983 | 830 | 0.502 7360 | 1143 | 0.321 4816 | 911 |
| 0.0231 | 452 6490 | 1034 | .281 4153 | 830 | .502 8503 | 1143 | .321 5727 | 910 |
| 0.0232 | 452 5456 | 1034 | .281 3323 | 830 | .502 9646 | 1143 | .321 6637 | 911 |
| 0.0233 | 452 4422 | 1033 | .281 2493 | 830 | .503 0789 | 1143 | .321 7548 | 912 |
| 0.0234 | 452 3389 | 1033 | .281 1663 | 830 | .503 1932 | 1144 | .321 8460 | 911 |
| 0.0235 | 0.452 2356 | 1033 | 0.281 0833 | 829 | 0.503 3076 | 1144 | 0.321 9371 | 911 |
| 0.0236 | 452 1323 | 1033 | .281 0004 | 830 | .503 4220 | 1144 | .322 0282 | 912 |
| 0.0237 | 452 0290 | 1032 | .280 9174 | 829 | .503 5364 | 1144 | .322 1194 | 912 |
| 0.0238 | 451 9258 | 1032 | .280 8345 | 829 | .503 6508 | 1145 | .322 2106 | 912 |
| 0.0239 | 451 8226 | 1032 | .280 7516 | 829 | .503 7653 | 1145 | .322 3018 | 912 |
| 0.0240 | 451 7194 | | .280 6687 | | .503 8798 | | .322 3930 | |

TABLE XVII.
For special Perturbations.

| ϕ, ϕ', ϕ'' | For positive values of the Argument. | | | | For negative values of the Argument. | | | |
|-----------------------|--------------------------------------|-------|----------------------|-------|--------------------------------------|-------|----------------------|-------|
| | log f | Diff. | log f' , log f'' | Diff. | log f | Diff. | log f' , log f'' | Diff. |
| 0.0240 | 0.451 7194 | 1032 | 0.280 6687 | 829 | 0.503 8798 | 1145 | 0.322 3930 | 913 |
| 0.0241 | 0.451 6162 | 1032 | .280 5858 | 828 | 0.503 9943 | 1146 | .322 4843 | 913 |
| 0.0242 | 0.451 5130 | 1031 | .280 5030 | 829 | 0.504 1089 | 1146 | .322 5756 | 912 |
| 0.0243 | 0.451 4099 | 1031 | .280 4201 | 828 | 0.504 2235 | 1146 | .322 6668 | 913 |
| 0.0244 | 0.451 3068 | 1031 | .280 3373 | 828 | 0.504 3381 | 1146 | .322 7581 | 914 |
| 0.0245 | 0.451 2037 | 1031 | 0.280 2545 | 828 | 0.504 4527 | 1147 | 0.322 8495 | 913 |
| 0.0246 | 0.451 1006 | 1031 | .280 1717 | 828 | 0.504 5674 | 1147 | .322 9408 | 914 |
| 0.0247 | 0.450 9975 | 1030 | .280 0889 | 827 | 0.504 6821 | 1147 | .323 0322 | 914 |
| 0.0248 | 0.450 8945 | 1030 | .280 0062 | 828 | 0.504 7968 | 1147 | .323 1236 | 914 |
| 0.0249 | 0.450 7915 | 1030 | .279 9234 | 827 | 0.504 9115 | 1148 | .323 2150 | 914 |
| 0.0250 | 0.450 6885 | 1030 | 0.279 8407 | 827 | 0.505 0263 | 1148 | 0.323 3064 | 914 |
| 0.0251 | 0.450 5855 | 1030 | .279 7580 | 827 | 0.505 1411 | 1148 | .323 3978 | 915 |
| 0.0252 | 0.450 4825 | 1029 | .279 6753 | 827 | 0.505 2559 | 1148 | .323 4893 | 915 |
| 0.0253 | 0.450 3796 | 1029 | .279 5926 | 827 | 0.505 3707 | 1149 | .323 5808 | 915 |
| 0.0254 | 0.450 2767 | 1029 | .279 5099 | 826 | 0.505 4856 | 1149 | .323 6723 | 915 |
| 0.0255 | 0.450 1738 | 1029 | 0.279 4273 | 827 | 0.505 6005 | 1149 | 0.323 7638 | 915 |
| 0.0256 | 0.450 0709 | 1028 | .279 3446 | 826 | 0.505 7154 | 1149 | .323 8553 | 916 |
| 0.0257 | 0.449 9681 | 1028 | .279 2620 | 826 | 0.505 8303 | 1150 | .323 9469 | 915 |
| 0.0258 | 0.449 8653 | 1028 | .279 1794 | 826 | 0.505 9453 | 1150 | .324 0384 | 916 |
| 0.0259 | 0.449 7625 | 1028 | .279 0968 | 825 | 0.506 0603 | 1150 | .324 1300 | 917 |
| 0.0260 | 0.449 6597 | 1028 | 0.279 0143 | 826 | 0.506 1753 | 1150 | 0.324 2217 | 916 |
| 0.0261 | 0.449 5569 | 1027 | .278 9317 | 825 | 0.506 2903 | 1151 | .324 3133 | 916 |
| 0.0262 | 0.449 4542 | 1027 | .278 8492 | 826 | 0.506 4054 | 1151 | .324 4049 | 917 |
| 0.0263 | 0.449 3515 | 1027 | .278 7666 | 825 | 0.506 5205 | 1151 | .324 4966 | 917 |
| 0.0264 | 0.449 2488 | 1027 | .278 6841 | 825 | 0.506 6356 | 1152 | .324 5883 | 917 |
| 0.0265 | 0.449 1461 | 1026 | 0.278 6016 | 825 | 0.506 7508 | 1152 | 0.324 6800 | 917 |
| 0.0266 | 0.449 0435 | 1026 | .278 5191 | 824 | 0.506 8660 | 1153 | .324 7717 | 918 |
| 0.0267 | 0.448 9409 | 1026 | .278 4367 | 825 | 0.506 9813 | 1152 | .324 8635 | 918 |
| 0.0268 | 0.448 8383 | 1026 | .278 3542 | 824 | 0.507 0965 | 1152 | .324 9553 | 917 |
| 0.0269 | 0.448 7357 | 1026 | .278 2718 | 824 | 0.507 2117 | 1153 | .325 0470 | 919 |
| 0.0270 | 0.448 6331 | 1026 | 0.278 1894 | 824 | 0.507 3270 | 1153 | 0.325 1389 | 918 |
| 0.0271 | 0.448 5305 | 1025 | .278 1070 | 824 | 0.507 4423 | 1154 | .325 2307 | 918 |
| 0.0272 | 0.448 4280 | 1025 | .278 0246 | 824 | 0.507 5577 | 1154 | .325 3225 | 919 |
| 0.0273 | 0.448 3255 | 1025 | .277 9422 | 823 | 0.507 6731 | 1154 | .325 4144 | 919 |
| 0.0274 | 0.448 2230 | 1025 | .277 8599 | 824 | 0.507 7885 | 1154 | .325 5063 | 919 |
| 0.0275 | 0.448 1205 | 1024 | 0.277 7775 | 823 | 0.507 9039 | 1155 | 0.325 5982 | 919 |
| 0.0276 | 0.448 0181 | 1024 | .277 6952 | 823 | 0.508 0194 | 1155 | .325 6901 | 920 |
| 0.0277 | 0.447 9157 | 1024 | .277 6129 | 823 | 0.508 1349 | 1155 | .325 7821 | 919 |
| 0.0278 | 0.447 8133 | 1024 | .277 5306 | 823 | 0.508 2504 | 1155 | .325 8740 | 920 |
| 0.0279 | 0.447 7109 | 1024 | .277 4483 | 822 | 0.508 3659 | 1155 | .325 9660 | 920 |
| 0.0280 | 0.447 6085 | 1023 | 0.277 3661 | 823 | 0.508 4814 | 1156 | 0.326 0580 | 920 |
| 0.0281 | 0.447 5062 | 1023 | .277 2838 | 822 | 0.508 5970 | 1156 | .326 1500 | 921 |
| 0.0282 | 0.447 4039 | 1023 | .277 2016 | 822 | 0.508 7126 | 1156 | .326 2421 | 920 |
| 0.0283 | 0.447 3016 | 1023 | .277 1194 | 822 | 0.508 8282 | 1157 | .326 3341 | 921 |
| 0.0284 | 0.447 2993 | 1023 | .277 0372 | 822 | 0.508 9439 | 1157 | .326 4262 | 921 |
| 0.0285 | 0.447 0970 | 1022 | 0.276 9550 | 822 | 0.509 0596 | 1157 | 0.326 5183 | 921 |
| 0.0286 | 0.446 9948 | 1022 | .276 8728 | 821 | 0.509 1753 | 1157 | .326 6104 | 922 |
| 0.0287 | 0.446 8926 | 1022 | .276 7907 | 821 | 0.509 2910 | 1158 | .326 7026 | 921 |
| 0.0288 | 0.446 7904 | 1022 | .276 7086 | 822 | 0.509 4068 | 1158 | .326 7947 | 922 |
| 0.0289 | 0.446 6882 | 1021 | .276 6264 | 821 | 0.509 5226 | 1158 | .326 8869 | 922 |
| 0.0290 | 0.446 5861 | 1021 | 0.276 5443 | 821 | 0.509 6384 | 1159 | 0.326 9791 | 922 |
| 0.0291 | 0.446 4840 | 1021 | .276 4622 | 820 | 0.509 7543 | 1159 | .327 0713 | 922 |
| 0.0292 | 0.446 3819 | 1021 | .276 3802 | 821 | 0.509 8702 | 1159 | .327 1635 | 923 |
| 0.0293 | 0.446 2798 | 1021 | .276 2981 | 820 | 0.509 9861 | 1159 | .327 2558 | 923 |
| 0.0294 | 0.446 1777 | 1021 | .276 2161 | 821 | 0.510 1020 | 1159 | .327 3481 | 923 |
| 0.0295 | 0.446 0756 | 1020 | 0.276 1340 | 820 | 0.510 2179 | 1160 | 0.327 4404 | 923 |
| 0.0296 | 0.445 9736 | 1020 | .276 0520 | 820 | 0.510 3339 | 1160 | .327 5327 | 923 |
| 0.0297 | 0.445 8716 | 1020 | .275 9700 | 820 | 0.510 4499 | 1160 | .327 6250 | 924 |
| 0.0298 | 0.445 7696 | 1020 | .275 8880 | 819 | 0.510 5659 | 1160 | .327 7174 | 923 |
| 0.0299 | 0.445 6676 | 1019 | .275 8061 | 820 | 0.510 6819 | 1161 | .327 8097 | 924 |
| 0.0300 | 0.445 5657 | | .275 7241 | | 0.510 7980 | | .327 9021 | |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | τ | h | m | s | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-----|-----|-----|-------|----------|-----|----------|----------|-------------|--------------|
| 1 | 66 Jan. 14, | 14 | 5 | 0 | 325 | 0 | 0 | 1.0 | 9.6480 | Retrograde. | Hind. |
| 2 | 141 March 29, | 2 | 0 | 0 | 251 | 55 | 0 | 1.0 | 9.8573 | " | " |
| 3 | 240 Nov. 10, | 0 | 0 | 0 | 271 | 0 | 0 | 1.0 | 9.5700 | Direct. | Burckhardt. |
| 4 | 539 Oct. 20, | 15 | 0 | 0 | 313 | 30 | 0 | 1.0 | 9.5307 | " | " |
| 5 | 565 July 9, | 0 | 0 | 0 | 88 | 0 | 0 | 1.0 | 9.85686 | Retrograde. | " |
| 6 | 568 Aug. 28, | 6 | 28 | 49 | 316 | 47 | 0 | 1.0 | 9.9491 | Direct. | Hind. |
| 7 | 574 April 7, | 6 | 43 | 14 | 143 | 39 | 0 | 1.0 | 9.9836 | " | " |
| 8 | 770 June 6, | 14 | 6 | 1 | 357 | 7 | 0 | 1.0 | 9.807664 | Retrograde. | Laugier. |
| 9 | 837 March 1, | 0 | 0 | 0 | 289 | 3 | 0 | 1.0 | 9.763428 | " | Pingré. |
| 10 | 961 Dec. 30, | 3 | 50 | 25 | 268 | 3 | 0 | 1.0 | 9.7418 | " | Hind. |
| 11 | 989 Sept. 12, | 0 | 0 | 0 | 264 | 0 | 0 | 1.0 | 9.7546 | " | Burckhardt. |
| 12 | 1066 April 1, | 0 | 0 | 0 | 264 | 55 | 0 | 1.0 | 9.8573 | " | Hind. |
| 13 | 1092 Feb. 15, | 0 | 0 | 0 | 156 | 20 | 0 | 1.0 | 9.9076 | Direct. | " |
| 14 | 1097 Sept. 21, | 21 | 26 | 39 | 332 | 30 | 0 | 1.0 | 9.86832 | " | Burckhardt. |
| 15 | 1281 Jan. 30, | 7 | 12 | 40 | 134 | 48 | 0 | 1.0 | 9.9767 | " | Pingré. |
| 16 | 1264 July 12, | 13 | 31 | 0 | 241 | 38 | 0 | 1.0 | 9.4938 | " | Hoek. |
| 17 | 1299 March 31, | 7 | 29 | 0 | 3 | 20 | 0 | 1.0 | 9.50233 | Retrograde. | Pingré. |
| 18 | 1337 June 15, | 1 | 46 | 0 | 2 | 20 | 0 | 1.0 | 9.01815 | " | Laugier. |
| 19 | 1366 Oct. 13, | 0 | 0 | 0 | 66 | 0 | 0 | 1.0 | 9.98140 | Direct. | Peirce. |
| 20 | 1378 Nov. 8, | 18 | 19 | 27 | 299 | 31 | 0 | 1.0 | 9.76604 | Retrograde. | Laugier. |
| 21 | 1385 Oct. 16, | 6 | 14 | 25 | 101 | 47 | 0 | 1.0 | 9.88860 | " | Hind. |
| 22 | 1433 Nov. 4, | 10 | 9 | 51 | 281 | 2 | 0 | 1.0 | 9.53079 | " | Laugier. |
| 23 | 1436 June 8, | 22 | 0 | 40 | 301 | 0 | 0 | 1.0 | 9.76754 | " | Pingré. |
| 24 | 1468 Oct. 7, | 9 | 49 | 40 | 356 | 3 | 0 | 1.0 | 9.93109 | " | Laugier. |
| 25 | 1472 Feb. 28, | 5 | 13 | 13 | 48 | 3 | 0 | 1.0 | 9.51718 | " | " |
| 26 | 1490 Dec. 24, | 11 | 16 | 50 | 58 | 40 | 0 | 1.0 | 9.8678 | Direct. | Hind. |
| 27 | 1491 Jan. 4, | 21 | 35 | 0 | 113 | 0 | 0 | 1.0 | 9.8780 | Retrograde. | Peirce. |
| 28 | 1506 Sept. 3, | 15 | 52 | 34 | 250 | 37 | 0 | 1.0 | 9.86565 | " | Laugier. |
| 29 | 1531 Aug. 25, | 19 | 0 | 40 | 301 | 12 | 0 | 0.967391 | 9.763380 | " | Halley. |
| 30 | 1532 Oct. 19, | 14 | 53 | 0 | 135 | 44 | 0 | 1.0 | 9.787141 | Direct. | Méchain. |
| 31 | 1533 June 14, | 21 | 11 | 25 | 217 | 40 | 0 | 1.0 | 9.514362 | " | Olbers. |
| 32 | 1556 April 22, | 0 | 25 | 0 | 274 | 15 | 0 | 1.0 | 9.70323 | " | Hind. |
| 33 | 1558 Aug. 10, | 12 | 24 | 45 | 329 | 49 | 0 | 1.0 | 9.76140 | Retrograde. | Olbers. |
| 34 | 1577 Oct. 26, | 22 | 44 | 36 | 129 | 42 | 0 | 1.0 | 9.24920 | " | Woldstedt. |
| 35 | 1580 Nov. 28, | 13 | 6 | 39 | 108 | 29 | 20 | 0.998631 | 9.77982 | Direct. | Schjellerup. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | π | h m s | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-------------------|-----------|-----------|----------|-------------|-----------|-------------|----------------------|
| 36 | 1582 May 6, | 9 51 22 | 256 15 18 | 229 18 1 | 60 47 3 | 1.0 | 9.226156 | Retrograde. | D'Arrest. |
| 37 | 1585 Oct. 8, | 0 38 44 | 8 8 26 | 37 41 15 | 6 5 52 | 1.0 | 0.0393531 | Direct. | Peters and Sawitsch. |
| 38 | 1590 Feb. 8, | 0 39 4 | 217 57 12 | 165 36 56 | 29 29 44 | 1.0 | 9.7541386 | Retrograde. | Hind. |
| 39 | 1593 July 18, | 13 39 0 | 176 19 0 | 164 15 0 | 87 58 0 | 1.0 | 8.949940 | Direct. | La Caille. |
| 40 | 1596 July 25, | 5 8 38 | 270 54 35 | 330 20 49 | 51 58 10 | 1.0 | 9.7537024 | Retrograde. | Hind. |
| 41 | 1607 Oct. 26, | 17 10 58 | 301 38 10 | 48 40 28 | 17 12 17 | 0.9670888 | 9.769358 | " | Bessel. |
| 42 | 1618 Aug. 17, | 3 2 0 | 318 20 0 | 293 25 0 | 21 18 0 | 1.0 | 9.710100 | Direct. | Pingré. |
| 43 | 1618 Nov. 8, | 8 25 1 | 3 5 21 | 75 44 10 | 37 11 31 | 1.0 | 9.590556 | " | Bessel. |
| 44 | 1652 Nov. 12, | 15 41 0 | 28 18 40 | 88 10 0 | 79 28 0 | 1.0 | 9.928140 | " | Halley. |
| 45 | 1661 Jan. 26, | 21 9 0 | 115 16 8 | 81 54 0 | 33 0 55 | 1.0 | 9.646131 | " | Méchain. |
| 46 | 1664 Dec. 4, | 11 36 5 | 130 33 15 | 81 15 52 | 21 18 12 | 1.0 | 0.010949 | Retrograde. | Lindelöf. |
| 47 | 1665 April 24, | 5 16 0 | 71 54 30 | 228 2 0 | 76 5 0 | 1.0 | 9.027309 | " | Halley. |
| 48 | 1668 Feb. 24, | 18 46 6 | 40 9 0 | 193 26 0 | 27 7 0 | 1.0 | 9.39990 | Direct. | Henderson. |
| 49 | 1672 March 1, | 8 38 0 | 46 59 30 | 297 30 30 | 83 22 10 | 1.0 | 9.843476 | " | Halley. |
| 50 | 1677 May 6, | 0 38 0 | 137 37 5 | 236 49 10 | 79 3 15 | 1.0 | 9.448072 | Retrograde. | " |
| 51 | 1678 Aug. 18, | 7 34 0 | 322 47 37 | 163 20 0 | 2 52 0 | 0.626970 | 0.058919 | Direct. | Le Verrier. |
| 52 | 1680 Dec. 17, | 23 46 9 | 262 49 5 | 272 9 29 | 60 40 16 | 0.999985417 | 7.7939551 | " | Encke. |
| 53 | 1682 Sept. 14, | 19 4 53 | 301 55 37 | 51 11 18 | 17 44 45 | 0.96792019 | 9.7655898 | Retrograde. | Rosenberger. |
| 54 | 1683 July 13, | 17 25 15 | 86 31 15 | 173 17 48 | 83 47 46 | 0.9832470 | 9.7430148 | " | Clausen. |
| 55 | 1684 June 8, | 10 17 0 | 238 52 0 | 268 15 0 | 65 48 40 | 1.0 | 9.982339 | Direct. | Halley. |
| 56 | 1686 Sept. 16, | 14 34 0 | 77 0 30 | 350 34 40 | 31 21 40 | 1.0 | 9.511883 | " | " |
| 57 | 1689 Nov. 29, | 4 48 1 | 269 41 0 | 90 25 0 | 59 5 0 | 1.0 | 8.27720 | Retrograde. | Vogel. |
| 58 | 1695 Nov. 9, | 17 0 0 | 60 0 0 | 216 0 0 | 22 0 0 | 1.0 | 9.9261 | Direct. | Burkhardt. |
| 59 | 1698 Oct. 18, | 16 58 0 | 270 51 15 | 267 44 15 | 11 46 0 | 1.0 | 9.839660 | Retrograde. | Halley. |
| 60 | 1699 Jan. 13, | 8 23 0 | 212 31 6 | 321 45 35 | 69 20 0 | 1.0 | 9.871570 | " | La Caille. |
| 61 | 1701 Oct. 17, | 9 51 0 | 133 41 0 | 298 41 0 | 41 39 0 | 1.0 | 9.77278 | " | Burkhardt. |
| 62 | 1702 March 13, | 14 33 22 | 138 46 34 | 188 59 10 | 4 24 44 | 1.0 | 9.810790 | Direct. | " |
| 63 | 1706 Jan. 30, | 4 57 0 | 72 36 25 | 13 11 23 | 55 14 5 | 1.0 | 9.630291 | " | Struyck. |
| 64 | 1707 Dec. 11, | 23 30 0 | 79 54 56 | 52 46 35 | 88 36 0 | 1.0 | 9.934368 | " | La Caille. |
| 65 | 1718 Jan. 14, | 21 44 16 | 121 39 55 | 127 55 29 | 31 8 6 | 1.0 | 0.010908 | Retrograde. | Anglander. |
| 66 | 1723 Sept. 27, | 15 4 9 | 42 52 35 | 14 14 17 | 50 0 18 | 0.9994743 | 9.9994743 | " | Spoerer. |
| 67 | 1729 June 13, | 6 19 27 | 320 31 22 | 310 38 0 | 77 5 18 | 1.0050334 | 0.6067570 | Direct. | Burkhardt. |
| 68 | 1737 Jan. 30, | 8 21 0 | 325 55 0 | 226 22 0 | 18 20 45 | 1.0 | 9.347960 | " | Bradley. |
| 69 | 1737 June 8, | 7 39 0 | 262 36 39 | 123 53 43 | 39 14 5 | 1.0 | 9.93802 | " | Dausy. |
| 70 | 1739 June 17, | 10 0 0 | 102 38 40 | 207 25 14 | 55 42 44 | 1.0 | 9.828388 | Retrograde. | La Caille. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | h m s | π ° ' " | Ω ° ' " | i ° ' " | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-------------------|----------------|-------------------|--------------|------------|-----------|-------------|--------------|
| 71 | 1742 Feb. 8, | 4 21 14 | 217 33 44 | 185 34 45 | 67 4 11 | 1.0 | 9.883976 | Retrograde. | Sturyc. |
| 72 | 1743 Jan. 10, | 20 20 16 | 92 57 51 | 67 31 57 | 2 16 16 | 1.0 | 9.923338 | Direct. | Olbers. |
| 73 | 1743 Sept. 20, | 14 11 13 | 247 15 37 | 6 15 29 | 45 38 10 | 1.0 | 9.719016 | Retrograde. | D'Arrest. |
| 74 | 1744 March 1, | 7 55 39 | 197 13 58 | 45 47 54 | 47 7 41 | 1.0 | 9.346842 | Direct. | Wolters. |
| 75 | 1747 March 3, | 9 57 19 | 277 2 5 | 147 18 42 | 79 6 45 | 1.0 | 0.342144 | Retrograde. | Maraldi. |
| 76 | 1748 April 28, | 19 25 24 | 215 0 50 | 232 52 16 | 85 26 57 | 1.0 | 9.924626 | " | " |
| 77 | 1748 June 18, | 21 18 1 | 278 47 10 | 33 8 29 | 67 3 28 | 1.0 | 9.976128 | Direct. | Bessel. |
| 78 | 1757 Oct. 21, | 7 54 39 | 122 58 0 | 214 12 50 | 12 50 20 | 1.0 | 9.528328 | " | Bradley. |
| 79 | 1758 June 11, | 3 17 39 | 267 38 0 | 230 50 0 | 68 19 0 | 1.0 | 9.333148 | " | Pingé. |
| 80 | 1759 March 12, | 13 14 34 | 303 10 28 | 53 50 27 | 17 36 52 | 0.96768436 | 9.7667989 | Retrograde. | Rosenberger. |
| 81 | 1759 Nov. 27, | 0 33 58 | 53 38 4 | 139 40 15 | 79 3 19 | 1.0 | 9.904218 | Direct. | Chappe. |
| 82 | 1759 Dec. 16, | 12 48 51 | 139 3 52 | 79 20 24 | 4 42 10 | 1.0 | 9.983064 | Retrograde. | " |
| 83 | 1762 May 28, | 8 1 42 | 104 2 0 | 348 33 5 | 85 38 13 | 1.0 | 0.003912 | Direct. | Burckhardt. |
| 84 | 1763 Nov. 1, | 20 54 58 | 84 57 27 | 356 17 38 | 72 34 10 | 0.9954268 | 9.6974946 | " | Lexell. |
| 85 | 1764 Feb. 12, | 13 42 15 | 15 14 52 | 120 4 33 | 52 53 31 | 1.0 | 9.744462 | Retrograde. | Pingé. |
| 86 | 1766 Feb. 17, | 8 41 0 | 143 15 25 | 244 10 50 | 40 50 20 | 1.0 | 9.703570 | " | " |
| 87 | 1766 April 26, | 23 43 55 | 251 13 0 | 74 11 0 | 8 1 45 | 0.864000 | 9.6009521 | Direct. | Burckhardt. |
| 88 | 1769 Oct. 7, | 14 53 22 | 144 11 29 | 175 3 59 | 40 45 50 | 0.99924901 | 9.089039 | " | Bessel. |
| 89 | 1770 Aug. 14, | 0 38 36 | 356 16 27 | 131 59 34 | 1 34 31 | 0.786839 | 9.8288597 | " | Le Verrier. |
| 90 | 1770 Nov. 22, | 5 39 0 | 208 22 44 | 108 42 10 | 31 25 55 | 1.0 | 9.722833 | Retrograde. | Pingé. |
| 91 | 1771 April 19, | 5 6 19 | 104 3 16 | 57 51 55 | 11 15 19 | 1.0093698 | 9.9559104 | Direct. | Encke. |
| 92 | 1772 Feb. 16, | 15 43 40 | 110 8 35 | 257 15 38 | 17 3 8 | 0.724510 | 9.993890 | " | Hubbard. |
| 93 | 1773 Sept. 5, | 14 1 50 | 75 17 0 | 121 8 20 | 61 15 11 | 1.0024901 | 0.052420 | " | Lexell. |
| 94 | 1774 Aug. 15, | 19 55 21 | 317 27 40 | 180 44 34 | 83 20 26 | 1.0282955 | 0.1562065 | " | Burckhardt. |
| 95 | 1779 Jan. 4, | 2 4 20 | 87 14 27 | 25 4 10 | 32 30 57 | 1.0 | 9.853186 | " | Zach. |
| 96 | 1780 Sept. 30, | 22 13 53 | 246 35 59 | 123 41 18 | 54 23 12 | 0.9999460 | 8.9836418 | Retrograde. | Clüver. |
| 97 | 1780 Nov. 28, | 20 21 0 | 246 52 0 | 142 1 0 | 72 3 30 | 1.0 | 9.712041 | " | Olbers. |
| 98 | 1781 July 7, | 4 31 59 | 239 11 25 | 83 0 38 | 81 43 26 | 1.0 | 9.889784 | Direct. | Méchain. |
| 99 | 1781 Nov. 29, | 12 33 25 | 16 3 7 | 77 22 55 | 27 12 4 | 1.0 | 9.982723 | Retrograde. | Legendre. |
| 100 | 1783 Nov. 19, | 11 50 50 | 50 3 8 | 55 45 20 | 44 53 24 | 0.5395345 | 0.1626829 | Direct. | Burckhardt. |
| 101 | 1784 Jan. 21, | 4 47 26 | 80 44 24 | 56 49 21 | 51 9 12 | 1.0 | 9.849946 | Retrograde. | Méchain. |
| 102 | 1785 Jan. 27, | 7 48 43 | 109 51 56 | 264 12 15 | 70 14 12 | 1.0 | 0.058198 | Direct. | " |
| 103 | 1785 April 8, | 8 58 51 | 297 29 33 | 64 33 36 | 87 31 54 | 1.0 | 9.630733 | Retrograde. | " |
| 104 | 1786 Jan. 30, | 20 57 51 | 156 38 0 | 334 8 0 | 13 36 0 | 0.84836 | 9.524810 | Direct. | Encke. |
| 105 | 1786 July 8, | 13 37 10 | 158 38 30 | 195 23 32 | 50 58 33 | 1.0 | 9.595763 | " | Reggio. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | h m s | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-------------------|-----------|-----------|----------|------------|-----------|-------------|--------------|
| 106 | 1787 May 10, | 19 48 39 | 0 | 106 51 35 | 48 15 51 | 1.0 | 9.542714 | Retrograde. | Saron. |
| 107 | 1788 Nov. 10, | 7 25 26 | 7 44 9 | 156 56 43 | 12 27 40 | 1.0 | 0.026538 | " | Méchain. |
| 108 | 1788 Nov. 20, | 7 15 39 | 22 49 54 | 352 24 26 | 64 30 24 | 1.0 | 9.879276 | Direct. | " |
| 109 | 1790 Jan. 16, | 18 58 9 | 58 24 45 | 172 50 2 | 29 44 7 | 1.0 | 9.873516 | Retrograde. | Saron. |
| 110 | 1790 Jan. 28, | 7 36 9 | 111 44 37 | 267 8 37 | 56 58 13 | 1.0 | 0.026650 | Direct. | Méchain. |
| 111 | 1790 May 21, | 5 46 54 | 273 43 27 | 33 11 2 | 63 52 27 | 1.0 | 9.901981 | Retrograde. | " |
| 112 | 1792 Jan. 13, | 12 50 15 | 36 20 32 | 190 42 9 | 39 45 47 | 1.0 | 0.111456 | " | Zach. |
| 113 | 1792 Dec. 27, | 7 47 9 | 135 52 35 | 283 14 44 | 49 7 14 | 1.0 | 9.985350 | " | Piazzi. |
| 114 | 1793 Nov. 4, | 20 12 0 | 228 42 0 | 108 29 0 | 60 21 0 | 1.0 | 9.605736 | " | Saron. |
| 115 | 1793 Nov. 20, | 5 6 21 | 71 54 3 | 2 0 12 | 51 31 10 | 0.9734211 | 0.1746744 | Direct. | D'Arrest. |
| 116 | 1795 Dec. 21, | 10 35 1 | 156 41 20 | 334 39 22 | 13 42 30 | 0.8488828 | 9.5243046 | " | Encke. |
| 117 | 1796 April 2, | 19 47 42 | 192 44 13 | 17 2 16 | 64 54 33 | 1.0 | 0.198151 | Retrograde. | Olbers. |
| 118 | 1797 July 9, | 2 31 10 | 49 27 8 | 329 15 37 | 50 40 34 | 1.0 | 9.721489 | " | " |
| 119 | 1797 April 4, | 11 58 16 | 105 6 57 | 122 12 21 | 43 44 42 | 1.0 | 9.685370 | Direct. | Burckhardt. |
| 120 | 1798 Dec. 31, | 13 17 3 | 34 27 27 | 249 30 30 | 42 26 4 | 1.0 | 9.891829 | Retrograde. | " |
| 121 | 1799 Sept. 7, | 5 50 36 | 3 38 16 | 99 23 3 | 51 2 27 | 1.0 | 9.924437 | " | Wahl. |
| 122 | 1799 Dec. 25, | 18 3 46 | 190 22 46 | 326 30 18 | 77 5 4 | 1.0 | 9.795483 | " | " |
| 123 | 1801 Aug. 8, | 13 23 0 | 183 49 0 | 44 28 0 | 21 20 0 | 1.0 | 9.417804 | " | Burckhardt. |
| 124 | 1802 Sept. 9, | 21 23 5 | 332 9 4 | 310 15 39 | 57 0 47 | 1.0 | 0.039061 | " | Olbers. |
| 125 | 1804 Feb. 13, | 14 6 55 | 148 44 51 | 176 47 58 | 56 28 40 | 1.0 | 0.029858 | Direct. | Gauss. |
| 126 | 1805 Nov. 21, | 11 59 50 | 156 47 24 | 334 20 10 | 13 33 30 | 0.84617529 | 9.5320168 | " | Encke. |
| 127 | 1806 Jan. 1, | 22 1 10 | 109 28 25 | 251 16 19 | 13 36 34 | 0.7457068 | 9.9576440 | " | Hubbard. |
| 128 | 1806 Dec. 28, | 22 9 2 | 97 3 24 | 322 23 16 | 35 2 33 | 1.010182 | 0.034189 | Retrograde. | Hensel |
| 129 | 1807 Sept. 18, | 17 43 59 | 270 54 42 | 266 47 11 | 63 10 28 | 0.99548781 | 9.8103158 | Direct. | Bessel. |
| 130 | 1808 May 12, | 22 52 4 | 69 12 57 | 322 58 36 | 45 43 7 | 1.0 | 9.59091 | Retrograde. | Encke. |
| 131 | 1808 July 12, | 4 0 58 | 252 38 50 | 24 11 15 | 39 18 59 | 1.0 | 9.783870 | " | Bessel. |
| 132 | 1810 Sept. 29, | 2 23 31 | 52 44 42 | 310 21 2 | 61 11 15 | 1.0 | 9.983355 | Direct. | Tricenecker. |
| 133 | 1811 Sept. 12, | 6 10 32 | 75 0 34 | 140 24 44 | 73 2 21 | 0.99509330 | 0.0131178 | Retrograde. | Argelander. |
| 134 | 1811 Nov. 10, | 23 46 17 | 47 27 27 | 93 1 52 | 31 17 11 | 0.98271088 | 0.1992359 | Direct. | Nicolai. |
| 135 | 1812 Sept. 15, | 7 31 31 | 92 18 44 | 253 1 2 | 72 57 3 | 0.9545412 | 9.8904995 | " | Encke. |
| 136 | 1813 March 4, | 12 38 10 | 69 56 8 | 60 48 24 | 21 13 33 | 1.0 | 9.8445579 | Retrograde. | Nicollet. |
| 137 | 1813 May 19, | 10 1 7 | 197 43 8 | 42 40 15 | 81 2 12 | 1.0 | 0.0849212 | " | Gerling. |
| 138 | 1815 April 25, | 23 48 42 | 149 1 56 | 83 28 34 | 44 29 55 | 0.93121968 | 0.0838109 | Direct. | Bessel. |
| 139 | 1816 March 1, | 8 18 0 | 267 35 33 | 323 14 56 | 43 5 26 | 1.0 | 8.685769 | " | Burckhardt. |
| 140 | 1818 Feb. 7, | 10 55 0 | 95 7 0 | 250 4 | 20 2 24 | 1.0 | 9.865260 | " | Pogson. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | λ | m | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|------------|------------|-----|-------|------------|-----|------|-----------|----------|-------------------|
| | | $^{\circ}$ | $'$ | $''$ | $^{\circ}$ | $'$ | $''$ | | | |
| 141 | 1818 Feb. | 25, | 23 | 0 49 | 182 | 45 | 22 | 70 26 11 | 89 43 48 | Encke |
| 142 | 1818 Dec. | 4, | 2 | 10 2 | 103 | 7 | 5 | 90 7 29 | 62 40 50 | Bessel. |
| 143 | 1819 Jan. | 27, | 6 | 8 53 | 156 | 59 | 12 | 334 33 19 | 13 36 54 | Encke. |
| 144 | 1819 June | 27, | 16 | 52 9 | 287 | 5 | 5 | 273 43 44 | 80 45 53 | Brinkley. |
| 145 | 1819 July | 18, | 21 | 36 18 | 274 | 40 | 51 | 113 10 46 | 10 42 48 | Encke. |
| 146 | 1819 Nov. | 20, | 5 | 53 34 | 67 | 18 | 48 | 77 13 57 | 9 1 16 | " |
| 147 | 1821 March | 21, | 12 | 52 39 | 239 | 29 | 25 | 48 40 56 | 73 33 7 | Rosenberger. |
| 148 | 1822 May | 5, | 13 | 34 52 | 192 | 47 | 45 | 177 25 4 | 53 35 34 | Gambart. |
| 149 | 1822 May | 23, | 23 | 6 40 | 157 | 11 | 44 | 334 25 9 | 13 20 17 | Encke. |
| 150 | 1822 July | 16, | 0 | 35 2 | 219 | 53 | 48 | 97 51 23 | 37 43 4 | v. Heiligenstein. |
| 151 | 1822 Oct. | 23, | 18 | 28 29 | 271 | 40 | 17 | 92 44 42 | 52 39 10 | Encke. |
| 152 | 1823 Dec. | 9, | 10 | 39 29 | 274 | 34 | 30 | 303 3 0 | 76 11 57 | " |
| 153 | 1824 July | 11, | 12 | 18 40 | 260 | 16 | 32 | 234 19 9 | 54 34 9 | Rümker. |
| 154 | 1824 Sept. | 29, | 1 | 23 58 | 4 | 31 | 7 | 279 15 39 | 54 36 59 | Encke. |
| 155 | 1825 May | 30, | 13 | 6 39 | 273 | 55 | 1 | 20 6 8 | 56 41 6 | Clausen. |
| 156 | 1825 Aug. | 18, | 17 | 3 55 | 10 | 14 | 25 | 192 56 10 | 89 41 47 | " |
| 157 | 1825 Sept. | 16, | 6 | 33 18 | 157 | 14 | 31 | 334 27 30 | 13 21 24 | Encke. |
| 158 | 1825 Dec. | 10, | 16 | 7 28 | 318 | 46 | 39 | 215 43 22 | 33 32 53 | Hubbard. |
| 159 | 1826 March | 18, | 10 | 43 9 | 109 | 48 | 47 | 251 27 19 | 13 33 54 | " |
| 160 | 1826 April | 21, | 23 | 27 46 | 117 | 11 | 14 | 197 30 19 | 39 57 24 | Nicolai. |
| 161 | 1826 April | 29, | 0 | 56 13 | 35 | 48 | 13 | 40 29 13 | 5 17 2 | Oliver. |
| 162 | 1826 Oct. | 8, | 22 | 51 14 | 57 | 48 | 24 | 44 6 28 | 25 57 18 | Argander. |
| 163 | 1826 Nov. | 18, | 9 | 47 55 | 315 | 29 | 39 | 235 6 11 | 89 22 9 | Gambart. |
| 164 | 1827 Feb. | 4, | 22 | 7 4 | 33 | 30 | 16 | 184 27 49 | 77 35 35 | v. Heiligenstein. |
| 165 | 1827 June | 7, | 20 | 11 15 | 297 | 31 | 42 | 318 10 28 | 43 38 45 | " |
| 166 | 1827 Sept. | 11, | 16 | 37 44 | 250 | 57 | 12 | 149 39 11 | 54 4 42 | Clüver. |
| 167 | 1829 Jan. | 9, | 17 | 54 7 | 157 | 17 | 53 | 334 29 32 | 13 20 34 | Encke. |
| 168 | 1830 April | 9, | 6 | 43 30 | 212 | 11 | 38 | 206 21 36 | 21 16 27 | Carlini. |
| 169 | 1830 Dec. | 27, | 15 | 50 58 | 310 | 59 | 19 | 337 53 7 | 44 45 30 | Wolfer. |
| 170 | 1832 May | 3, | 23 | 24 45 | 157 | 21 | 1 | 334 32 9 | 13 22 9 | " |
| 171 | 1832 Sept. | 25, | 12 | 38 58 | 227 | 54 | 36 | 72 26 49 | 43 18 41 | Peters. |
| 172 | 1832 Nov. | 26, | 9 | 36 44 | 109 | 56 | 24 | 248 11 49 | 13 11 48 | Santini. |
| 173 | 1833 Sept. | 10, | 9 | 29 30 | 224 | 21 | 23 | 323 28 17 | 7 18 17 | Hartwig. |
| 174 | 1834 April | 2, | 15 | 55 11 | 276 | 33 | 49 | 226 48 52 | 5 56 52 | Petersen. |
| 175 | 1835 March | 30, | 16 | 29 51 | 206 | 9 | 24 | 58 55 57 | 9 2 42 | Rümker. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | h | m | s | π | Ω | i | δ | $\log q$ | Motion. | Computed by |
|-----|----------------|-----|-----|-----|-----------|-----------|----------|-------------|-----------|-------------|-----------------------|
| 176 | 1835 Aug. 26, | 8 | 39 | 32 | 157 23 29 | 334 34 59 | 13 21 15 | 0.8450356 | 9.5371089 | Direct. | Encke. |
| 177 | 1835 Nov. 16, | 22 | 32 | 1 | 304 31 32 | 55 9 59 | 17 45 13 | 0.96739091 | 9.7683194 | Retrograde. | Westphalen. |
| 178 | 1838 Dec. 19, | 0 | 17 | 38 | 157 27 4 | 334 36 41 | 13 21 18 | 0.8451775 | 9.5366085 | Direct. | Encke. |
| 179 | 1840 Jan. 4, | 10 | 13 | 42 | 192 11 50 | 119 57 46 | 53 5 52 | 1.0002050 | 9.7913017 | " | Peters and O. Struve. |
| 180 | 1840 March 12, | 23 | 46 | 32 | 80 18 10 | 236 49 6 | 59 13 20 | 0.9978836 | 0.0868593 | Retrograde. | Plantamour. |
| 181 | 1840 April 12, | 11 | 53 | 27 | 324 12 27 | 186 2 45 | 79 51 52 | 1.0 | 9.8740948 | Direct. | Rümker. |
| 182 | 1840 Nov. 13, | 15 | 27 | 55 | 22 31 40 | 248 56 22 | 57 57 23 | 0.96985265 | 0.1705070 | " | Goetze. |
| 183 | 1842 April 12, | 0 | 26 | 9 | 157 29 27 | 334 39 10 | 13 20 26 | 0.8447904 | 9.5378361 | " | Encke. |
| 184 | 1842 Dec. 15, | 22 | 57 | 39 | 327 16 13 | 207 49 1 | 73 33 37 | 1.0 | 9.7026605 | Retrograde. | Laugier. |
| 185 | 1843 Feb. 27, | 9 | 51 | 9 | 278 40 17 | 1 14 55 | 35 40 39 | 0.999915717 | 7.7433765 | " | Hubbard. |
| 186 | 1843 May 6, | 1 | 20 | 33 | 281 29 43 | 157 14 54 | 52 44 46 | 1.0001798 | 0.2085316 | Direct. | Goetze. |
| 187 | 1843 Oct. 17, | 3 | 33 | 46 | 49 33 52 | 209 29 36 | 11 22 33 | 0.5558997 | 0.2284632 | " | Möller. |
| 188 | 1844 Sept. 2, | 11 | 22 | 36 | 342 30 50 | 63 49 0 | 2 54 50 | 0.6176539 | 0.0742308 | " | Brünnow. |
| 189 | 1844 Oct. 17, | 8 | 15 | 15 | 179 35 57 | 31 39 6 | 48 36 1 | 0.9996083 | 9.9321644 | Retrograde. | Plantamour. |
| 190 | 1844 Dec. 13, | 16 | 11 | 42 | 296 2 18 | 118 19 22 | 45 38 47 | 1.00035303 | 9.4009126 | Direct. | Bond. |
| 191 | 1845 Jan. 8, | 3 | 58 | 19 | 91 20 22 | 336 44 13 | 46 50 39 | 1.0 | 9.9567652 | " | Hind. |
| 192 | 1845 April 21, | 0 | 44 | 37 | 192 33 19 | 347 6 45 | 56 23 36 | 0.0985330 | 0.0085330 | " | Faye. |
| 193 | 1845 June 5, | 16 | 9 | 44 | 262 2 56 | 337 48 56 | 48 41 59 | 0.9898742 | 9.603823 | Retrograde. | D'Arrest. |
| 194 | 1845 Aug. 9, | 15 | 1 | 50 | 157 44 21 | 334 19 33 | 13 7 34 | 0.8474362 | 9.5291008 | Direct. | Encke. |
| 195 | 1846 Jan. 22, | 2 | 15 | 11 | 89 6 22 | 111 8 26 | 47 26 6 | 0.9924026 | 0.1704680 | " | Jelinek. |
| 196 | 1846 Feb. 10, | 22 | 10 | 22 | 109 2 54 | 245 54 17 | 12 34 55 | 0.7566060 | 9.9327096 | " | Hubbard. |
| 197 | 1846 Feb. 25, | 8 | 58 | 39 | 116 28 15 | 102 40 58 | 30 55 53 | 0.7933880 | 9.8129825 | " | Brünnow. |
| 198 | 1846 March 5, | 13 | 5 | 18 | 90 27 0 | 77 33 33 | 85 5 42 | 0.9628911 | 9.8219813 | " | Van Deïuse. |
| 199 | 1846 May 27, | 19 | 44 | 55 | 82 39 20 | 161 18 29 | 57 36 24 | 1.0 | 0.182020 | Retrograde. | Graham. |
| 200 | 1846 June 1, | 5 | 5 | 53 | 240 7 35 | 260 28 59 | 30 24 24 | 0.7213385 | 0.1842997 | Direct. | C. H. F. Peters. |
| 201 | 1846 June 5, | 11 | 30 | 5 | 162 5 40 | 261 52 51 | 29 18 47 | 0.9899389 | 9.8018857 | Retrograde. | Oudemans. |
| 202 | 1846 Oct. 29, | 6 | 49 | 57 | 98 47 15 | 4 38 18 | 49 39 3 | 0.9933127 | 9.9187601 | Direct. | Quirling. |
| 203 | 1847 March 30, | 21 | 59 | 29 | 276 2 22 | 21 41 52 | 48 39 50 | 0.99991293 | 8.6293024 | " | Hornstein. |
| 204 | 1847 June 12, | 9 | 1 | 39 | 137 41 34 | 173 25 50 | 80 16 57 | 1.0 | 0.3257617 | Retrograde. | D'Arrest. |
| 205 | 1847 Aug. 9, | 8 | 50 | 44 | 246 45 11 | 338 16 57 | 83 26 15 | 0.9985879 | 0.2470052 | " | Mauvais. |
| 206 | 1847 Aug. 9, | 6 | 13 | 10 | 21 20 41 | 76 42 10 | 32 38 24 | 0.9974348 | 0.1715154 | " | Schweizer. |
| 207 | 1847 Sept. 9, | 13 | 1 | 31 | 79 12 6 | 309 48 49 | 19 8 25 | 0.972560 | 9.688297 | Direct. | D'Arrest. |
| 208 | 1847 Nov. 14, | 9 | 36 | 39 | 274 12 57 | 190 49 53 | 71 50 56 | 1.0001326 | 9.5172334 | Retrograde. | Rümker. |
| 209 | 1848 Sept. 8, | 1 | 20 | 40 | 310 34 36 | 211 34 36 | 84 28 22 | 1.0 | 9.5048748 | " | " |
| 210 | 1848 Nov. 26, | 2 | 44 | 10 | 157 47 8 | 334 22 12 | 13 8 36 | 0.8478280 | 9.5276718 | Direct. | Encke. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | h | m | s | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-----|-----|-----|-----------|-----------|----------|------------|-----------|-------------|--------------|
| | | | | | ° ' " | ° ' " | ° ' " | | | | |
| 211 | 1849 Jan. 19, | 8 | 31 | 37 | 63 14 56 | 215 12 51 | 85 2 13 | 1.0 | 9.9820756 | Direct. | Hensel. |
| 212 | 1849 May 26, | 12 | 37 | 26 | 235 45 15 | 202 33 27 | 67 7 50 | 0.9978863 | 0.0641120 | " | Weyer. |
| 213 | 1849 June 8, | 4 | 53 | 15 | 267 6 8 | 30 32 0 | 66 55 19 | 0.997830 | 9.951525 | " | D'Arrest. |
| 214 | 1850 July 28, | 12 | 40 | 16 | 273 25 5 | 92 53 28 | 68 11 24 | 0.9988519 | 0.0340060 | " | Carrington. |
| 215 | 1850 Oct. 19, | 8 | 14 | 20 | 89 16 3 | 205 59 31 | 40 8 53 | 1.0 | 9.7522749 | " | Mauvais. |
| 216 | 1851 April 1, | 22 | 25 | 17 | 49 42 10 | 209 31 16 | 11 21 38 | 0.5549601 | 0.2304281 | " | Möller. |
| 217 | 1851 July 9, | 2 | 39 | 15 | 322 55 55 | 148 24 51 | 13 55 8 | 0.6592674 | 0.0694270 | " | Schulze. |
| 218 | 1851 Aug. 26, | 5 | 37 | 52 | 310 58 49 | 223 40 33 | 38 9 2 | 0.9968586 | 9.9931272 | " | Brösen. |
| 219 | 1851 Sept. 30, | 19 | 8 | 58 | 338 46 26 | 44 21 30 | 73 58 37 | 1.0 | 9.1521784 | " | Klinkerfues. |
| 220 | 1852 March 14, | 19 | 6 | 25 | 157 51 2 | 334 23 21 | 13 7 55 | 0.8476726 | 9.5282054 | " | Encke. |
| 221 | 1852 April 19, | 15 | 15 | 6 | 278 42 18 | 317 29 30 | 49 11 8 | 1.0525041 | 9.9604040 | Retrograde. | Hartwig. |
| 222 | 1852 Sept. 22, | 22 | 38 | 25 | 109 8 16 | 245 51 28 | 12 33 19 | 0.7558650 | 9.9348124 | Direct. | Hubbard. |
| 223 | 1852 Oct. 12, | 18 | 0 | 57 | 43 13 42 | 346 10 0 | 40 55 0 | 0.91891698 | 0.0968963 | " | Westphal. |
| 224 | 1853 Feb. 28, | 23 | 55 | 49 | 153 44 19 | 69 33 36 | 20 13 20 | 0.990412 | 0.0381820 | Retrograde. | Hartwig. |
| 225 | 1853 May 9, | 19 | 39 | 59 | 201 44 37 | 40 57 37 | 57 49 3 | 0.9893194 | 9.9584172 | " | Rümker. |
| 226 | 1853 Sept. 1, | 16 | 54 | 26 | 310 56 59 | 140 31 22 | 61 30 11 | 0.7294246 | 9.4871354 | Direct. | Stockwell. |
| 227 | 1853 Oct. 16, | 14 | 31 | 44 | 302 14 53 | 220 5 52 | 60 59 44 | 1.0012289 | 9.2372363 | Retrograde. | D'Arrest. |
| 228 | 1854 Jan. 2, | 17 | 19 | 36 | 56 38 52 | 227 0 44 | 66 0 44 | 1.0 | 0.3108246 | " | Klinkerfues. |
| 229 | 1854 March 24, | 0 | 20 | 41 | 213 49 14 | 315 27 27 | 82 32 43 | 1.0 | 9.4425551 | " | Mathieu. |
| 230 | 1854 June 22, | 2 | 1 | 43 | 272 58 6 | 347 48 45 | 71 8 21 | 1.0 | 9.8111244 | " | Bruhns. |
| 231 | 1854 Oct. 27, | 12 | 13 | 4 | 94 24 18 | 354 28 31 | 40 54 38 | 0.9933246 | 9.902384 | Direct. | Lesser. |
| 232 | 1854 Dec. 15, | 17 | 11 | 27 | 165 9 25 | 238 7 54 | 14 8 50 | 0.9864041 | 0.1327551 | " | Adam. |
| 233 | 1855 Feb. 5, | 1 | 8 | 11 | 226 37 34 | 189 43 33 | 51 24 19 | 0.9651850 | 0.3411427 | Retrograde. | Tiele. |
| 234 | 1855 May 29, | 10 | 58 | 4 | 239 28 46 | 260 10 48 | 23 9 54 | 0.039970 | 9.751970 | " | Schulze. |
| 235 | 1855 July 1, | 4 | 40 | 0 | 157 53 12 | 334 26 24 | 13 8 9 | 0.8477869 | 9.5277600 | Direct. | Encke. |
| 236 | 1855 Nov. 25, | 9 | 8 | 58 | 86 2 13 | 51 34 31 | 10 11 19 | 0.997255 | 0.090728 | Retrograde. | Hoek. |
| 237 | 1857 March 21, | 8 | 43 | 38 | 74 43 59 | 313 9 37 | 87 56 13 | 0.9992144 | 9.8878700 | Direct. | Schulze. |
| 238 | 1857 March 28, | 16 | 4 | 19 | 115 46 25 | 101 45 15 | 29 48 53 | 0.8022946 | 9.7928091 | " | Bruhns. |
| 239 | 1857 July 17, | 23 | 33 | 10 | 249 36 1 | 23 41 28 | 58 57 51 | 0.9989984 | 9.5652331 | Retrograde. | Villarcieu. |
| 240 | 1857 Aug. 23, | 23 | 54 | 59 | 21 46 51 | 200 49 16 | 32 46 24 | 0.9803714 | 9.8732267 | Direct. | Möller. |
| 241 | 1857 Sept. 30, | 21 | 7 | 5 | 250 7 38 | 14 57 48 | 56 3 21 | 0.9969135 | 9.7504285 | Retrograde. | Linsser. |
| 242 | 1857 Nov. 19, | 1 | 42 | 31 | 44 13 16 | 139 18 42 | 37 48 55 | 0.9969918 | 0.003889 | " | Anwers. |
| 243 | 1857 Nov. 28, | 19 | 36 | 14 | 323 3 9 | 148 27 7 | 13 56 1 | 0.6598094 | 0.0633373 | Direct. | Schulze. |
| 244 | 1858 Feb. 23, | 12 | 34 | 20 | 115 51 35 | 269 3 13 | 54 24 10 | 0.820903 | 0.010940 | " | Bruhns. |
| 245 | 1858 May 2, | 1 | 24 | 32 | 275 39 54 | 113 30 59 | 10 47 55 | 0.7541036 | 9.8858281 | " | Hänsel. |

TABLE XVIII.

Elements of the Orbits of Comets which have been observed.

| No. | T | h | m | s | π | Ω | i | e | $\log q$ | Motion. | Computed by |
|-----|----------------|-----|-----|-----|-------|----------|-----|-------------|-----------|-------------|------------------|
| 246 | 1858 May 2, | 7 | 42 | 37 | 195 | 58 | 44 | 1.0 | 0.0826760 | Direct. | Watson. |
| 247 | 1858 June 5, | 7 | 5 | 39 | 226 | 6 | 5 | 1.0 | 9.7358072 | Retrograde. | Anwers. |
| 248 | 1858 Sept. 20, | 23 | 8 | 51 | 36 | 12 | 31 | 0.99629326 | 9.7622804 | " | Hill. |
| 249 | 1858 Oct. 18, | 8 | 41 | 33 | 157 | 57 | 30 | 0.8463915 | 9.5324034 | Direct. | Encke. |
| 250 | 1858 Sept. 13, | 21 | 26 | 37 | 49 | 51 | 54 | 0.5577360 | 0.2291239 | " | Möller. |
| 251 | 1858 Oct. 12, | 19 | 26 | 46 | 4 | 13 | 18 | 1.0 | 0.1544245 | Retrograde. | Weiss. |
| 252 | 1859 May 29, | 5 | 25 | 38 | 75 | 20 | 31 | 1.0 | 9.303265 | " | Hertzsprung. |
| 253 | 1860 Feb. 16, | 16 | 9 | 30 | 173 | 45 | 21 | 1.0 | 0.078219 | Direct. | Liais. |
| 254 | 1860 March 5, | 17 | 12 | 25 | 50 | 16 | 5 | 1.0 | 0.1167062 | " | Seeling. |
| 255 | 1860 June 16, | 0 | 20 | 56 | 161 | 31 | 10 | 0.997240 | 9.465570 | " | Liais. |
| 256 | 1860 Sept. 28, | 6 | 49 | 0 | 111 | 59 | 0 | 1.0 | 9.9795 | Retrograde. | Valz. |
| 257 | 1861 June 1, | 9 | 21 | 0 | 243 | 22 | 2 | 0.983463143 | 9.9641181 | Direct. | Oppolzer. |
| 258 | 1861 June 11, | 12 | 17 | 7 | 249 | 4 | 27 | 0.9853832 | 9.9150740 | " | Sawitsch. |
| 259 | 1861 Dec. 7, | 4 | 17 | 18 | 173 | 30 | 30 | 1.0 | 9.923813 | Retrograde. | Pape. |
| 260 | 1862 Feb. 6, | 4 | 7 | 49 | 158 | 0 | 10 | 0.8467094 | 9.5313486 | Direct. | Encke. |
| 261 | 1862 June 22, | 0 | 43 | 59 | 229 | 20 | 27 | 1.0 | 9.991818 | Retrograde. | Seeling. |
| 262 | 1862 Aug. 22, | 21 | 53 | 32 | 344 | 41 | 26 | 0.9612708 | 9.9834648 | " | Oppolzer. |
| 263 | 1862 Dec. 28, | 8 | 33 | 28 | 125 | 9 | 43 | 1.0 | 9.904475 | " | Engelmann. |
| 264 | 1863 Feb. 8, | 11 | 47 | 16 | 191 | 22 | 45 | 0.9999470 | 9.9002349 | Direct. | " |
| 265 | 1863 April 4, | 21 | 42 | 13 | 247 | 15 | 25 | 1.0 | 0.0286067 | Retrograde. | Frtschaut. |
| 266 | 1863 April 20, | 20 | 39 | 7 | 395 | 31 | 6 | 1.0 | 9.798266 | Direct. | Karlinski. |
| 267 | 1863 Nov. 9, | 11 | 35 | 16 | 94 | 43 | 17 | 1.0 | 9.849171 | " | Stampfer. |
| 268 | 1863 Dec. 27, | 18 | 19 | 44 | 60 | 24 | 28 | 1.0 | 9.887344 | " | Weiss. |
| 269 | 1863 Dec. 29, | 4 | 0 | 45 | 183 | 7 | 18 | 1.0006499 | 0.1183045 | " | Rosén. |
| 270 | 1864 July 27, | 19 | 50 | 29 | 185 | 31 | 54 | 1.0 | 9.822162 | Retrograde. | Valentin. |
| 271 | 1864 Aug. 15, | 13 | 46 | 54 | 394 | 11 | 52 | 0.9967771 | 9.9587003 | " | Kowalczyk. |
| 272 | 1864 Oct. 11, | 9 | 41 | 54 | 159 | 18 | 2 | 0.99995324 | 9.9690407 | " | von Asten. |
| 273 | 1864 Dec. 22, | 11 | 7 | 31 | 321 | 42 | 31 | 1.0 | 9.886982 | Direct. | Tiefen. |
| 274 | 1864 Dec. 27, | 17 | 16 | 20 | 162 | 23 | 36 | 1.0 | 0.0471352 | Retrograde. | Valentuer. |
| 275 | 1865 Jan. 14, | 8 | 10 | 23 | 141 | 15 | 37 | 1.0 | 8.4152071 | " | Tebbutt. |
| 276 | 1866 Jan. 11, | 3 | 12 | 47 | 60 | 28 | 48 | 0.9054198 | 9.9896813 | " | Oppolzer. |
| 277 | 1866 Feb. 14, | 0 | 29 | 48 | 49 | 56 | 55 | 0.5575382 | 0.2258707 | Direct. | Möller. |
| 278 | 1867 Jan. 19, | 20 | 39 | 15 | 75 | 52 | 15 | 0.8490551 | 0.1965869 | " | Searle. |
| 279 | 1867 Feb. 27, | 20 | 17 | 25 | 162 | 40 | 17 | 1.0 | 0.050900 | " | C. F. W. Peters. |
| 280 | | | | | | | | | | | |

TABLE XIX

Elements of the Orbits of the Minor Planets.

| No. | Name. | Epoch and Mean Equinox. Berlin Mean Time. | M | | | π | | | Ω | | | i | | | ϕ | | | μ | log a | Date of Discovery. | Discoverer. |
|-----|-------------|--|-----|----|------|-------|----|------|----------|------|------|-----|----|------|--------|------|------------|-----------|---------------|--------------------|--------------|
| | | | ° | ' | " | ° | ' | " | ° | ' | " | ° | ' | " | ° | ' | " | " | | | |
| 1 | Ceres. | 1866 Jan. 21.0 | 337 | 10 | 35.7 | 148 | 22 | 8.4 | 0 | 50 | 7.2 | 34 | 42 | 38.8 | 4 | 36 | 2.8 | 771.02418 | 0.4419590 | 1801 Jan. | Piazzi. |
| 2 | Pallas. | 1866 June 19.0 | 11 | 56 | 49.0 | 122 | 1 | 16.8 | 172 | 43 | 55.6 | 34 | 42 | 38.8 | 13 | 53 | 17.5 | 769.80066 | 0.442415 | 1802 March 28 | Others. |
| 3 | Juno. | 1865 Nov. 3.0 | 329 | 20 | 8.9 | 54 | 56 | 31.9 | 170 | 49 | 36.8 | 13 | 1 | 21.3 | 14 | 54 | 25.8 | 814.0068 | 0.4262524 | 1804 Sept. | Harding. |
| 4 | Vesta. | 1810 Jan. 0.0 | 216 | 42 | 25.8 | 249 | 19 | 28.6 | 103 | 11 | 22.1 | 7 | 8 | 5.0 | 5 | 36.3 | 977.638563 | 0.3732203 | 1807 March 29 | Others. | |
| 5 | Astræa. | 1866 Sept. 19.0 | 234 | 23 | 32.5 | 135 | 14 | 48.1 | 141 | 26 | 18.3 | 5 | 19 | 9.0 | 10 | 48 | 30.1 | 857.60520 | 0.4111462 | 1845 Dec. | Hencke. |
| 6 | Hebe. | 1866 June 30.0 | 283 | 17 | 20.5 | 15 | 6 | 12.7 | 138 | 39 | 17.3 | 14 | 46 | 43.9 | 11 | 41 | 34.8 | 939.08225 | 0.3848687 | 1847 July 1 | Hencke. |
| 7 | Iris. | 1850 Jan. 0.0 | 166 | 7 | 9.0 | 41 | 23 | 21.1 | 259 | 47 | 55.8 | 5 | 28 | 3.0 | 13 | 20 | 50.2 | 902.58662 | 0.3777130 | 1847 Aug. | Hind. |
| 8 | Flora. | 1848 Jan. 1.0 | 53 | 54 | 3.6 | 32 | 54 | 28.3 | 110 | 17 | 48.6 | 5 | 35 | 8.0 | 9 | 56.3 | 1086.33998 | 0.3426903 | 1847 Oct. | Hind. | |
| 9 | Motis. | 1858 June 30.0 | 57 | 4 | 34.7 | 71 | 3 | 52.1 | 68 | 31 | 35.2 | 5 | 36 | 0.3 | 7 | 5 | 24 | 902.33898 | 0.3777857 | 1848 April 25 | Graham. |
| 10 | Hygeia. | 1864 Feb. 22.0 | 199 | 13 | 22.0 | 235 | 10 | 29.2 | 286 | 43 | 1.8 | 3 | 49 | 0.2 | 5 | 44 | 56.4 | 634.31180 | 0.4984692 | 1849 April 12 | Gasparis. |
| 11 | Parthenope. | 1865 March 27.0 | 239 | 14 | 15.1 | 317 | 14 | 31.4 | 125 | 7 | 27.9 | 4 | 37 | 1.6 | 5 | 42 | 54.1 | 924.15366 | 0.3895083 | 1850 May 11 | Gasparis. |
| 12 | Victoria. | 1851 Jan. 0.0 | 66 | 2 | 39.9 | 301 | 39 | 25.0 | 235 | 34 | 47.7 | 8 | 23 | 17.7 | 12 | 38 | 44.9 | 994.83472 | 0.3681389 | 1850 Sept. | Hind. |
| 13 | Egeria. | 1866 Aug. 29.0 | 220 | 54 | 41.3 | 120 | 5 | 15.0 | 43 | 15 | 56.3 | 16 | 30 | 48.8 | 4 | 59 | 47.1 | 857.87961 | 0.411054 | 1850 Nov. | 2 Gasparis. |
| 14 | Irene. | 1864 Nov. 28.0 | 134 | 55 | 9.2 | 179 | 52 | 0.8 | 80 | 42 | 23.7 | 9 | 7 | 37.5 | 9 | 33 | 23.7 | 853.20824 | 0.4126344 | 1851 May 19 | Hind. |
| 15 | Eunomia. | 1854 Jan. 0.0 | 122 | 5 | 31.5 | 27 | 52 | 0.5 | 293 | 52 | 14.5 | 11 | 44 | 17.4 | 10 | 47 | 32.2 | 825.45550 | 0.422209 | 1851 July 29 | Gasparis. |
| 16 | Psyche. | 1867 Jan. 0.0 | 115 | 10 | 46.6 | 15 | 26 | 27.0 | 150 | 33 | 17.6 | 3 | 3 | 57.2 | 7 | 47 | 0.3 | 709.7603 | 0.465930 | 1852 March 17 | Gasparis. |
| 17 | Thetis. | 1866 July 1.5 | 177 | 17 | 24.1 | 260 | 24 | 17.6 | 125 | 23 | 4.2 | 5 | 36 | 6.2 | 7 | 20 | 12.3 | 912.06563 | 0.393320 | 1852 April 17 | Luther. |
| 18 | Melpomene. | 1854 Jan. 0.0 | 80 | 4 | 37.0 | 15 | 5 | 31.0 | 150 | 3 | 49.7 | 10 | 9 | 16.9 | 12 | 34 | 20.2 | 1020.1108 | 0.360903 | 1852 June 22 | Hind. |
| 19 | Fortuna. | 1863 June 24.0 | 258 | 15 | 3.3 | 30 | 57 | 54.2 | 211 | 22 | 29.1 | 1 | 32 | 44.8 | 9 | 7 | 30.2 | 930.2787 | 0.3875957 | 1852 Aug. 22 | Hind. |
| 20 | Massalia. | 1866 June 15.5 | 161 | 43 | 44.2 | 98 | 29 | 31.8 | 206 | 45 | 6.7 | 0 | 41 | 9.5 | 8 | 15 | 13.7 | 948.55878 | 0.381962 | 1852 Sept. 19 | Gasparis. |
| 21 | Lutetia. | 1853 Jan. 2.0 | 74 | 20 | 51.1 | 327 | 3 | 8.4 | 80 | 27 | 7.2 | 3 | 5 | 9.5 | 9 | 19 | 44.6 | 933.55438 | 0.3865780 | 1852 Nov. 15 | Goldschmidt. |
| 22 | Calliope. | 1866 Aug. 30.5 | 289 | 37 | 46.9 | 58 | 15 | 36.1 | 66 | 35 | 39.8 | 13 | 43 | 47.4 | 5 | 39 | 18.6 | 714.5187 | 0.463998 | 1852 Nov. 16 | Hind. |
| 23 | Thalia. | 1867 Jan. 0.0 | 68 | 14 | 16.3 | 123 | 49 | 41.6 | 67 | 40 | 47.2 | 10 | 13 | 26.6 | 13 | 25 | 8.8 | 833.1087 | 0.419537 | 1852 Dec. 15 | Hind. |
| 24 | Themis. | 1864 Aug. 20.0 | 40 | 14 | 0.7 | 140 | 8 | 26.5 | 3 | 12 | 12.6 | 0 | 48 | 52.1 | 6 | 42 | 52.9 | 936.76345 | 0.4973523 | 1853 April 5 | Gasparis. |
| 25 | Phocæa. | 1865 Nov. 12.0 | 79 | 17 | 21.8 | 302 | 49 | 53.4 | 214 | 5 | 7.3 | 21 | 34 | 36.3 | 14 | 43 | 58.4 | 953.82327 | 0.380360 | 1853 April 6 | Chacornac. |
| 26 | Proserpina. | 1853 June 11.0 | 351 | 5 | 55.6 | 236 | 25 | 15.0 | 45 | 54 | 59.3 | 3 | 35 | 47.7 | 5 | 0 | 37.3 | 819.68468 | 0.4242399 | 1853 May 5 | Luther. |
| 27 | Euterpe. | 1866 May 26.5 | 149 | 7 | 51.3 | 87 | 35 | 3.6 | 93 | 48 | 1.5 | 1 | 35 | 29.8 | 9 | 58 | 29.2 | 986.99048 | 0.370403 | 1853 Nov. 8 | Hind. |
| 28 | Bellona. | 1862 March 24.0 | 303 | 8 | 27.8 | 122 | 55 | 29.6 | 144 | 41 | 9.9 | 9 | 21 | 26.3 | 8 | 37 | 57.5 | 766.12228 | 0.4438057 | 1854 March 1 | Luther. |
| 29 | Amphitrite. | 1866 March 10.0 | 104 | 21 | 32.3 | 56 | 56 | 1.8 | 356 | 30 | 5.2 | 6 | 7 | 49.3 | 4 | 14 | 35.3 | 869.33444 | 0.497214 | 1854 March 1 | Marth. |
| 30 | Urania. | 1865 Aug. 18.0 | 306 | 32 | 25.0 | 31 | 28 | 57.9 | 308 | 9 | 39.2 | 2 | 6 | 6.9 | 7 | 16 | 6.3 | 975.27438 | 0.373920 | 1854 July 22 | Hind. |
| 31 | Euphrosyne. | 1867 Jan. 0.0 | 11 | 8 | 2.0 | 93 | 42 | 6.6 | 31 | 45.9 | | 26 | 27 | 5.0 | 12 | 44 | 10.3 | 633.8508 | 0.498680 | 1854 Sept. 1 | Ferguson. |
| 32 | Pomona. | 1855 Jan. 5.0 | 222 | 54 | 24.0 | 191 | 21 | 32.1 | 220 | 48 | 14.5 | 5 | 29 | 5.0 | 4 | 43 | 43.7 | 852.5769 | 0.4128487 | 1854 Oct. 28 | Goldschmidt. |
| 33 | Polyhymnia. | 1866 Feb. 11.0 | 144 | 10 | 45.7 | 342 | 31 | 6.7 | 9 | 5 | 54.9 | 1 | 56 | 19.9 | 19 | 47 | 58.6 | 731.6866 | 0.457121 | 1854 Oct. 28 | Chacornac. |
| 34 | Circæ. | 1865 Aug. 22.0 | 170 | 13 | 2.5 | 150 | 3 | 19.2 | 184 | 48 | 36.5 | 5 | 26 | 28.9 | 6 | 9 | 44.1 | 805.85537 | 0.4291663 | 1855 April 6 | Chacornac. |
| 35 | Leucothea. | 1866 June 22.0 | 47 | 39 | 33.7 | 201 | 40 | 29.0 | 355 | 51 | 35.8 | 8 | 10 | 47.6 | 12 | 22 | 51.7 | 680.7841 | 0.477998 | 1855 April 19 | Lut. et. |

Elements of the Orbits of the Minor Planets.

| No. | Name. | Epoch and Mean Equinox. Berlin Mean Time. | M | | π | | Ω | | i | | φ | | μ | log a | Date of Discovery. | Discoverer. | | | | | |
|-----|------------|---|-----|----|------|-----|----|------|-----|----|------|----|----|-------|-----------------------|-------------|------|-----------|---------------|----|------------------|
| | | | ° | ' | ° | ' | ° | ' | ° | ' | ° | ' | " | | | | | | | | |
| 36 | Atalanta. | 1866 Feb. 21.0 | 74 | 52 | 38.3 | 42 | 47 | 47.7 | 359 | 11 | 14.9 | 18 | 42 | 14.8 | 17 | 31 | 53.2 | 0.438721 | 1855 Oct. | 5 | Goldschmidt. |
| 37 | Fides. | 1863 Oct. 5.0 | 266 | 46 | 29.0 | 66 | 20 | 17.3 | 8 | 12 | 29.4 | 3 | 7 | 12.3 | 10 | 10 | 46.1 | 0.4218268 | 1855 Oct. | 5 | Luther. |
| 38 | Leda. | 1866 Jan. 0.0 | 12 | 6 | 43.3 | 100 | 51 | 44.3 | 296 | 27 | 34.9 | 6 | 58 | 25.3 | 8 | 56 | 30.7 | 0.4377740 | 1856 Jan. | 12 | Chacornac. |
| 39 | Lætiia. | 1866 May 2.0 | 231 | 39 | 4.8 | 2 | 30 | 27.3 | 157 | 21 | 11.5 | 10 | 22 | 5.1 | 6 | 35 | 2.2 | 0.4420219 | 1856 Feb. | 8 | Chacornac. |
| 40 | Harmonia. | 1866 March 3.0 | 160 | 34 | 22.3 | 1 | 27 | 26.1 | 93 | 35 | 58.8 | 4 | 15 | 54.8 | 2 | 41 | 7.6 | 0.3554678 | 1856 March 31 | 1 | Goldschmidt. |
| 41 | Daphne. | 1866 July 29.5 | 55 | 45 | 17.5 | 220 | 12 | 14.1 | 179 | 6 | 58.7 | 15 | 59 | 12.1 | 15 | 25 | 19.7 | 0.442346 | 1856 May | 22 | Goldschmidt. |
| 42 | Isis. | 1860 Jan. 0.0 | 289 | 29 | 25.4 | 318 | 0 | 48.7 | 84 | 30 | 40.4 | 8 | 34 | 33.0 | 13 | 2 | 20.6 | 0.3874006 | 1856 May | 23 | Pogson. |
| 43 | Aradne. | 1866 Jan. 1.0 | 184 | 54 | 15.1 | 277 | 48 | 9.6 | 264 | 37 | 43.9 | 9 | 38 | 30.7 | 9 | 38 | 30.7 | 0.3435683 | 1857 April | 15 | Pogson. |
| 44 | Nysa. | 1866 Oct. 9.0 | 283 | 21 | 50.5 | 112 | 5 | 31.5 | 131 | 3 | 31.2 | 3 | 41 | 57.6 | 8 | 40 | 17.9 | 0.3841674 | 1857 May | 27 | Goldschmidt. |
| 45 | Eugenia. | 1866 June 4.0 | 19 | 22 | 1.6 | 230 | 50 | 34.9 | 148 | 6 | 3.7 | 6 | 35 | 25.0 | 4 | 35 | 2.2 | 0.434762 | 1857 June | 27 | Goldschmidt. |
| 46 | Hestia. | 1865 July 26.0 | 322 | 11 | 46.6 | 354 | 10 | 34.9 | 181 | 26 | 45.3 | 2 | 17 | 32.1 | 9 | 26 | 55.7 | 0.4025124 | 1857 Aug. | 16 | Pogson. |
| 47 | Aglata. | 1859 June 17.0 | 162 | 29 | 40.5 | 314 | 3 | 45.0 | 4 | 12 | 34.2 | 5 | 0 | 88.5 | 7 | 35 | 15.7 | 0.4595800 | 1857 Sept. | 15 | Luther. |
| 48 | Doris. | 1862 July 25.0 | 235 | 11 | 27.8 | 74 | 20 | 42.4 | 185 | 5 | 29.6 | 6 | 29 | 28.2 | 4 | 23 | 15.7 | 0.4926769 | 1857 Sept. | 19 | Goldschmidt. |
| 49 | Pales. | 1863 Nov. 14.0 | 20 | 0 | 30.8 | 32 | 14 | 49.7 | 290 | 32 | 17.4 | 3 | 8 | 46.4 | 13 | 43 | 18.3 | 0.4889025 | 1857 Sept. | 19 | Goldschmidt. |
| 50 | Virginia. | 1863 Jan. 18.0 | 83 | 27 | 18.9 | 9 | 53 | 21.4 | 173 | 31 | 59.2 | 2 | 47 | 48.4 | 16 | 40 | 22.5 | 0.4235097 | 1857 Oct. | 4 | Ferguson. |
| 51 | Nemusa. | 1865 Jan. 17.0 | 316 | 39 | 29.6 | 174 | 52 | 0.6 | 175 | 43 | 6.3 | 9 | 56 | 52.8 | 3 | 47 | 40.7 | 0.3739602 | 1858 Jan. | 22 | Laurent. |
| 52 | Europa. | 1858 Jan. 0.0 | 34 | 25 | 7.3 | 101 | 56 | 14.8 | 129 | 57 | 16.0 | 7 | 24 | 41.0 | 5 | 49 | 14.1 | 0.4913564 | 1858 Feb. | 6 | Goldschmidt. |
| 53 | Calypso. | 1866 Jan. 4.0 | 7 | 11 | 44.0 | 92 | 53 | 30.3 | 144 | 1 | 9.0 | 5 | 6 | 39.0 | 11 | 45 | 54.8 | 0.4182540 | 1858 April | 4 | Luther. |
| 54 | Alexandra. | 1863 Nov. 14.0 | 83 | 37 | 8.2 | 295 | 27 | 8.7 | 314 | 5 | 8.4 | 11 | 46 | 41.9 | 11 | 21 | 24.0 | 0.4333401 | 1858 Sept. | 10 | Goldschmidt. |
| 55 | Pandora. | 1863 Oct. 25.0 | 35 | 42 | 11.7 | 11 | 9 | 47.8 | 10 | 52 | 9.6 | 7 | 13 | 49.8 | 8 | 19 | 19.2 | 0.4407624 | 1858 Sept. | 10 | Searle. |
| 56 | Melete. | 1865 June 20.0 | 344 | 40 | 12.6 | 293 | 29 | 25.0 | 194 | 27 | 23.7 | 8 | 1 | 40.9 | 13 | 44 | 9.5 | 0.4142944 | 1857 Sept. | 9 | Goldschmidt. |
| 57 | Mnemosyne. | 1860 Jan. 1.0 | 335 | 30 | 22.2 | 53 | 7 | 9.9 | 200 | 5 | 31.5 | 15 | 8 | 8.6 | 5 | 28 | 17.1 | 0.4992106 | 1859 Sept. | 22 | Luther. |
| 58 | Concordia. | 1865 Jan. 7.0 | 21 | 50 | 58.8 | 188 | 41 | 55.0 | 161 | 19 | 35.6 | 5 | 1 | 53.2 | 2 | 56 | 15.2 | 0.4314112 | 1860 March 22 | 1 | Luther. |
| 59 | Elpis. | 1865 Jan. 7.0 | 334 | 18 | 42.6 | 18 | 18 | 54.2 | 170 | 20 | 28.8 | 8 | 37 | 13.5 | 6 | 44 | 1.3 | 0.4343669 | 1860 Sept. | 12 | Chacornac. |
| 60 | Echo. | 1866 Jan. 0.0 | 65 | 44 | 37.1 | 98 | 33 | 32.6 | 192 | 2 | 9.0 | 10 | 38 | 18.5 | 10 | 38 | 45.8 | 0.3879508 | 1860 Sept. | 15 | Ferguson. |
| 61 | Danaë. | 1865 Aug. 19.0 | 345 | 54 | 41.2 | 341 | 25 | 28.5 | 334 | 11 | 50.0 | 18 | 15 | 25.6 | 9 | 17 | 59.3 | 0.4749112 | 1860 Sept. | 19 | Goldschmidt. |
| 62 | Erato. | 1865 May 7.0 | 279 | 40 | 20.8 | 34 | 8 | 29.1 | 126 | 11 | 42.1 | 2 | 12 | 17.5 | 9 | 46 | 43.3 | 0.4954961 | 1860 Sept. | 14 | Forster, Lesser. |
| 63 | Ausonia. | 1865 April 17.0 | 307 | 24 | 5.0 | 269 | 32 | 49.0 | 338 | 6 | 58.3 | 5 | 47 | 16.3 | 7 | 13 | 45.3 | 0.3792095 | 1861 Feb. | 10 | Gasparis. |
| 64 | Angelina. | 1865 Jan. 7.0 | 355 | 46 | 41.1 | 123 | 37 | 49.1 | 311 | 9 | 7.2 | 1 | 19 | 52.0 | 7 | 21 | 58.9 | 0.4282872 | 1861 March 4 | 8 | Tempel. |
| 65 | Cybele. | 1861 Jan. 0.0 | 281 | 57 | 34.7 | 258 | 20 | 36.9 | 158 | 53 | 34.8 | 3 | 28 | 9.8 | 6 | 54 | 36.4 | 0.534092 | 1861 March 8 | 8 | Tempel. |
| 66 | Maia. | 1865 Jan. 27.0 | 87 | 7 | 3.2 | 44 | 25 | 0.6 | 8 | 15 | 23.7 | 3 | 4 | 15.1 | 9 | 5 | 46.9 | 0.4234510 | 1861 April | 9 | Tuttle. |
| 67 | Asia. | 1865 Jan. 7.0 | 296 | 2 | 14.0 | 306 | 8 | 6.9 | 202 | 43 | 29.0 | 5 | 59 | 35.9 | 10 | 39 | 58.6 | 0.3841270 | 1861 April | 17 | Pogson. |
| 68 | Leto. | 1863 Dec. 20.0 | 93 | 53 | 22.4 | 345 | 4 | 58.2 | 44 | 53 | 11.4 | 7 | 57 | 34.9 | 10 | 37 | 34.9 | 0.4441008 | 1861 April | 29 | Luther. |
| 69 | Hesperia. | 1861 June 3.0 | 54 | 46 | 56.9 | 109 | 6 | 25.4 | 167 | 1 | 7.5 | 8 | 28 | 19.2 | 10 | 0 | 38.3 | 0.4733004 | 1861 April | 29 | Schiaparelli. |
| 70 | Panopæa. | 1861 May 28.0 | 308 | 41 | 11.5 | 300 | 3 | 30.3 | 48 | 14 | 42.6 | 11 | 38 | 30.2 | 10 | 33 | 7.5 | 0.417184 | 1861 May | 5 | Goldschmidt. |

TABLE XIX. Elements of the Orbits of the Minor Planets.

| No. | Name. | Epoch and Mean Equinox. Berlin Mean Time. | M | | π | | Ω | | i | | ϕ | | μ | $\log a$ | Date of Discovery. | Discoverer. |
|-----|--------------|---|-----|-----------|-------|---------|----------|---------|------------|------------|------------|------------|-----------|---------------|-----------------------|-------------|
| | | | ° | ' | ° | ' | ° | ' | ° | ' | ° | ' | " | | | |
| 71 | Niobe. | 1864 Jan. 23.0 | 283 | 10 47.6 | 222 | 4 26.8 | 316 | 19 7.0 | 23 18 51.2 | 10 0 15.1 | 10 0 15.1 | 775.73290 | 0.4401964 | 1861 Aug. 13 | Luther. | |
| 72 | Peronia. | 1866 Jan. 0.0 | 31 | 17 25.1 | 307 | 54 49.5 | 207 | 44 59.6 | 5 23 54.5 | 6 52 45.9 | 6 52 45.9 | 1040.14680 | 0.3552747 | 1861 May 29 | Peters. | |
| 73 | Clytia. | 1864 Oct. 4.0 | 325 | 18 55.8 | 59 | 59 11.0 | 7 | 34 19.1 | 2 24 39.5 | 2 27 0.5 | 2 27 0.5 | 814.84338 | 0.425955 | 1862 April 7 | Tuttle. | |
| 74 | Galatea. | 1866 Jan. 0.0 | 249 | 23 12.1 | 7 | 22 10.2 | 197 | 58 59.3 | 3 58 54.6 | 13 46 49.1 | 13 46 49.1 | 766.4390 | 0.4436860 | 1862 Aug. 29 | Tempel. | |
| 75 | Eurydice. | 1864 Feb. 2.0 | 133 | 39 40.8 | 334 | 27 46.0 | 359 | 56 43.4 | 5 0 4.2 | 17 51 2.1 | 17 51 2.1 | 812.9517 | 0.426628 | 1862 Sept. 22 | Peters. | |
| 76 | Freia. | 1863 July 27.0 | 355 | 31 36.1 | 93 | 13 58.1 | 212 | 58 21.4 | 2 1 50.8 | 10 49 12.0 | 10 49 12.0 | 569.0505 | 0.5299038 | 1862 Oct. 21 | D'Arrest. | |
| 77 | Eriga. | 1866 Jan. 0.0 | 228 | 36 16.6 | 58 | 11 32.0 | 2 | 9 27.6 | 2 27 56.6 | 7 48 20.4 | 7 48 20.4 | 812.40096 | 0.4268241 | 1862 Nov. 12 | Peters. | |
| 78 | Dianna. | 1865 Oct. 4.0 | 256 | 20 50.5 | 121 | 42 47.5 | 333 | 55 48.4 | 8 38 39.9 | 11 51 34.5 | 11 51 34.5 | 835.35315 | 0.4187577 | 1863 March 15 | Luther. | |
| 79 | Euryome. | 1864 Jan. 1.0 | 1 | 30 56.7 | 44 | 17 58.1 | 206 | 42 42.6 | 4 36 46.5 | 11 14 53.1 | 11 14 53.1 | 929.1286 | 0.3879539 | 1863 Sept. 14 | Watson. | |
| 80 | Sappho. | 1865 Oct. 7.0 | 50 | 11 57.355 | 5 | 12.5 | 218 | 31 45.0 | 8 36 51.3 | 11 33 5.6 | 11 33 5.6 | 1019.6804 | 0.3610284 | 1864 May 2 | Pogson. | |
| 81 | Terpsichore. | 1864 Oct. 6.0 | 333 | 26 18.1 | 43 | 33 7.9 | 2 | 32 1.6 | 7 55 40.8 | 12 13 31.5 | 12 13 31.5 | 735.0244 | 0.4558032 | 1864 Sept. 30 | Tempel. | |
| 82 | Alemene. | 1865 Feb. 16.0 | 332 | 33 22.9 | 131 | 18 19.7 | 26 | 56 51.5 | 5 21 15.0 | 13 3 43.1 | 13 3 43.1 | 773.711 | 0.440952 | 1864 Nov. 27 | Luther. | |
| 83 | Beatrix. | 1865 May 4.0 | 17 | 1 59.0 | 188 | 28 0.9 | 27 | 34 9.1 | 5 2 11.3 | 4 49 38.8 | 4 49 38.8 | 937.415 | 0.383583 | 1865 April 26 | Gasparis. | |
| 84 | Clio. | 1865 Nov. 13.0 | 14 | 36 45.5 | 339 | 12 0.1 | 327 | 22 1.5 | 9 22 16.0 | 13 39 34.8 | 13 39 34.8 | 977.5422 | 0.3732474 | 1865 Aug. 25 | Luther. | |
| 85 | Io. | 1866 Jan. 0.0 | 56 | 49 20.9 | 322 | 32 28.9 | 203 | 52 33.3 | 11 53 12.8 | 11 0 53.1 | 11 0 53.1 | 820.7120 | 0.4238772 | 1865 Sept. 19 | Peters. | |
| 86 | Semele. | 1866 Jan. 8.0 | 8 | 23 14.6 | 28 | 39 3.9 | 87 | 55 49.6 | 4 47 44.6 | 11 49 36.5 | 11 49 36.5 | 652.9848 | 0.490069 | 1866 Jan. 4 | Tietjen. | |
| 87 | Sylvia. | 1866 May 16.5 | 274 | 4 2.3 | 337 | 21 48.6 | 76 | 23 59.0 | 10 51 22.0 | 4 39 22.6 | 4 39 22.6 | 543.5800 | 0.5431620 | 1866 May 16 | Pogson. | |
| 88 | Thisbe. | 1866 Aug. 4.5 | 356 | 5 1.4 | 308 | 55 30.5 | 277 | 44 7.8 | 5 14 58.1 | 9 29 55.7 | 9 29 55.7 | 769.561 | 0.442509 | 1866 June 15 | Peters. | |
| 89 | | 1866 Sept. 1.0 | 339 | 44 19.2 | 349 | 30 29.8 | 311 | 31 7.5 | 16 32 38.0 | 9 37 22.2 | 9 37 22.2 | 872.656 | 0.406109 | 1866 Aug. 6 | Stephan. | |
| 90 | Antiope. | 1866 Oct. 18.0 | 52 | 6 9.2 | 294 | 3 7.3 | 71 | 0 54.0 | 2 17 25.2 | 11 39 2.7 | 11 39 2.7 | 632.35913 | 0.4993618 | 1866 Oct. 1 | Luther. | |
| 91 | | 1866 Dec. 21.0 | 336 | 46 5.4 | 75 | 16 23.5 | 11 | 19 10.4 | 2 9 25.0 | 5 4 27.2 | 5 4 27.2 | 867.0876 | 0.4079624 | 1866 Nov. 4 | Stephan. | |

TABLE XX. Elements of the Orbits of the Major Planets.

| Name. | Epoch and Mean Equinox. Greenwich Mean Time. | L | | π | | $\Delta\pi$ | | Ω | | $\Delta\Omega$ | | i | | Δi | e | Δe | a |
|----------|---|-----|---------|-------|---------|-------------|---------|-------------|--------------|----------------|--------------|-----------|--------|------------|--------------|--------------|------------|
| | | ° | ' | ° | ' | ° | ' | ° | ' | ° | ' | ° | ' | " | | | |
| Mercury. | 1801 Jan. 1.0 | 166 | 0 43.2 | 74 | 21 37.2 | + 9 44 | + 9 44 | 45 58 20.2 | - 13 2 | - 13 2 | - 13 2 | 7 0 4.5 | + 18.1 | 0.2056003 | + 0.0000037 | + 0.0000037 | 0.3870984 |
| Venus. | " | 11 | 33 30.2 | 128 | 43 53.1 | - 19 41 | - 19 41 | 74 54 12.9 | - 21 11 | - 21 11 | - 21 11 | 3 23 28.5 | + 4.5 | 0.0068607 | - 0.00006275 | - 0.00006275 | 0.7233316 |
| Earth. | " | 100 | 39 10.2 | 99 | 30 5.0 | + 19 41 | + 19 41 | 48 0 3.5 | - 38 49 | - 38 49 | - 38 49 | 1 51 6.2 | - 0.3 | 0.0167836 | + 0.0004359 | + 0.0004359 | 1.0000000 |
| Mars. | " | 64 | 22 55.5 | 332 | 23 56.6 | + 26 22 | + 26 22 | 98 26 18.9 | - 26 21 | - 26 21 | - 26 21 | 1 18 51.3 | - 22.0 | 0.0933070 | + 0.0009019 | + 0.0009019 | 1.5236923 |
| Jupiter. | " | 112 | 15 23.0 | 11 | 8 34.6 | + 11 5 | + 11 5 | 111 56 37.4 | - 32 22 | - 32 22 | - 32 22 | 2 29 35.7 | - 15.5 | 0.0481621 | + 0.00016036 | + 0.00016036 | 5.2037760 |
| Saturn. | " | 135 | 20 6.5 | 89 | 9 29.8 | + 32 17 | + 32 17 | 72 59 35.3 | - 59 59 | - 59 59 | - 59 59 | 1 47 1.7 | - 45.0 | 0.03401505 | - 0.00031240 | - 0.00031240 | 9.5387861 |
| Uranus. | " | 177 | 48 23.0 | 167 | 31 16.1 | + 4 0 | + 4 0 | 130 7 31.8 | - 130 7 31.8 | - 130 7 31.8 | - 130 7 31.8 | 1 47 1.7 | - 45.0 | 0.0466794 | - 0.00002521 | - 0.00002521 | 19.1823900 |
| Neptune. | 1850 Jan. 0.0 | 335 | 5 38.9 | 43 | 17 30.3 | + 4 0 | + 4 0 | 130 7 31.8 | - 130 7 31.8 | - 130 7 31.8 | - 130 7 31.8 | 1 47 1.7 | - 45.0 | 0.0084962 | - 0.00002521 | - 0.00002521 | 30.0705520 |

TABLE XXI. Constants, &c.

| | | | |
|--|--------------------------|-----|-----------------|
| Base of Napierian logarithms | $e = 2.71828183$ | log | 0.43429448 |
| Modulus of the common logarithms | $\lambda_0 = 0.43429448$ | | 9.63778431 — 10 |
| Radius of a Circle in seconds | $r = 206264.806$ | | 5.31442513 |
| " " " " minutes | $r = 3437.7468$ | | 3.53627388 |
| " " " " degrees | $r = 57.29578$ | | 1.75812263 |
| Circumference of a Circle in seconds | 1296000 | | 6.11260500 |
| " " " " when $r = 1$ | $\pi = 3.14159265$ | | 0.49714987 |
| Sine of 1 second | 0.000004848137 | | 4.68557487 |

| | | | |
|---|------------------|--|------------|
| Equatorial horizontal parallax of the sun, according to | | | |
| Encke | 8".57116 | | 0.9330396 |
| Length of the sidereal year, according to Hansen and | | | |
| Olufsen | 365.2563582 days | | 2.56259778 |
| Length of the tropical year, according to Hansen and | | | |
| Olufsen | 365.2422008 " | | 2.56258095 |
| This value of the length of the tropical year is for 1850.0. The annual variation is — 0.000000624. | | | |

| | | | |
|---|-------------------|--|-----------------|
| Time occupied by the passage of light over a distance | | | |
| equal to the mean distance of the earth from the | | | |
| sun, according to Struve | 497.827 | | 2.6970785 |
| Attractive force of the sun, according to Gauss | $k = 0.017202099$ | | 8.23558144 — 10 |
| " " " " " " " " in se- | | | |
| conds of arc | 3548.18761 | | 3.55000657 |

| | | | |
|---|--|--|----------|
| Constant of Aberration, according to Struve | | | 20".4451 |
| " " Nutation, " " Peters | | | 9".2231 |

| | | | |
|---|--|--|--|
| Mean Obliquity of the ecliptic for 1750 + t , | | | |
| according to Bessel | $23^\circ 28' 18''.00 - 0''.48368t - 0''.00000272295t$ | | |
| Mean Obliquity of the ecliptic for 1800 + t , | | | |
| according to Struve and Peters | $23^\circ 27' 54''.22 - 0''.4738t - 0''.0000014t^2$ | | |

| | |
|--|--------------------------------|
| General Precession for the year 1750 + t , according to Bessel | $50''.21129 + 0''.0002442966t$ |
| " " " " " " " " Struve | $50''.22980 + 0''.000226t$ |

MASSSES OF THE PLANETS, THE MASS OF THE SUN BEING THE UNIT.

| | | | |
|-------------------|-------------------------|-------------------|--------------------------|
| Mercury | $m = \frac{1}{4865751}$ | Jupiter | $m = \frac{1}{1047.879}$ |
| Venus | $\frac{1}{390000}$ | Saturn | $\frac{1}{3501.6}$ |
| Earth | $\frac{1}{354936}$ | Uranus | $\frac{1}{24905}$ |
| Mars | $\frac{1}{3107713}$ | Neptune | $\frac{1}{18780}$ |



EXPLANATION OF THE TABLES.

TABLE I. contains the values of the *angle of the vertical* and of the logarithm of the earth's radius, with the geographical latitude as the argument. The adopted elements are those derived by Bessel. Denoting by ρ the radius of the earth, by φ the geographical latitude, and by φ' the geocentric latitude, we have

$$\begin{aligned}\varphi &= \varphi - 11' 30''.65 \sin 2\varphi + 1''.16 \sin 4\varphi - \&c., \\ \log \rho &= 9.9992747 + 0.0007271 \cos 2\varphi - 0.0000018 \cos 4\varphi + \&c.,\end{aligned}$$

ρ being expressed in parts of the equatorial radius as the unit. These quantities are required in the determination of the parallax of a heavenly body. The formulæ for the parallax in right ascension and in declination are given in Art. 61.

TABLE II. gives the intervals of sidereal time corresponding to given intervals of mean time. It is required for the conversion of mean solar into sidereal time.

TABLE III. gives the intervals of mean time corresponding to given intervals of sidereal time. It is required for the conversion of sidereal into mean solar time.

TABLE IV. furnishes the numbers required in converting hours, minutes, and seconds into decimals of a day. Thus, to convert $13h\ 19m\ 43.5s$ into the decimal of a day, we find from the Table

$$\begin{aligned}13h &= 0.5416667 \\ 19m &= 0.0131944 \\ 43s &= 0.0004977 \\ 0.5s &= 0.0000058\end{aligned}$$

$$\text{Therefore } 13h\ 19m\ 43.5s = 0.5553646$$

The decimal corresponding to $0.5s$ is found from that for $5s$ by changing the place of the decimal point.

TABLE V. serves to find, for any instant, the number of days from the beginning of the year. Thus, for 1863 Sept. 14, $15h\ 53m\ 37.2s$, we have

Sept. 0.0 = 243.00000 days from the beginning of the year.

$14d\ 15h\ 53m\ 37.2s = 14.66224$

Required number of days = 257.66224

TABLE VI. contains the values of $M = 75 \tan \frac{1}{2}v + 25 \tan^3 \frac{1}{2}v$ for values of v at intervals of one minute from 0° to 180° . For an explanation of its construction and use, see Articles 22, 27, 29, 41, and 72.

In the case of parabolic motion the formulæ are

$$m = \frac{C_0}{q^{\frac{3}{2}}}, \quad M = m(t - T),$$

wherein $\log C_0 = 9.9601277$. From these, by means of the Table, v may be found when $t - T$ is given, or $t - T$ when v is known. From $v = 30^\circ$ to $v = 180^\circ$ the Table contains the values of $\log M$.

TABLE VII., the construction of which is explained in Art. 23, serves to determine, in the case of parabolic motion, the true anomaly or the time from the perihelion when v approaches near to 180° . The formulæ are

$$\sin w = \sqrt[3]{\frac{200}{M}}, \quad v = w + \Delta_0, \quad t - T = \frac{200}{C_0} \cdot \frac{q^{\frac{3}{2}}}{\sin^3 w},$$

w being taken in the second quadrant. The Table gives the values of Δ_0 with w as the argument. As an example, let it be required to find the true anomaly corresponding to the values $t - T = 22.5$ days and $\log q = 7.902720$. From these we derive

$$\log M = 4.4582302.$$

Table VI. gives for this value of $\log M$, taking into account the second differences,

$$v = 168^\circ\ 59'\ 32''.49;$$

but, using Table VII., we have

$$w = 168^\circ\ 59'\ 29''.11, \quad \Delta_0 = 3''.37,$$

and hence

$$v = w + \Delta_0 = 168^\circ 59' 32''.48,$$

the two results agreeing completely.

TABLE VIII. serves to find the time from the perihelion in the case of parabolic motion. For an explanation of its construction and use, see Articles 24, 69, and 72.

TABLE IX. is used in the determination of the true anomaly or the time from the perihelion in the case of orbits of great eccentricity. Its construction is fully explained in Art. 28, and its use in Art. 41.

TABLE X. serves to find the value of v or of $t - T$ in the case of elliptic or hyperbolic orbits. The construction of this Table is explained in Art. 29. The first part gives the values of $\log B$ and $\log C$, with A as the argument, for the ellipse and the hyperbola. In the case of $\log C$ there are given also \log I. Diff. and \log half II. Diff., expressed in units of the seventh decimal place, by means of which the interpolation is facilitated. Thus, if we denote by $\log (C)$ the value which the Table gives directly for the argument next less than the given value of A , and by ΔA the difference between this argument and the given value of A , expressed in units of the second decimal place, we have, for the required value,

$$\log C = \log (C) + \Delta A \times \text{I. Diff.} + \Delta A^2 \times \text{half II. Diff.}$$

For example, let it be required to find the value of $\log C$ corresponding to $A = 0.02497944$, and the process will be:—

| | | (1) | (2) |
|-------------------------|------------------------|----------------------------------|--------------------------------------|
| Arg. 0.02, | $\log (C) = 0.0034986$ | $\log \text{I. Diff.} = 4.24585$ | $\log \text{half II. Diff.} = 1.778$ |
| | (1) = 8770.6 | $\log \Delta A = 9.69718$ | $2 \log \Delta A = -9.394$ |
| $\Delta A = 0.497944$, | (2) = 14.8 | 3.94303 | 1.172 |
| | $\log C = 0.0043771$ | | |

The second part of the Table gives the values of A corresponding to given values of τ .

TABLE XI. serves to determine the chord of the orbit when the extreme radii-vectores and the time of describing the parabolic arc are given. For an explanation of the construction and use of this Table, see Articles 68, 72, and 117.

TABLE XII. exhibits the limits of the real roots of the equation

$$\sin(z' - \zeta) = m_0 \sin^4 z'.$$

The construction and use of this table are fully explained in Articles 84 and 93.

TABLES XIII. and XIV. are used in finding the ratio of the sector included by two radii-vectores to the triangle included by the same radii-vectores and the chord joining their extremities. For an explanation of the construction and use of these Tables, see Articles 88, 89, 93, and 101.

TABLE XV. is used in the determination of the chord of the part of the orbit described in a given time in the case of very eccentric elliptic motion, and in the determination of the interval of time whenever the chord is known. For an explanation of its construction and use, see Articles 116, 117, and 119.

TABLE XVI. is used in finding the chord or the interval of time in the case of hyperbolic motion. See Articles 118 and 119 for an explanation of the use of the Table, and also the explanation of Table X. for an illustration of the use of the columns headed log I. Diff. and log half II. Diff.

TABLE XVII. is used in the computation of special perturbations when the terms depending on the squares and higher powers of the masses are taken into account. For an explanation of its construction and use, see Articles 157, 165, 166, 170, and 171.

TABLE XVIII. contains the elements of the orbits of the comets which have been observed. These elements are: T , the time of perihelion passage (mean time at Greenwich); π , the longitude of the perihelion; Ω , the longitude of the ascending node; i , the inclination of the orbit to the plane of the ecliptic; e , the eccentricity of the orbit; and q , the perihelion distance. The longitudes for Nos. 1, 2, 12, 16, 91, 92, 115, 127, 138, 155, 156, 159, 160, 162, 171, 173-175, 180, 181, 185, 191, 192, 195-199, 201, 203, 204, 207, 208, 212-215, 217-219, 221-228, 230, 233, 234, 237-248, 251-258, 261-267, 269-275, 277-279, are in each case measured from the mean equinox of the beginning of the year. In the case of Nos. 134, 146, 172, 182, 189, 190, 205, 231, 232, 236, 259, and 268, the longitudes are

measured from the mean equinox of the beginning of the next year. The longitudes for Nos. 19 and 27 are measured from the mean equinox of 1850.0; for No. 186, from the mean equinox of July 3; for No. 187, from the mean equinox of Nov. 9; for No. 200, from the mean equinox of July 1; for No. 202, from the mean equinox of Oct. 1; for No. 206, from the mean equinox of Oct. 7; for No. 211, from the mean equinox of 1848.0; for No. 216, from the mean equinox of Feb. 20; for No. 229, from the mean equinox of April 1; for No. 250, from the mean equinox of Oct. 1; and for No. 276, from the mean equinox of 1865 Oct. 4.0.

Nos. 1, 2, 11, 12, 20, 23, 29, 41, 53, 80, and 177 give the elements for the successive appearances of Halley's comet; Nos. 104, 116, 126, 143, 149, 157, 167, 170, 176, 178, 183, 194, 210, 220, 235, 249, and 260, those for Encke's comet, the longitudes being measured from the mean equinox for the instant of the perihelion passage. Nos. 92, 127, 159, 172, 196, and 222 give the elements for the successive appearances of Biela's comet; Nos. 187, 216, 250, and 276, those for Faye's comet; Nos. 197 and 238, those for Brorsen's comet; Nos. 217 and 243, those for D'Arrest's comet; and Nos. 145 and 245, those for Winnecke's comet. For epochs previous to 1583 the dates are given according to the old style.

This Table is useful for identifying a comet which may appear with one previously observed, by means of a similarity of the elements, its periodic character being otherwise unknown or at least uncertain. The elements given are those which appear to represent the observations most completely. For a collection of elements by various computers, and also for information in regard to the observations made and in regard to the place and manner of their publication, consult Carl's *Repertorium der Cometen-Astronomie* (Munich, 1864), or Galle's *Cometen-Verzeichniss* appended to the latest edition of Olbers's *Methode die Bahn eines Cometen zu berechnen*.

TABLE XIX. contains the elements of the orbits of the minor planets, derived chiefly from the *Berliner Astronomisches Jahrbuch für 1868*. The epoch is given in Berlin mean time; M denotes the mean anomaly, φ the angle of eccentricity, μ the mean daily motion, and a the semi-transverse axis. The elements of Vesta, Iris, Flora, Metis, Victoria, Eunomia, Melpomene, Lutetia, Proserpina, and Pomona are mean elements; the others are osculating for the epoch. The date of the discovery of the planet, and the name of the discoverer, are also added.

TABLE XX. contains the mean elements of the orbits of the major planets, together with the amount of their variations during a period of one hundred years. The epoch is expressed in Greenwich mean time, and L denotes the mean longitude of the planet.

TABLE XXI. gives the values of the masses of the major planets, and also various constants which are used in astronomical calculations.

APPENDIX.

A. Precession.—If we adopt the values for the precession and for the variation of the position of the plane of the ecliptic given in Art. 40, and put

$$M = 171^\circ 36' 10'' + 39''.79 (t - 1750),$$

the formulæ for the annual precession in longitude (λ) and latitude (β) become, for the instant t ,

$$\begin{aligned} \frac{d\lambda}{dt} &= 50''.2113 + 0''.0002443 (t - 1750) \\ &\quad + (0''.4889 - 0''.00000614 (t - 1750)) \cos (\lambda - M) \tan \beta, \\ \frac{d\beta}{dt} &= - (0''.4889 - 0''.00000614 (t - 1750)) \sin (\lambda - M). \end{aligned} \quad (1)$$

If we denote the planetary precession by a , the luni-solar precession by l , and the obliquity of the fixed ecliptic, at the time $1750 + \tau$, by ϵ_0 , we have, according to Bessel,

$$\begin{aligned} \frac{da}{dt} &= 0''.17926 - 0''.0005320786 \tau, \\ \frac{dl}{dt} &= 50''.37572 - 0''.000243589 \tau, \\ \epsilon_0 &= 23^\circ 28' 18''.0 + 0''.0000098423 \tau^2, \end{aligned}$$

and if we put

$$\cos \epsilon_0 \frac{dl}{dt} - \frac{da}{dt} = m, \quad \sin \epsilon_0 \frac{dl}{dt} = n,$$

the formulæ for the annual precession in right ascension (α) and declination (δ) become

$$\frac{d\alpha}{dt} = m + n \tan \delta \sin \alpha, \quad \frac{d\delta}{dt} = n \cos \alpha, \quad (2)$$

and the numerical values of m and n are, for the instant t ,

$$\begin{aligned} m &= 46''.02824 + 0''.0003086450 (t - 1750), \\ n &= 20''.06442 - 0''.0000970204 (t - 1750). \end{aligned}$$

To determine the precession during the interval $t' - t$, we compute the annual variation for the instant $\frac{1}{2}(t' + t)$ and this variation multiplied by $t' - t$ furnishes the required result.

B. Nutation.—The expressions for the equation of the equinoxes and for the nutation of the obliquity of the ecliptic are, according to Peters,

$$\begin{aligned} \Delta\lambda &= -17''.2405 \sin \Omega + 0''.2073 \sin 2\Omega - 0''.2041 \sin 2\zeta + 0''.0677 \sin (\zeta - \Gamma') \\ &\quad - 1''.2694 \sin 2\odot + 0''.1279 \sin (\odot - \Gamma) \\ &\quad - 0''.0213 \sin (\odot + \Gamma), \\ \Delta\epsilon &= +9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2\zeta \\ &\quad + 0''.5510 \cos 2\odot + 0''.0093 \cos (\odot + \Gamma), \end{aligned} \quad (3)$$

for the year 1800, and

$$\begin{aligned} \Delta\lambda &= -17''.2577 \sin \Omega + 0''.2073 \sin 2\Omega - 0''.2041 \sin 2\zeta + 0''.0677 \sin (\zeta - \Gamma') \\ &\quad - 1''.2695 \sin 2\odot + 0''.1275 \sin (\odot - \Gamma) \\ &\quad - 0''.0213 \sin (\odot + \Gamma), \\ \Delta\epsilon &= +9''.2240 \cos \Omega - 0''.0896 \cos 2\Omega + 0''.0885 \cos 2\zeta \\ &\quad + 0''.5507 \cos 2\odot + 0''.0092 \cos (\odot + \Gamma), \end{aligned}$$

for the year 1900. In these equations Ω denotes the longitude of the ascending node of the moon's orbit, referred to the mean equinox, ζ the true longitude of the moon, \odot the true longitude of the sun, Γ the true longitude of the sun's perigee, and Γ' the true longitude of the moon's perigee. The values of these quantities may be derived from the solar and lunar tables, and thus the required values of $\Delta\lambda$ and $\Delta\epsilon$ may be found. The equations give the corrections for the reduction from the mean equinox to the true equinox.

To find the nutation in right ascension and in declination, if we consider only the terms of the first order, we have

$$\begin{aligned} \Delta\alpha &= \frac{da}{d\lambda} \Delta\lambda + \frac{da}{d\epsilon} \Delta\epsilon, \\ \Delta\delta &= \frac{d\delta}{d\lambda} \Delta\lambda + \frac{d\delta}{d\epsilon} \Delta\epsilon. \end{aligned} \quad (4)$$

The values of $\Delta\lambda$ and $\Delta\epsilon$ are found from the preceding equations, and for the differential coefficients we have

$$\begin{aligned}
 \frac{d\alpha}{d\lambda} &= \cos \varepsilon + \sin \varepsilon \tan \delta \sin \alpha, & \frac{d\delta}{d\lambda} &= \cos \alpha \sin \varepsilon, \\
 \frac{d\alpha}{d\varepsilon} &= -\cos \alpha \tan \delta, & \frac{d\delta}{d\varepsilon} &= \sin \alpha.
 \end{aligned}
 \tag{5}$$

The terms of the second order are of sensible magnitude only when the body is very near the pole, and in this case by computing the second differential coefficients the complete values may be found.

In the reduction of the place of a planet or comet from the mean equinox of one date t to the true equinox of another date t' , the determination of the correction for precession and of that for nutation may be effected simultaneously. Thus, let τ denote the interval $t' - t$ expressed in parts of a year, and the sum of the corrections for precession and nutation gives

$$\begin{aligned}
 \Delta\alpha &= m\tau + \Delta\lambda \cos \varepsilon + (n\tau + \Delta\lambda \sin \varepsilon) \sin \alpha \tan \delta - \Delta\varepsilon \cos \alpha \tan \delta, \\
 \Delta\delta &= (n\tau + \Delta\lambda \sin \varepsilon) \cos \alpha + \Delta\varepsilon \sin \alpha.
 \end{aligned}
 \tag{6}$$

Let us now put

$$\begin{aligned}
 m\tau + \Delta\lambda \cos \varepsilon &= f, \\
 n\tau + \Delta\lambda \sin \varepsilon &= g \cos G, \\
 -\Delta\varepsilon &= g \sin G,
 \end{aligned}
 \tag{7}$$

and the equations (6) become

$$\begin{aligned}
 \Delta\alpha &= f + g \sin (G + \alpha) \tan \delta, \\
 \Delta\delta &= g \cos (G + \alpha),
 \end{aligned}
 \tag{8}$$

as already given in Art. 40.

The astronomical ephemerides give at intervals of a few days the values of the quantities f , g , and G for the reduction of the place of the body from the mean equinox of the beginning of the year to the true equinox of the date; and, in order to obtain uniformity and accuracy, the beginning of the year is taken at the instant when the mean longitude of the sun is 280° . When these tables are not available, the values of f , g , and G may be found directly by means of the equations (7). The reduction from the true equinox of t' to the mean equinox of t will be obtained by changing the signs of the corrections.

C. Aberration.—The aberration in the case of the planets and comets may be considered in three different modes:—

1. If we subtract from the observed time the interval occupied by

the light in passing to the earth, the result will be the time for which the true place is identical with the apparent place for the observed time.

2. If we compute the time occupied by light in traversing the distance between the body and the earth, and, by means of the rate of the variation of the geocentric spherical co-ordinates, compute the motion during this interval, we may derive the true place at the instant of observation.

3. We may consider the observed place corrected for the aberration of the fixed stars as the true place at the instant when the light was emitted, but as seen from the place of the earth at the instant of observation.

The formulæ for the actual aberration of the fixed stars are—

$$\begin{aligned}\Delta\lambda &= -20''.4451 \cos(\lambda - \odot) \sec \beta - 0''.3429 \cos(\lambda - \Gamma) \sec \beta, \\ \Delta\beta &= +20''.4451 \sin(\lambda - \odot) \sin \beta + 0''.3429 \sin(\lambda - \Gamma) \sin \beta,\end{aligned}\quad (9)$$

in the case of the longitude and latitude, and

$$\begin{aligned}\Delta\alpha &= -20''.4451 (\cos \odot \cos \epsilon \cos \alpha + \sin \odot \sin \alpha) \sec \delta \\ &\quad - 0''.3429 (\cos \Gamma \cos \epsilon \cos \alpha + \sin \Gamma \sin \alpha) \sec \delta, \\ \Delta\delta &= +20''.4451 \cos \odot (\sin \alpha \sin \delta \cos \epsilon - \cos \delta \sin \epsilon) \\ &\quad - 20''.4451 \sin \odot \cos \alpha \sin \delta \\ &\quad + 0''.3429 \cos \Gamma (\sin \alpha \sin \delta \cos \epsilon - \cos \delta \sin \epsilon) \\ &\quad - 0''.3429 \sin \Gamma \cos \alpha \sin \delta,\end{aligned}\quad (10)$$

in the case of the right ascension and declination. In these formulæ Γ denotes the longitude of the sun's perigee, and they give the corrections for the reduction from the true place to the apparent place.

D. Intensity of Light.—If we denote by r the distance of a planet or comet from the sun, by Δ its distance from the earth, and by C a constant quantity depending on the magnitude of the body and on its capacity for reflecting the light, the intensity of the light of the body as seen from the earth will be

$$I = \frac{C}{r^2 \Delta^2}. \quad (11)$$

When the constant C is unknown, we may determine the relative brilliancy of the comet at different times by means of the formula

$$B = \frac{r^2 \Delta^2}{r'^2 \Delta'^2}. \quad (12)$$

In the case of the planets we adopt as the unit of the intensity of light the value of I when the planet is in opposition and both it and the earth are at their mean distances from the sun. Thus we obtain

$$C = a^2 (a - 1)^2,$$

and hence

$$I = \frac{a^2 (a - 1)^2}{r^2 \Delta^2}. \quad (13)$$

Let us now denote by R the ratio of the intensities of the light for two consecutive stellar magnitudes; then, if we denote by M the apparent stellar magnitude of the planet when $I=1$, and by m the magnitude for any value of I , we shall have

$$I = \frac{R^M}{R^m},$$

and hence

$$m = M - \frac{\log I}{\log R}. \quad (14)$$

By means of photometric determinations of the relative brilliancy of the stars, it has been found that

$$R = 2.56,$$

and hence we derive

$$m = M - 2.45 \log I, \quad (15)$$

by means of which the apparent stellar magnitude of a planet may be determined, I being found by means of equation (13). The value of M must be determined for each planet by means of observed values of m .

EXAMPLE.—The value of M for *Eurynome* is 10.4; required the apparent stellar magnitude of the planet when $\log a = 0.38795$, $\log r = 0.2956$, and $\log \Delta = 9.9952$.

The equation (13) gives

$$\log I = 0.5129,$$

and from (15) we derive

$$m = 10.4 - 1.3 = 9.1.$$

For the values $\log r = 0.4338$, $\log \Delta = 0.2357$, we obtain

$$\log I = 9.7555 - 10,$$

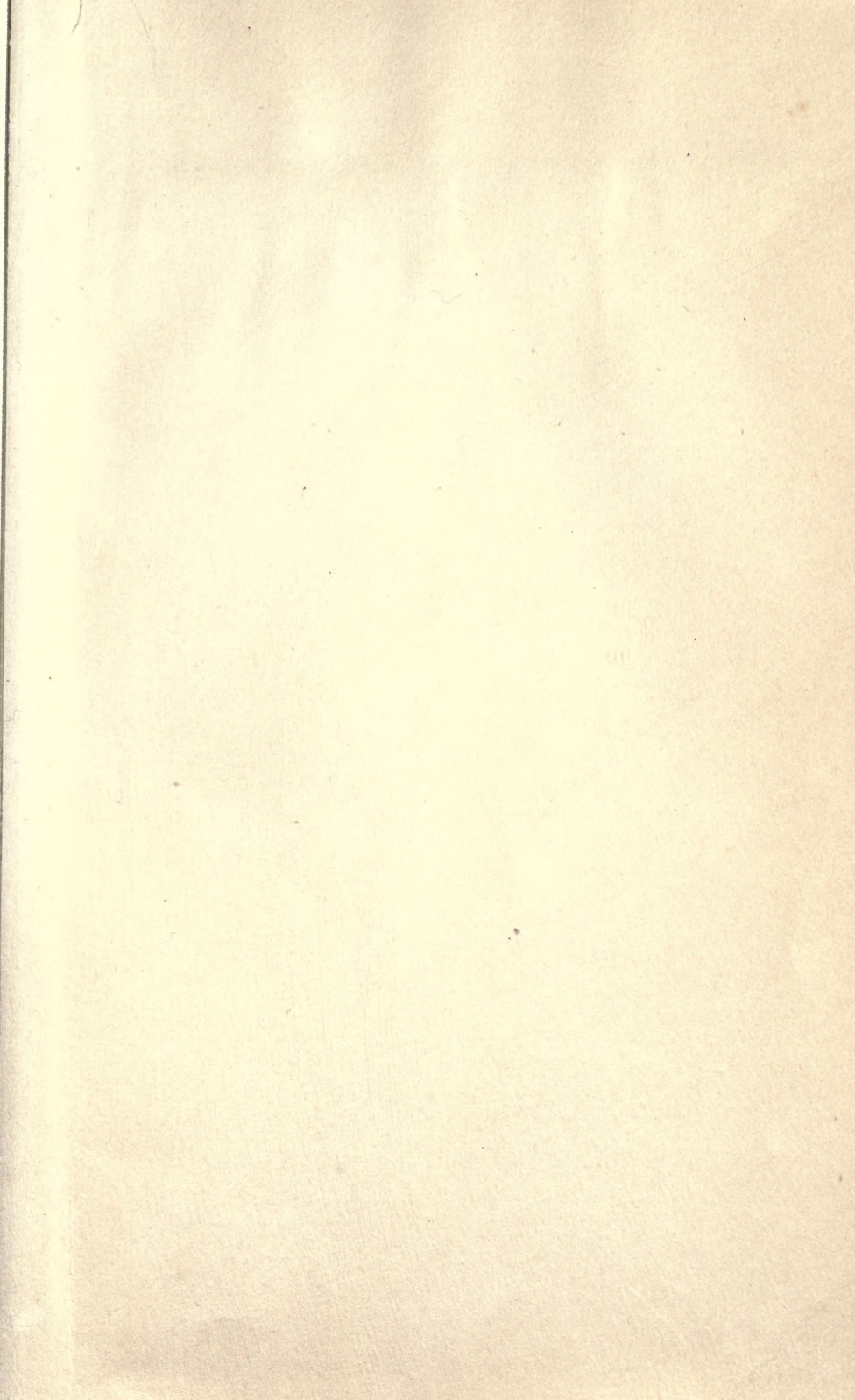
and

$$m = 10.4 + 2.45 \times 0.2445 = 11.0.$$

E. Numerical Calculations.—The extended numerical calculations required in many of the problems of Theoretical Astronomy, render it important that a judicious arrangement of the details should be effected. The beginner will not, in general, be able to effect such an arrangement at the outset; and it would only confuse to attempt to give any specific directions. Familiarity with the formulæ to be applied, and practice in the performance of calculations of this character, will speedily suggest those various devices of arrangement by which skillful computers expedite the mechanical part of the solution. There are, however, a few general suggestions which may be of service. Thus, it will always facilitate the calculation, when several values of a variable are to be computed, to arrange it so that the values of each function involved shall appear in the same vertical or horizontal column. The course of the differences will then indicate the existence of errors which might not otherwise be discovered until the greater part if not the entire calculation has been completed; and, besides, by carrying along the several parts simultaneously the use of the logarithmic and other tables will be facilitated. Numbers which are to be frequently used may be written on slips of paper and applied wherever they may be required; and by performing the addition or subtraction of two logarithms or of two numbers from left to right (which will be effected easily and certainly after a little practice), the sum or difference to be used as the argument in the tables may be retained in the memory, and thus the required number or arc may be written down directly. The number of the decimal figures of the logarithms to be used will depend on the character of the data as well as on the accuracy sought to be obtained, and the use of approximate formulæ will be governed by the same considerations. Whenever the formulæ furnish checks or tests of the accuracy of the numerical process, they should be applied; and whenever these are not provided, the use of differences for the same purpose should not be overlooked. By proper attention to these suggestions, much time and labor will be saved. The agreement of the several proofs will beget confidence, relieve the mind from much anxiety, and thus greatly facilitate the progress of the work.

THE END.





GENERAL LIBRARY
UNIVERSITY OF CALIFORNIA—BERKELEY

RETURN TO DESK FROM WHICH BORROWED

This book is due on the last date stamped below, or on the
date to which renewed.

Renewed books are subject to immediate recall.

4 May '54 MC

JUN 17 1954 LD

5 Jul '60 VD

REC'D LD

JUL 8 1960

25 Nov '62 RA

REC'D LD

NOV 24 1962

6 Jan '64 VE
IN STACKS

DEC 10 1963

REC'D LD

FEB 5 1964

19 Feb '64 DY

IN STACKS

FEB 5 1964

REC'D LD

JUN 9 '64 -1 PM

8 Nov '64 WC

8 Nov '64 WC
IN STACKS

OCT 25 1964

REC'D LD

FEB 4 '65 -10 AM

18 Feb '65 SS

IN STACKS

FEB 4 1965

REC'D LD

JUN 5 '65 -2 PM

Doc 7463 (L 1084) 1.00

LB 4 1965

Doc 7463 (L 1084) 1.00

Doc 7463 (L 1084) 1.00

Doc 7463 (L 1084) 1.00

Doc 7463 (L 1084) 1.00

Doc 7463 (L 1084) 1.00

Doc 7463 (L 1084) 1.00

